

THE GENERALIZED TODA LATTICE EQUATION ON SEMISIMPLE LIE ALGEBRAS

YUJI KODAMA* AND JIAN YE†

(September 20, 1995)

ABSTRACT. In this paper we present some results on the generalized nonperiodic Toda lattice equations. We start with an iso-spectral deformation of general matrix which is a natural generalization of the Toda lattice equation. This deformation is equivalent to the Cholesky flow, a continuous version of the Cholesky algorithm introduced by Watkins. We prove the integrability of the deformation, and give an explicit formula for the solution to the initial value problem. Using the formula, the solution to the LU factorization can be constructed explicitly. Based on the root spaces for simple Lie algebras, we consider several reductions of the equation. This leads to Toda equations related to other classical semisimple Lie algebras which include the integrable systems studied by Kostant. We show these systems can be solved explicitly in a unified way. The behaviors of the solutions are also studied. Generically, there are two types of solutions, having either sorting property or blowing up to infinity in finite time.

Mathematics Subject Classifications (1991). 58F07, 34A05

1. INTRODUCTION

In this paper we describe briefly some results on the generalized nonperiodic Toda equations. Details of the present results can be found in [13]. First we consider an iso-spectral deformation of an arbitrary diagonalizable matrix $L \in \mathfrak{M}(N, \mathbb{R})$. With the deformation parameter $t \in \mathbb{R}$, this is defined by

$$(1.1) \quad \frac{d}{dt}L = [P, L],$$

where P is the generating matrix of the deformation, and is given by a projection of L ,

$$(1.2) \quad P = \Pi(L) := (L)_{>0} - (L)_{<0}.$$

*Department of Mathematics, The Ohio State University, Columbus, OH 43210
E-mail address: kodama@math.ohio-state.edu

† Department of Mathematics, The Ohio State University, Columbus, OH 43210
E-mail address: ye@math.ohio-state.edu

Here $(L)_{>0}$ (<0) denotes the strictly upper (lower) triangular part of L . In terms of the standard basis of $\mathfrak{M}(N, \mathbb{R})$, i.e.

$$(1.3) \quad E_{ij} = (\delta_{ik}\delta_{j\ell})_{1 \leq k, \ell \leq N} ,$$

the matrices L and P are expressed as

$$(1.4) \quad L = \sum_{1 \leq i, j \leq N} a_{ij} E_{ij} ,$$

$$(1.5) \quad P = \sum_{1 \leq i < j \leq N} a_{ij} E_{ij} - \sum_{1 \leq j < i \leq N} a_{ij} E_{ij} .$$

Using the commutation relations for E_{ij} , i.e.

$$(1.6) \quad [E_{ij} , E_{k\ell}] = E_{i\ell}\delta_{jk} - E_{jk}\delta_{i\ell} ,$$

the equations for the components $a_{ij} = a_{ij}(t)$ are written in the form,

$$(1.7) \quad \begin{aligned} \frac{da_{ij}}{dt} &= 2 \left(\sum_{k=I+1}^N - \sum_{k=1}^{J-1} \right) a_{ik} a_{kj} \\ &+ (a_{II} - a_{JJ}) a_{ij} , \end{aligned}$$

where $I := \max(i, j)$ and $J := \min(i, j)$. The equation (1.1) is also defined as the compatibility of the following linear equations with iso-spectral property of L ,

$$(1.8) \quad L\Phi = \Phi\Lambda ,$$

$$(1.9) \quad \frac{d}{dt}\Phi = P\Phi ,$$

where Φ is the eigenmatrix, and Λ is the diagonal matrix of eigenvalues,

$$(1.10) \quad \Lambda = \text{diag}(\lambda_1, \dots, \lambda_N) .$$

The set of equations (1.8) and (1.9) is also referred as the inverse scattering transform for the system (1.1).

In the case of the original nonperiodic Toda lattice equation, L is given by a symmetric tridiagonal matrix [16]. The matrices L and P for this equation are commonly written as

$$(1.11) \quad L_T = \sum_{i=1}^N a_i E_{ii} + \sum_{i=1}^{N-1} b_i (E_{i,i+1} + E_{i+1,i}) ,$$

$$(1.12) \quad P_T = \sum_{i=1}^{N-1} b_i (E_{i,i+1} - E_{i+1,i}) .$$

The integrability of the Toda lattice equation has been shown by the inverse scattering method [8] [15] [16]. In this paper, we call (1.1) with (1.2) the “generalized Toda equation”.

Several generalizations of the Toda lattice equation have been considered. In [2], Bogoyavlensky extended the equation based on the simple roots of semi-simple Lie algebra $\mathfrak{g}$, where L and P were given by

$$(1.13) \quad L_B = \sum_{i=1}^r a_i h_i + \sum_{\alpha \in \Pi} b_\alpha (e_\alpha + e_{-\alpha}),$$

$$(1.14) \quad P_B = \sum_{\alpha \in \Pi} b_\alpha (e_\alpha - e_{-\alpha}).$$

Here the elements $h_i, e_\alpha, e_{-\alpha}$ are Cartan-Weyl bases in $\mathfrak{g}$ with $r = \text{rank}(\mathfrak{g})$ and $\Pi =$ the set of the simple roots. All of these equations associated with semi-simple Lie algebras are shown to be completely integrable hamiltonian systems. In [14] Kostant solved these by using the representation theory of semi-simple Lie algebras. In [1], Bloch et al. showed that these systems can be also written as gradient flow equations on an adjoint orbit of compact Lie group. They then showed that the generic flow assumes the “sorting property” (or convexity). Here the sorting property means that $L(t) \rightarrow \Lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$ as $t \rightarrow \infty$, with the eigenvalues being ordered by $\lambda_1 > \lambda_2 > \dots > \lambda_N$.

There are also other types of extensions: One of them is to extend L_T in (1.11) to a full symmetric matrix. The corresponding system, which we call the “full symmetric Toda equation”, was shown by Deift et al. [5] to be also a complete integrable hamiltonian system. In [11] Kodama and McLaughlin solved the initial value problem of the corresponding inverse scattering problem (1.8) and (1.9), and obtained an explicit formula of the solution in a determinant form. They also showed the sorting property in the generic solution. The full symmetric Toda equation gives a QR-flow defined by [17], and the solution is obtained by the QR factorization method. As a slight generalization of the full symmetric Toda equation, Kodama and Ye [12] considered a system with symmetrizable matrix L , which is expressed as $L_{KY} = L_S S$ with a full symmetric matrix L_S and a diagonal matrix S . A key feature of this system is that the eigenmatrix of L_{KY} can be taken as an element of noncompact group of matrices, such as $O(p, q)$, and defines an indefinite metric in the eigenspace. The integrability was also shown by a similar way as in [11], and the general solution now assumes either sorting property or blowing up to infinity in finite time as a result of the indefinite metric. This system is equivalent to the HR-flow, a continuous version of the HR algorithm introduced by Watkins [20].

In [7], Ercolani et al. proposed the equation (1.1) with matrices,

$$(1.15) \quad L_H = \sum_{i=1}^{N-1} E_{i,i+1} + \sum_{1 \leq j \leq i \leq N} b_{ij} E_{ij},$$

$$(1.16) \quad P_H = -2(L_H)_{<0} = -2 \sum_{1 \leq j < i \leq N} b_{ij} E_{ij},$$

which was called the “full Kostant-Toda equation”. This is also an extension of the Toda

equation (1.11) which can be written in the form,

$$(1.17) \quad \tilde{L}_T = \sum_{i=1}^{N-1} E_{i,i+1} + \sum_{i=1}^N a_i E_{ii} + \sum_{i=1}^{N-1} b_i^2 E_{i+1,i}.$$

As we will show in this paper, the transformation from (1.11) to (1.17) is given by a rescaling of the eigenvectors of L_T . Several group theoretical structure of the extended system were found. However the question whether the system is completely integrable still remains open in a sense of explicit integration.

It is immediate but important to observe that all of these extensions are special reductions of the generalized Toda equation (1.1). In fact, we show that these reductions are obtained more systematically as certain decompositions of the root spaces of semi-simple Lie algebras.

In terms of the matrix factorization, the generalized Toda equation (1.1) with (1.2) is equivalent to the Cholesky flow introduced by Watkins in [20]. In fact, writing P in (1.2) as $P = L - 2(L)_{<0} - (L)_0$, where $(L)_0$ denotes the diagonal part of L , we see that the equation (1.1) is the same as the Cholesky flow in [20] except a scale of t by 2. Deift et. al. showed [6] that the Cholesky flow is a completely integrable hamiltonian system, and it can be solved by the following LU-type of matrix factorization:

$$(1.18) \quad e^{tL(0)} = V(t)W(t),$$

where $V(t)$ and $W(t)$ are lower and upper matrices respectively with $\text{diag}(V(t)) = \text{diag}(W(t))$. Note that the usual LU factorization has a different normalization in the diagonal part, $\text{diag}(V(t)) = \text{diag}(1, \dots, 1)$. Then the solution $L(t)$ is given by

$$(1.19) \quad L(t) = V^{-1}(t)L(0)V(t) = W(t)L(0)W^{-1}(t).$$

The above solution is not explicit in the sense of (1.18). The explicit formula of the factorization is a direct consequence of our results.

In this paper we first show the complete integrability of (1.1) with (1.4) and (1.5) by means of the method of inverse scattering transform and give an explicit solution to the initial value problem. Then we prove the complete integrability of any reductions of (1.1) which include generalized Toda equations based on other classical semi-simple Lie algebras. The content of this paper is as follows: We start with a preliminary in Section 2 to give some background information necessary for analysis of the system (1.1) and the inverse scattering scheme (1.8) and (1.9).

In Section 3, we give solutions to the initial value problem of (1.9) for the general system (1.1). A key in the method is to use the orthonormalization procedure of Szegő, which is equivalent to the Gram-Schmidt orthogonalization method. This shows the complete integrability of the generalized Toda equation in the sense of inverse scattering transformation method. Based on our explicit solution, we then give an explicit solution to the Cholesky factorization (1.18).

In Section 4, we present reductions of (1.1) according to the classification of semi-simple Lie algebras. The matrix L here then contains “all” the root vectors, and it gives a generalization of the systems formulated by Bogoyavlensky [2]. A key ingredient here is to find a matrix representation of the algebra in a decomposition consisting of diagonal, strictly upper and lower matrices (Lie’s Theorem [10]). Then the integrability of these systems associated with semi-simple Lie algebras is a direct consequence of the result in Section 3.

Section 5 provides other reductions which include the full Kostant-Toda equation and system with a matrix L having band structure in the elements.

In Section 6, we discuss the behavior of the solutions. Generically, in addition to the sorting property, there are solutions blowing up to infinity in finite time, as in the case discussed in [12].

Finally we illustrate the results obtained in this paper with explicit examples in Section 7.

2. PRELIMINARY

Here we give some background information necessary for the inverse scattering method (1.8) and (1.9). As we will see in the next section, a key idea for solving these equations is to use an orthogonality of the eigenfunctions of (1.8). This is simply to consider a dual system of (1.8) and (1.9), which are written by

$$(2.1) \quad \Psi L = \Lambda \Psi ,$$

$$(2.2) \quad \frac{d}{dt} \Psi = - \Psi P ,$$

where the matrix Ψ is taken to be Φ^{-1} , and of course

$$(2.3) \quad \Psi \Phi = I, \quad \Phi \Psi = I .$$

In terms of the eigenvectors, these matrices can be expressed as

$$(2.4) \quad \Phi \equiv [\phi(\lambda_1), \dots, \phi(\lambda_N)] = [\phi_i(\lambda_j)]_{1 \leq i, j \leq N} .$$

$$(2.5) \quad \Psi \equiv [\psi^T(\lambda_1), \dots, \psi^T(\lambda_N)]^T = [\psi_j(\lambda_i)]_{1 \leq i, j \leq N} .$$

Note here that $\phi(\lambda_i)$ and $\psi(\lambda_i)$ are the column and row eigenvectors, respectively. Then the equations (2.3) give

$$(2.6) \quad \sum_{k=1}^N \psi_k(\lambda_i) \phi_k(\lambda_j) = \delta_{ij} ,$$

$$(2.7) \quad \sum_{k=1}^N \phi_i(\lambda_k) \psi_j(\lambda_k) = \delta_{ij} ,$$

which are called the “first and second orthogonality conditions”. With (2.7), one can define an inner product $\langle \cdot, \cdot \rangle$ for two functions f and g of λ as

$$(2.8) \quad \langle f, g \rangle := \sum_{k=1}^N f(\lambda_k) g(\lambda_k),$$

which we hereafter write as $\langle fg \rangle$. From $L = \Phi \Lambda \Psi$, the entries of L are then expressed by

$$(2.9) \quad a_{ij} := (L)_{ij} = \langle \lambda \phi_i \psi_j \rangle.$$

This gives a key equation for the inverse problem where we compute L from the eigenmatrix Φ (and Ψ) with the eigenvalues λ_i . So the eigenmatrix plays the role of the scattering data in the inverse scattering method. Then the method for solving the initial value problem of the equation (1.1) can be formulated as follows: First we solve the eigenvalue (or scattering) problem (1.8) at $t = 0$, and find the scattering data, $\Phi^0 := \Phi(0)$. Then we solve the time evolution of the eigenmatrix from (1.9), and with the solution $\Phi(t)$ we obtain $L(t)$ through the equation (2.9).

3. INVERSE SCATTERING METHOD

In this section, we construct an explicit solution formula for the initial value problem of the generalized Toda equation (1.1) by using the inverse scattering method. A key of this method is to generalize the orthogonalization procedure of Szegő with respect to the inner product (2.8). This is essentially an extension of the method developed in [11].

Following [11] we first “gauge” transform Φ and Ψ by

$$(3.1) \quad \Phi = G \tilde{\Phi}, \quad \Psi = \tilde{\Psi} G$$

where the matrix G is given by

$$G = \text{diag} \left[\langle \tilde{\phi}_1 \tilde{\psi}_1 \rangle^{-1/2}, \dots, \langle \tilde{\phi}_N \tilde{\psi}_N \rangle^{-1/2} \right].$$

Note that the gauge transform (3.1) includes a freedom in the choice of $\tilde{\phi}$ and $\tilde{\psi}$, that is, (3.1) is invariant under the scaling $\tilde{\phi}_i, \tilde{\psi}_i \rightarrow f_i(\mathbf{t}) \tilde{\phi}_i, f_i(\mathbf{t}) \tilde{\psi}_i$, with $\{f_i\}_{i=1}^N$ arbitrary functions of $\mathbf{t}$. With (3.1), the equations (1.8) and (1.9), as well as (2.1) and (2.2), become

$$(3.2) \quad (G^{-1} L G) \tilde{\Phi} = \tilde{\Phi} \Lambda, \quad \tilde{\Psi} (G L G^{-1}) = \Lambda \tilde{\Psi},$$

$$(3.3) \quad \frac{d}{dt} \tilde{\Phi} = (G^{-1} P G) \tilde{\Phi} - \left(\frac{d}{dt} \log G \right) \tilde{\Phi},$$

$$(3.4) \quad \frac{d}{dt} \tilde{\Psi} = - \tilde{\Psi} (G P G^{-1}) - \tilde{\Psi} \left(\frac{d}{dt} \log G \right).$$

Noting $G^{-1}(L)_{<0}G = (G^{-1}LG)_{<0}$ etc, we write

$$\begin{aligned} G^{-1}PG &= -2(G^{-1}LG)_{<0} + G^{-1}LG - \text{diag}(L) , \\ GPG^{-1} &= 2(GLG^{-1})_{>0} - GLG^{-1} + \text{diag}(L) , \end{aligned}$$

from which we obtain the equations for the column vectors $\tilde{\phi}(\lambda, t)$ in $\tilde{\Phi}$ and the row vectors $\tilde{\psi}(\lambda, t)$ in $\tilde{\Psi}$,

$$(3.5) \quad \frac{d\tilde{\phi}}{dt} = -2(G^{-1}LG)_{<0} \tilde{\phi} + \lambda \tilde{\phi} - \left(\text{diag}(L) + \frac{d}{dt} \log G \right) \tilde{\phi} ,$$

$$(3.6) \quad \frac{d\tilde{\psi}}{dt} = -2\tilde{\psi} (GLG^{-1})_{>0} + \lambda \tilde{\psi} - \tilde{\psi} \left(\text{diag}(L) + \frac{d}{dt} \log G \right) .$$

We here observe that (3.5) and (3.6) can be split into the following sets of equations by fixing the gauge freedom in the determination of ϕ and ψ . In the components, these are

$$(3.7) \quad \frac{d\tilde{\phi}_i}{dt} = -2 \sum_{j=1}^{i-1} \frac{\langle \lambda \tilde{\phi}_i \tilde{\psi}_j \rangle}{\langle \tilde{\phi}_j \tilde{\psi}_j \rangle} \tilde{\phi}_j + \lambda \tilde{\phi}_i ,$$

$$(3.8) \quad \frac{d\tilde{\psi}_j}{dt} = -2 \sum_{i=1}^{j-1} \tilde{\psi}_i \frac{\langle \lambda \tilde{\phi}_i \tilde{\psi}_j \rangle}{\langle \tilde{\phi}_i \tilde{\psi}_i \rangle} + \lambda \tilde{\psi}_j ,$$

$$(3.9) \quad \frac{1}{2} \frac{d}{dt} \log \langle \tilde{\phi}_i \tilde{\psi}_i \rangle = a_{ii} .$$

It is easy to check that (3.7) or (3.8) implies (3.9). It is also immediate from (3.7) and (3.8) that we have:

Proposition 1. *The solutions of (3.7) and (3.8) can be written in the following forms of separation of variables,*

$$(3.10) \quad \tilde{\phi}(\lambda, t) = M(t) \phi^0(\lambda) e^{\lambda t} ,$$

$$(3.11) \quad \tilde{\psi}(\lambda, t) = \psi^0(\lambda) N(t) e^{\lambda t} ,$$

where $M(t)$ and $N(t)$ are, respectively, lower and upper triangular matrices with $\text{diag}[M(t)] = \text{diag}[N(t)] = I$, the identity matrix.

Note here that the initial data for $\tilde{\phi}$ and $\tilde{\psi}$ are chosen as those of ϕ and ψ , i.e. $\tilde{\phi}(\lambda, 0) = \phi(\lambda, 0) := \phi^0(\lambda)$ and $\tilde{\psi}(\lambda, 0) = \psi(\lambda, 0) := \psi^0(\lambda)$. As a direct consequence of this proposition, and the orthogonality of the eigenvectors, (2.7), i.e. $\langle \tilde{\phi}_i \tilde{\psi}_j \rangle = 0$ for $i \neq j$, we have:

Corollary 1. (Orthogonality): *For each $i, j \in \{2, \dots, N\}$, we have for all $t \in \mathbb{R}$,*

$$(3.12) \quad \langle \tilde{\phi}_i \psi_\ell^0 e^{\lambda t} \rangle = 0 , \text{ for } \ell = 1, 2, \dots, i-1 ,$$

$$(3.13) \quad \langle \phi_k^0 \tilde{\psi}_j e^{\lambda t} \rangle = 0 , \text{ for } k = 1, 2, \dots, j-1 .$$

Now we obtain the formulae for the eigenvectors of L in terms of the initial data $\{\phi_i^0(\lambda)\}_{1 \leq i \leq N}$ and $\{\psi_j^0(\lambda)\}_{1 \leq j \leq N}$:

Theorem 1. *The solutions $\tilde{\phi}_i(\lambda, t)$ and $\tilde{\psi}_j(\lambda, t)$ of (3.7) and (3.8) are given by*

$$(3.14) \quad \tilde{\phi}_i(\lambda, t) = \frac{e^{\lambda t}}{D_{i-1}(t)} \begin{vmatrix} c_{11} & \dots & c_{1,i-1} & \phi_1^0(\lambda) \\ \vdots & \ddots & \vdots & \vdots \\ c_{i1} & \dots & c_{i,i-1} & \phi_i^0(\lambda) \end{vmatrix},$$

$$(3.15) \quad \tilde{\psi}_j(\lambda, t) = \frac{e^{\lambda t}}{D_{j-1}(t)} \begin{vmatrix} c_{11} & \dots & c_{1j} \\ \vdots & \ddots & \vdots \\ c_{j-1,1} & \dots & c_{j-1,j} \\ \psi_1^0(\lambda) & \dots & \psi_j^0(\lambda) \end{vmatrix}$$

where $c_{ij}(t) = \langle \phi_i^0 \psi_j^0 e^{2\lambda t} \rangle$, and $D_k(t)$ is the determinant of the $k \times k$ matrix with entries $c_{ij}(t)$, i.e.,

$$(3.16) \quad D_k(t) = \det \left[\left(c_{ij}(t) \right)_{1 \leq i, j \leq k} \right].$$

(Note here that $c_{ij}(0) = \delta_{ij}$ and $D_k(0) = 1$.)

We then note:

Corollary 2. *The gauge factors $\langle \tilde{\phi}_i \tilde{\psi}_i \rangle$ can be expressed by*

$$(3.17) \quad \langle \tilde{\phi}_i \tilde{\psi}_i \rangle(t) = \frac{D_i(t)}{D_{i-1}(t)}.$$

This yields the formulae for the normalized eigenfunctions

$$(3.18) \quad \phi_i(\lambda, t) = \frac{e^{\lambda t}}{\sqrt{D_i(t)D_{i-1}(t)}} \begin{vmatrix} c_{11} & \dots & c_{1,i-1} & \phi_1^0(\lambda) \\ \vdots & \ddots & \vdots & \vdots \\ c_{i1} & \dots & c_{i,i-1} & \phi_i^0(\lambda) \end{vmatrix},$$

$$(3.19) \quad \psi_j(\lambda, t) = \frac{e^{\lambda t}}{\sqrt{D_j(t)D_{j-1}(t)}} \begin{vmatrix} c_{11} & \dots & c_{1j} \\ \vdots & \ddots & \vdots \\ c_{j-1,1} & \dots & c_{j-1,j} \\ \psi_1^0(\lambda) & \dots & \psi_j^0(\lambda) \end{vmatrix}.$$

With the formula (3.18) and (3.19), we now have the solution (2.9) of the inverse scattering problem (1.8) and (1.9).

The above derivation of the eigenvectors is the same as the orthogonalization procedure of Szegő [19], which is equivalent to the Gram - Schmidt orthogonalization as observed in [11].

To see the connection with the LU factorization method (1.18), we have the following corollary from $\Phi(t)\Psi(t) = I$:

Corollary 3. *The matrices $V(t)$ and $W(t)$ in the LU-type factorization (1.18) can be explicitly represented by*

$$(3.20) \quad V(t) = M^{-1}(t)G^{-1}(t), \quad \text{and } W(t) = G^{-1}(t)N^{-1}(t),$$

where $G(t)$ is the gauge matrix in (3.1), $M(t)$ is the lower triangular matrix in (3.10) and $N(t)$ is the upper triangular matrix in (3.11). (Here we normalized $2t$ to t in $\exp(tL(0))$.)

Remark 1. The generalized Toda equation (1.1) with (1.2) possesses a hierarchy defined by

$$(3.21) \quad \frac{\partial}{\partial t_n} L = [P_n, L], \quad n = 1, 2, \dots,$$

where P_n is given by

$$(3.22) \quad P_n = \Pi(L^n) \equiv (L^n)_{>0} - (L^n)_{<0}.$$

The commutativity of these flows can be shown by the “zero” curvature conditions of P_n , i.e.

$$(3.23) \quad \frac{\partial P_m}{\partial t_n} - \frac{\partial P_n}{\partial t_m} + [P_m, P_n] = 0,$$

which is a direct consequence of the choice of (3.22) [12]. The solution for the hierarchy is then obtained by extending the argument λt in the solution formula to $\xi(\lambda, t) := \sum_{n=1}^{\infty} \lambda^n t_n$ [12].

Remark 2. The well known QR flow for a general matrix $L \in \mathfrak{M}(N, \mathbb{R})$ is in the same form as (1.1) with the following generating matrix P :

$$(3.24) \quad P = (L)_{>0} - (L^T)_{<0} = (L)_{>0} - [(L)_{>0}]^T.$$

It has been studied extensively in [17], [18], [4], [6] and [20]. They showed that this equation is completely integrable hamiltonian system and can be solved in the sense of a matrix factorization of QR type, and the solution converges to a matrix in the triangular form. Our method developed in this section can be also applied to this problem as follows: First we note that the product $\Phi^* \Phi$ of the eigenmatrix Φ and its adjoint $\Phi^* := \overline{\Phi}^T$ is invariant under this flow (1.1). Then we define a hermitian matrix $S = (s_{ij})_{1 \leq i, j \leq N}$ as the inverse of $\Phi^* \Phi$, i.e.

$$(3.25) \quad \Phi S \Phi^* = I.$$

The matrix S is determined from the initial eigenmatrix Φ^0 , and $S \Phi^*$ gives the inverse of Φ , that is, we have $S \Phi^*$ for Ψ in our method. Note that if L is symmetric, S is an identity matrix I and $\Phi \in O(N)$. In general, we see from the Binet-Cauchy theorem that S is positive definite. The equation (3.25) now gives the orthogonality relation,

$$(3.26) \quad \sum_{1 \leq k, \ell \leq N} \phi_i(\lambda_k) s_{k\ell} \overline{\phi_j(\lambda_\ell)} = \delta_{ij},$$

from which we define the following inner product as in (2.8),

$$(3.27) \quad \ll f, g \gg := \sum_{1 \leq k, \ell \leq N} f(\lambda_k) s_{k\ell} \overline{g(\lambda_\ell)} = \overline{\ll g, f \gg}.$$

This leads to a positive definite metric. Then following the procedure in this section with some modifications based on $\Psi = S\Phi^*$, we obtain the same result for the eigenvectors (3.18) except the quantities c_{ij} which is now given by

$$(3.28) \quad c_{ij} = \ll \phi_i^0 e^{\lambda_i t}, \phi_j^0 e^{\lambda_j t} \gg = \bar{c}_{ji}.$$

The solution $L(t)$ is then given by $L(t) = \Phi \Lambda S \Phi^*$, i.e.

$$(3.29) \quad a_{ij}(t) = \ll \lambda \phi_i, \phi_j \gg(t).$$

Thus, we can show explicitly the integrability of the equation (1.1) with the generator P given by (3.24) for arbitrary diagonal matrix L , and as a result of the positivity in the metric, the solution converges to a upper triangular matrix.

Remark 3. From Corollary 3, the usual LU factorization of $e^{tL(0)}$ can be written as

$$(3.30) \quad e^{tL(0)} = (M^{-1})(G^{-2}N^{-1}).$$

One verifies that M^{-1} is lower triangular with $\text{diag}(M^{-1}) = I$ and $G^{-2}N^{-1}$ is upper triangular with positive diagonal entries.

4. ISOSPECTRAL FLOWS ON SIMPLE LIE ALGEBRAS

In this section, we consider the generalized Toda equations (1.1) associated with simple Lie algebras $\mathfrak{g}$, and show their integrability. The matrices L and P here are given by a generalization of (1.13) and (1.14), i.e.

$$(4.1) \quad L_{\mathfrak{g}} = \sum_{i=1}^r a_i h_i + \sum_{\alpha \in \Delta^+} b_{\alpha} e_{\alpha} + \sum_{\beta \in \Delta^-} c_{\beta} e_{\beta},$$

$$(4.2) \quad P_{\mathfrak{g}} = \sum_{\alpha \in \Delta^+} b_{\alpha} e_{\alpha} - \sum_{\beta \in \Delta^-} c_{\beta} e_{\beta}.$$

Here h_i are the bases for the Cartan subalgebra with $r = \text{rank}(\mathfrak{g})$, Δ^+ and Δ^- are the sets of positive and negative roots with the corresponding root vectors e_{α} and $e_{\beta}(=e_{-\alpha})$. These vectors $\{h_i, e_{\alpha}\}$ form the Cartan-Weyl bases of the simple Lie algebra $\mathfrak{g}$ which satisfy for $i, j \in \{1, \dots, r\}$ and $\alpha, \beta \in \Delta := \Delta^+ \cup \Delta^-$

$$(4.3) \quad \begin{aligned} [h_i, h_j] &= 0, \quad [h_i, e_{\alpha}] = \alpha(h_i) e_{\alpha}, \\ [e_{\alpha}, e_{\beta}] &= N_{\alpha\beta} e_{\alpha+\beta}, \quad \text{if } \alpha + \beta \in \Delta, \\ [e_{\alpha}, e_{-\alpha}] &= h_{\alpha}, \quad \text{for } \alpha \in \Delta^+. \end{aligned}$$

Using representations of the Cartan-Weyl bases, we now express (4.1) and (4.2) in matrix form for each simple Lie algebra. Then we prove that the equation (1.1) with those $L_{\mathfrak{g}}$ and $P_{\mathfrak{g}}$ associated with the Lie algebra $\mathfrak{g}$ is completely integrable by the inverse scattering method developed in Section 3. The key ingredient in the proof is to show that for each

simple Lie algebra $\mathfrak{g}$ there exists a “permutation” matrix $O_{\mathfrak{g}}$ such that the matrices $L_{\mathfrak{g}}$ and $P_{\mathfrak{g}}$ are similar to L and P in (1.1) with P defined by (1.2), i.e.

$$(4.4) \quad L = O_{\mathfrak{g}} L_{\mathfrak{g}} O_{\mathfrak{g}}^T,$$

$$(4.5) \quad P = O_{\mathfrak{g}} P_{\mathfrak{g}} O_{\mathfrak{g}}^T = \Pi(L).$$

In another word, we look for a similarity transform such that the matrix representations of e_{α} for $\alpha \in \Delta^+$ and e_{β} for $\beta \in \Delta^-$ are transformed to strictly upper and lower triangular matrices, respectively. The existence of such representations is due to Lie's theorem [10]. Then the result in Section 3 implies the integrability of the system (1.1) with $L_{\mathfrak{g}}$ and $P_{\mathfrak{g}}$. Note here that the generalized Toda equation is invariant under the similarity transform. Here we consider all the classical simple Lie algebras, A_n, B_n, C_n and D_n . The system associated with the exceptional algebra can be treated as the same way. For convenient matrix representations of the Cartan-Weyl bases, we follow the notations in [3] and [10].

A_n : Let E_{ij} be the $(n+1) \times (n+1)$ matrix defined in (1.3). We then take an element of the Cartan subalgebra as $h = \sum_{i=1}^{n+1} \lambda_i E_{ii}$ with $\sum_{i=1}^{n+1} \lambda_i = 0$. Using (1.6) for E_{ij} , we have

$$(4.6) \quad [h, E_{ij}] = (\lambda_i - \lambda_j) E_{ij}.$$

Thus E_{ij} gives a root vector corresponding to the root $\alpha(h) = \lambda_i - \lambda_j$. The simple roots are defined as

$$(4.7) \quad \alpha_k(h) = \lambda_k - \lambda_{k+1}, \text{ for } k = 1, \dots, n.$$

Then the positive (negative) roots are given by $\lambda_i - \lambda_j$ with $i < j$ ($i > j$). This implies that the choice of the P_{A_n} is the same as that in (1.2). Note also that adding some constant to the Cartan subalgebra, one can choose h_i of the basis to be E_{ii} . Namely, the generalized Toda equation (1.1) with (1.4) and (1.5) can be considered as an iso-spectral flow on the simple Lie algebra A_n .

C_m : The element of this algebra is given by a $2m \times 2m$ matrix X satisfying $X^T J + JX = 0$ where J is defined by

$$(4.8) \quad J = \begin{pmatrix} 0_m & I_m \\ -I_m & 0_m \end{pmatrix}.$$

Here 0_m is the $m \times m$ 0-matrix, and I_m is the $m \times m$ identity matrix. We then choose the following bases with the $2m \times 2m$ matrix E_{ij} defined in (1.3),

$$(4.9) \quad \begin{aligned} e_{ij}^1 &= E_{ij} - E_{j+m, i+m}, \quad 1 \leq i, j \leq m, \\ e_{ij}^2 &= E_{i, j+m} + E_{j, i+m}, \quad 1 \leq i \leq j \leq m, \\ e_{ij}^3 &= E_{i+m, j} + E_{j+m, i}, \quad 1 \leq i \leq j \leq m. \end{aligned}$$

Writing $h = \sum_{i=1}^m \lambda_i e_{ii}^1$ as a general element of the Cartan subalgebra, we have

$$(4.10) \quad \begin{aligned} [h, e_{ij}^1] &= (\lambda_i - \lambda_j) e_{ij}^1, \quad i \neq j, \\ [h, e_{ij}^2] &= (\lambda_i + \lambda_j) e_{ij}^2, \quad i \leq j, \\ [h, e_{ij}^3] &= -(\lambda_i + \lambda_j) e_{ij}^3, \quad i \leq j. \end{aligned}$$

The simple roots are taken by

$$(4.11) \quad \begin{aligned} \alpha_k(h) &= \lambda_k - \lambda_{k+1}, \text{ for } 1 \leq k \leq m-1, \\ \alpha_m(h) &= 2\lambda_m, \end{aligned}$$

from which the sets of positive and negative root vectors $\Sigma_{C_m}^+$ and $\Sigma_{C_m}^-$ are given by

$$(4.12) \quad \Sigma_{C_m}^+ = \{ e_{ij}^1, e_{k\ell}^2 \mid 1 \leq i < j \leq m, 1 \leq k \leq \ell \leq m \},$$

$$(4.13) \quad \Sigma_{C_m}^- = \{ e_{ij}^1, e_{k\ell}^3 \mid 1 \leq j < i \leq m, 1 \leq k \leq \ell \leq m \}.$$

Then the matrix L_{C_m} can be represented by

$$(4.14) \quad L_{C_m} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

where $A_1, \dots, A_4$ are the $m \times m$ matrices satisfying the relations

$$(4.15) \quad A_1^T = -A_4, \quad A_2 = A_2^T, \quad A_3 = A_3^T.$$

The matrix P_{C_m} is now given by

$$(4.16) \quad P_{C_m} = \begin{pmatrix} \Pi(A_1) & A_2 \\ -A_3 & -\Pi(A_4) \end{pmatrix}.$$

We then obtain:

Proposition 2. *With the permutation matrix O_{C_m} , we have the generalized Toda equation (1.1) on C_m with L - P pair given by*

$$(4.17) \quad L = O_{C_m} L_{C_m} O_{C_m}^T,$$

$$(4.18) \quad P = O_{C_m} P_{C_m} O_{C_m}^T = \Pi(L),$$

where O_{C_m} is given by

$$(4.19) \quad O_{C_m} = \begin{pmatrix} I_m & 0_m \\ 0_m & Q_m \end{pmatrix},$$

with the $m \times m$ matrix Q_m

$$(4.20) \quad Q_m = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ 0 & \vdots & \vdots & 0 \\ 1 & \dots & 0 & 0 \end{pmatrix} = Q_m^T.$$

Example 1: We take the simplest case C_2 . The matrices L_{C_2} and P_{C_2} are represented as

$$(4.21) \quad L_{C_2} = \begin{pmatrix} a_1 & b_1 & b_2 & b_4 \\ c_1 & a_2 & b_4 & b_3 \\ c_2 & c_4 & -a_1 & -c_1 \\ c_4 & c_3 & -b_1 & -a_2 \end{pmatrix},$$

and

$$(4.22) \quad P_{C_2} = \begin{pmatrix} 0 & b_1 & b_2 & b_4 \\ -c_1 & 0 & b_4 & b_3 \\ -c_2 & -c_4 & 0 & c_1 \\ -c_4 & -c_3 & -b_1 & 0 \end{pmatrix}.$$

Under the similarity transformation with O_{C_2} defined in (4.19), L_{C_2} and P_{C_2} becomes

$$(4.23) \quad L = O_{C_2} L_{C_2} O_{C_2}^T = \begin{pmatrix} a_1 & b_1 & b_4 & b_2 \\ c_1 & a_2 & b_3 & b_4 \\ c_4 & c_3 & -a_2 & -b_1 \\ c_2 & c_4 & -c_1 & -a_1 \end{pmatrix},$$

and

$$(4.24) \quad P = O_{C_2} P_{C_2} O_{C_2}^T = \begin{pmatrix} a_1 & b_1 & b_4 & b_2 \\ -c_1 & a_2 & b_3 & b_4 \\ -c_4 & -c_3 & -a_2 & -b_1 \\ -c_2 & -c_4 & c_1 & -a_1 \end{pmatrix}.$$

Note here that under the similarity transformation the root space is decomposed into the diagonal, upper and lower triangular parts of the matrix (Lie's theorem).

Similarly, for D_m and B_m , we have the following two propositions:

Proposition 3. *With the permutation matrix $O_{D_m} = O_{C_m}$ given in (4.19), we have*

$$(4.25) \quad L = O_{D_m} L_{D_m} O_{D_m}^T,$$

$$(4.26) \quad P = O_{D_m} P_{D_m} O_{D_m}^T = \Pi(L).$$

where L_{D_m} is a $2m \times 2m$ matrix expressed as

$$(4.27) \quad L_{D_m} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

with the $m \times m$ matrices $A_1, \dots, A_4$ satisfy

$$(4.28) \quad A_1^T = -A_4, \quad A_2 = -A_2^T, \quad A_3 = -A_3^T,$$

and P_{D_m} is given by

$$(4.29) \quad P_{D_m} = \begin{pmatrix} \Pi(A_1) & A_2 \\ -A_3 & -\Pi(A_4) \end{pmatrix}.$$

Proposition 4. *With the $(2m+1) \times (2m+1)$ permutation matrix O_{B_m} , we have*

$$(4.30) \quad L = O_{B_m} L_{B_m} O_{B_m}^T,$$

$$(4.31) \quad P = O_{B_m} P_{B_m} O_{B_m}^T = \Pi(L).$$

where O_{B_m} is given by

$$(4.32) \quad O_{B_m} = \begin{pmatrix} \mathbf{0} & I_m & 0_m \\ 1 & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0} & 0_m & Q_m \end{pmatrix},$$

L_{B_m} is a $(2m+1) \times (2m+1)$ matrix expressed as

$$(4.33) \quad L_{B_m} = \begin{pmatrix} 0 & \mathbf{b}_1^T & \mathbf{b}_2^T \\ -\mathbf{b}_2 & A_1 & A_2 \\ -\mathbf{b}_1 & A_3 & A_4 \end{pmatrix},$$

where $\mathbf{b}_1, \mathbf{b}_2$ are the m -column vectors, and the $m \times m$ matrices $A_1, \dots, A_4$ satisfy the same relations as (4.28). and P_{B_m} is given by

$$(4.34) \quad P_{B_m} = \begin{pmatrix} 0 & -\mathbf{b}_1^T & \mathbf{b}_2^T \\ -\mathbf{b}_2 & \Pi(A_1) & A_2 \\ \mathbf{b}_1 & -A_3 & -\Pi(A_4) \end{pmatrix}.$$

5. REDUCTIONS ON ROOT SPACES

As we have explained in the introduction, several generalizations of the Toda equation may be obtained by taking reductions of the generalized Toda equation (1.1) with general matrix L . We then showed in the previous section that the equations on simple Lie algebras studied in [2] are generalized by taking all the root vectors in the algebras. In this section, we consider reductions of these equations by restricting the set of roots in the sums in (4.1).

Let S^+ and S^- be subsets of positive and negative roots of a simple Lie algebra $\mathfrak{g}$ defined by, for $\forall \alpha_0 \in S^+$ and $\forall \beta_0 \in S^-$,

$$(5.1) \quad S^+ := \{ \alpha \in \Delta^+ \mid \alpha \prec \alpha_0 \},$$

$$(5.2) \quad S^- := \{ \beta \in \Delta^- \mid \beta \succ \beta_0 \}.$$

Here “ $\prec$ ” and “ $\succ$ ” are the standard partial orderings between roots. We then consider the equation (1.1) with the matrices $\hat{L}$ and $\hat{P}$ given by

$$(5.3) \quad \hat{L} = \sum_{i=1}^n a_i h_i + \sum_{\alpha \in S^+} b_\alpha e_\alpha + \sum_{\beta \in S^-} c_\beta e_\beta,$$

$$(5.4) \quad \hat{P} = \sum_{\alpha \in S^+} b_\alpha e_\alpha - \sum_{\beta \in S^-} c_\beta e_\beta,$$

where $n = \text{rank}(\mathfrak{g})$. We have:

Proposition 5. *The equation (1.1) with $\hat{L}$ and $\hat{P}$ is a reduction of the generalized Toda equation on $\mathfrak{g}$.*

Example 3: The generalized Toda equation with band matrix L .

This example can be obtained as the following reduction on A_{N-1} : Consider the subsets of the roots S^+ and S^- given by

$$(5.5) \quad S^+ = \{ (i, j) \in \Delta^+ \mid 0 < j - i \leq M^+ \leq N - 1 \},$$

$$(5.6) \quad S^- = \{ (i, j) \in \Delta^- \mid 0 < i - j \leq M^- \leq N - 1 \},$$

where M^+ and M^- are some positive integers. Then the corresponding matrix $\hat{L}$ which we denote $L_{(M^+, M^-)}$ becomes

$$(5.7) \quad L_{(M^+, M^-)} = \begin{pmatrix} a_{11} & \dots & a_{1,1+M^+} & 0 & \dots & 0 \\ \vdots & \ddots & \dots & \ddots & \dots & \vdots \\ a_{1+M^-,1} & \dots & \dots & \dots & \ddots & 0 \\ 0 & \ddots & \dots & \ddots & \dots & a_{N-M^+,N} \\ \vdots & \dots & \ddots & \dots & \ddots & \vdots \\ 0 & \dots & 0 & a_{N,N-M^-} & \dots & a_{NN} \end{pmatrix}.$$

As a special case of this example, we now construct the full Kostant-Toda equation having L_H - P_H pair given in (1.15) and (1.16). Here we choose S^+ and S^- to be the sets of the simple roots (i.e. $M^+ = 1$) and of all the negative roots (i.e. $M^- = N - 1$), respectively. Thus the corresponding matrix is expressed as

$$(5.8) \quad L_{(1, N-1)} = \begin{pmatrix} a_{11} & b_1 & 0 & \dots & 0 \\ a_{21} & a_{22} & b_2 & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \dots & \dots & a_{N-1, N-1} & b_{N-1} \\ a_{N1} & \dots & \dots & a_{N, N-1} & a_{NN} \end{pmatrix}.$$

We have:

Proposition 6. *The full Kostant-Toda equation with L_H in (1.15) and $P_H = -2(L_H)_{<0}$ in (1.16) can be obtained from the generalized Toda equation (1.1) with $L_{(1, N-1)}$ and $P_{(1, N-1)} := \Pi(L_{(1, N-1)})$ through a similarity transform $L_H = H L_{(1, N-1)} H^{-1}$, where H is given by*

$$(5.9) \quad H = \text{diag} \left[1, b_1, b_1 b_2, \dots, \prod_{i=1}^{N-1} b_i \right].$$

Thus the full Kostant-Toda equation can be solved through the generalized Toda equation with the $L_{(1, N-1)}$ - $P_{(1, N-1)}$ pair as the reduction on A_{N-1} , that is, with the solution $L_{(1, N-1)}$, $L_H = H L_{(1, N-1)} H^{-1}$. The similarity transform H in (5.9) was introduced by Kostant [14] to write the original nonperiodic Toda in the Hessenberg matrix form.

Remark 4. In [20], Watkins introduced the LU flow as a continuous version of the LU algorithm. Deift et. al. [6] then showed that it is a completely integrable hamiltonian system. The flow on a general matrix $L_W \in \mathfrak{M}(N, \mathbb{R})$ is in the same form as (1.1) with the generating matrix P_W :

$$(5.10) \quad P_W = -2(L_W)_{<0}.$$

Namely the full Kostant-Toda equation is a special case of the LU flow. Then the LU flow with L_W - P_W pair can be obtained from the generalized Toda equation (1.1) with L - P pair in (1.4) and (1.5) through a similarity transform $L_W = H L H^{-1}$ where H is given by the form (5.9) with the *new additional* variables b_i s satisfying

$$(5.11) \quad \frac{db_i}{dt} = (a_{i+1,i+1} - a_{ii})b_i \text{ and } b_i(0) = 1.$$

This immediately implies the solvability of the LU flow through our method, and the explicit solution is given by the LU factorization in (3.30).

6. BEHAVIORS OF THE SOLUTIONS

Here we study the behavior of the solution of the generalized Toda equation obtained in Section 3 by following the approach in [12]. Many results obtained in [12] are valid for this more general situation. First we note:

Lemma 1. *The determinants D_i for $i = 1, 2, \dots, N$ in (3.16) are real functions.*

We also have:

Lemma 2. *Suppose $D_i(t_0) = 0$ for some $t_0 < \infty$ and some i . Then $L(t)$ blows up to infinity at t_0 .*

To study the asymptotic behavior of D_i for large t , we have the following expansion:

Lemma 3. *The determinants D_i with $i = 1, 2, \dots, N$ can be expressed as*

$$(6.1) \quad D_i(t) = \sum_{J_{iN}} e^{2 \sum_{k=1}^i \lambda_{j_k} t} \begin{vmatrix} \phi_1^0(\lambda_{j_1}) & \dots & \phi_1^0(\lambda_{j_i}) \\ \vdots & \ddots & \vdots \\ \phi_i^0(\lambda_{j_1}) & \dots & \phi_i^0(\lambda_{j_i}) \end{vmatrix} \begin{vmatrix} \psi_1^0(\lambda_{j_1}) & \dots & \psi_i^0(\lambda_{j_1}) \\ \vdots & \ddots & \vdots \\ \psi_1^0(\lambda_{j_i}) & \dots & \psi_i^0(\lambda_{j_i}) \end{vmatrix},$$

where $J_{iN} = (j_1, \dots, j_i)$ represents all possible combinations for $1 \leq j_1 < \dots < j_i \leq N$. In particular $D_0(t) \equiv 1$, and $D_N(t) = \exp(2 \sum_{i=1}^N \lambda_i t)$.

We now obtain:

Theorem 2. Let the eigenvalues of L be all real and ordered as $\lambda_1 > \lambda_2 > \dots > \lambda_N$. Suppose that $\det(\Phi_k^0) \neq 0$ and $\det(\Psi_k^0) \neq 0$ for $k = 1, \dots, N$, where Φ_k^0 and Ψ_k^0 are the k -th leading principal submatrices of Φ^0 and Ψ^0 , respectively. Then as $t \rightarrow \infty$, the eigenfunctions $\phi_i(\lambda_j, t)$ and $\psi_j(\lambda_i, t)$ satisfy

$$(6.2) \quad \phi_i(\lambda_j, t) \rightarrow \delta_{ij} \times \frac{\det(\Phi_i^0) \det(\Psi_{i-1}^0)}{\sqrt{\det(\Phi_i^0 \Psi_i^0) \det(\Phi_{i-1}^0 \Psi_{i-1}^0)}},$$

$$(6.3) \quad \psi_j(\lambda_i, t) \rightarrow \delta_{ij} \times \frac{\det(\Phi_{i-1}^0) \det(\Psi_i^0)}{\sqrt{\det(\Phi_i^0 \Psi_i^0) \det(\Phi_{i-1}^0 \Psi_{i-1}^0)}},$$

which implies the sorting property as $t \rightarrow \infty$, that is, $L(t) = \Phi(t) \Lambda \Psi(t) \rightarrow \Lambda$.

This theorem implies that if all the eigenvalues of L are real, then generic solutions have the “sorting property” in the asymptotic sense. It should be however noted that $D_i(t)$ might be zero for some “finite” times, where the solution blows up (Lemma 2). Next theorem provides sufficient conditions for the solutions to blow up to infinity in finite time.

Theorem 3. Suppose some eigenvalues of L are not real, $\det \Phi_n^0 \neq 0$ and $\det \Psi_n^0 \neq 0$, for $n = 1, \dots, N$. Then $L(t)$ blows up to infinity in finite time.

Remark 5 All the results in this section remain valid for the full Kostant-Toda equation defined by (1.15) and (1.16). To see this, from Proposition 6, we solve $L_{(1,N-1)}$ with $L_{(1,N-1)}(0) = L_H(0)$. Then $L_H(t)$ is related to $L_{(1,N-1)}(t)$ through $L_H = H L_{(1,N-1)} H^{-1}$ where H is defined in (5.9) with $b_i(0) = 1$, $i = 1, \dots, N-1$. In the case $L_{(1,N-1)}(t)$ has the sorting property, since b_i s all go to zero, one verifies L_H also has the sorting property. Thus Theorem 2 holds. In the case of blowing-up, since the transformation with H (5.9) does not change the diagonal elements, Lemma 2 holds, and thus Theorem 3 is valid. In [9], the solution behavior of tridiagonal Kostant-Toda equation is considered, and a necessary and sufficient condition for blowing-up solution is obtained.

7. EXAMPLE

In this section, we demonstrate the results obtained in this paper by taking an explicit form of the matrix L . The main purpose here is to solve the generalized Toda equation (1.1) for this explicit matrix, and discuss the behavior of the solution.

Let us consider a 2×2 matrix $L(t) = (a_{ij})_{1 \leq i,j \leq 2}$. The generalized Toda equation then gives

$$(7.1) \quad \frac{d}{dt} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 2a_{12}a_{21} & a_{12}(a_{22} - a_{11}) \\ a_{21}(a_{22} - a_{11}) & -2a_{21}a_{12} \end{pmatrix}.$$

The initial data of $L(t)$ is assumed to be

$$(7.2) \quad L(0) = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix},$$

where a and b are arbitrary constants. The eigenvalues of $L(0)$, λ_1 and λ_2 , are

$$(7.3) \quad \lambda_{1,2} = \frac{1}{2} \left(b \pm \sqrt{b^2 + 4a} \right).$$

Then the initial eigenmatrices Φ^0 and Ψ^0 are expressed by

$$(7.4) \quad \Phi^0 = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix},$$

$$(7.5) \quad \Psi^0 = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{pmatrix}.$$

In order to compute the solutions $\Phi(t)$ and $\Psi(t)$ from (3.18) and (3.19), we need the quantities $c_{ij} = \langle \phi^0 \psi^0 e^{2\lambda t} \rangle$. From (7.4) and (7.5), they are

$$(7.6) \quad \begin{aligned} c_{11}(t) &= \frac{1}{\lambda_2 - \lambda_1} (\lambda_2 e^{2\lambda_1 t} - \lambda_1 e^{2\lambda_2 t}), \\ c_{12}(t) &= \frac{1}{\lambda_2 - \lambda_1} (-e^{2\lambda_1 t} + e^{2\lambda_2 t}), \\ c_{21}(t) &= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} (e^{2\lambda_1 t} - e^{2\lambda_2 t}), \\ c_{22}(t) &= \frac{1}{\lambda_2 - \lambda_1} (-\lambda_1 e^{2\lambda_1 t} + \lambda_2 e^{2\lambda_2 t}), \end{aligned}$$

from which the determinants $D_i(t)$ in (3.16) become

$$(7.7) \quad D_1(t) = c_{11}(t), \quad D_2(t) = \begin{vmatrix} c_{11}(t) & c_{12}(t) \\ c_{21}(t) & c_{22}(t) \end{vmatrix} = e^{2(\lambda_1 + \lambda_2)t}.$$

We now have the solutions (Theorem 1),

$$(7.8) \quad \Phi(t) = \frac{1}{\sqrt{D_1(t)}} \begin{pmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} \\ \lambda_1 e^{\lambda_2 t} & \lambda_2 e^{\lambda_1 t} \end{pmatrix},$$

$$(7.9) \quad \Psi(t) = \frac{1}{(\lambda_2 - \lambda_1) \sqrt{D_1(t)}} \begin{pmatrix} \lambda_2 e^{\lambda_1 t} & -e^{\lambda_2 t} \\ -\lambda_1 e^{\lambda_2 t} & e^{\lambda_1 t} \end{pmatrix}.$$

The solution $L(t)$ of the generalized Toda equation is then obtained from (2.9), $a_{ij}(t) = \langle \lambda \phi_i \psi_j \rangle(t)$,

$$(7.10) \quad L(t) = \frac{1}{\lambda_2 e^{2\lambda_1 t} - \lambda_1 e^{2\lambda_2 t}} \begin{pmatrix} \lambda_1 \lambda_2 (e^{2\lambda_1 t} - e^{2\lambda_2 t}) & (\lambda_2 - \lambda_1) e^{(\lambda_1 + \lambda_2)t} \\ -\lambda_1 \lambda_2 (\lambda_2 - \lambda_1) e^{(\lambda_1 + \lambda_2)t} & \lambda_2^2 e^{2\lambda_1 t} - \lambda_1^2 e^{2\lambda_2 t} \end{pmatrix}.$$

Now let us discuss the solution behavior for $t > 0$. First we assume both eigenvalues λ_1 and λ_2 to be real. With the choice of the eigenvalues in (7.3), we have $\lambda_1 \geq \lambda_2$. Then

if $\lambda_1 \lambda_2 \leq 0$, then the function $D_1(t)$ does not vanish for all t . This implies the sorting property (Theorem 2). For the case of $\lambda_1 > \lambda_2 > 0$, the D_1 vanishes and we have the blowing up in the solution at the time $t = t_B > 0$,

$$(7.11) \quad t_B = \frac{1}{2(\lambda_1 - \lambda_2)} \log \frac{\lambda_1}{\lambda_2}.$$

This formula also implies that for $0 > \lambda_1 > \lambda_2$ we have the sorting result for $t > 0$. Note here that the blowing up occurs at one time $t = t_B$ (7.11), and then the solution $L(t)$ will be sorted as $t \rightarrow \infty$, with the asymptotic forms of the eigenmatrices, i.e. (6.2) and (6.3),

$$(7.12) \quad \Phi(t) \rightarrow \sqrt{\frac{\lambda_2 - \lambda_1}{\lambda_2}} \begin{pmatrix} 1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

$$(7.13) \quad \Psi(t) \rightarrow \frac{1}{\sqrt{\lambda_2(\lambda_2 - \lambda_1)}} \begin{pmatrix} \lambda_2 & 0 \\ 0 & 1 \end{pmatrix}.$$

For the case of the complex eigenvalue $\lambda_1 = \bar{\lambda}_2 := \alpha + i\beta$, $D_1(t)$ is expressed as

$$(7.14) \quad D_1(t) = e^{2\alpha t} \sec \theta \cos(2\beta t + \theta)$$

with $\tan \theta = \alpha/\beta$. This indicates the blowing up (Theorem 3).

In the case of degenerate eigenvalues $\lambda_1 = \lambda_2$ (i.e. $b^2 + 4a = 0$), we take the limit $\lambda_2 \rightarrow \lambda_1 := \lambda_0$ in (7.10), and obtain

$$(7.15) \quad L(t) = \frac{1}{1 - 2\lambda_0 t} \begin{pmatrix} -2\lambda_0^2 t & 1 \\ -\lambda_0^2 & 2\lambda_0(1 - \lambda_0 t) \end{pmatrix}.$$

which shows the “sorting property” as $t \rightarrow \infty$, i.e. $L(t) \rightarrow \lambda_0 I_2$. It should be noted however that $L(0)$ with the degenerate eigenvalues is not similar to the “diagonal” matrix $\lambda_0 I_2$.

Acknowledgment The work of Y. Kodama is partially supported by an NSF grant DMS9403597.

REFERENCES

1. A.M. Bloch, R.W. Brockett, and T.S. Ratiu. Completely Integrable Gradient Flow. *Comm. Math. Phys.*, **147** (1992), 57-74.
2. O.I. Bogoyavlensky. On Perturbations of the Periodic Toda Lattice. *Comm. Math. Phys.*, **51** (1976), 201-209.
3. R.N. Cahn. *Semi-simple Lie Algebras and their Representations*. Benjamin/Cummings 1994.
4. M. T. Chu, and L. K. Norris. Isospectral Flows and Abstract Matrix Factorizations. *SIAM J. Numer. Anal.*, **25** (1988), 1383-1391.
5. P. Deift, L.C. Li, T. Nanda, and C. Tomei. The Toda Flow on a Generic Orbit is Integrable. *Comm. Pure Appl. Math.*, **39** (1986), 183-232.
6. P. Deift, L.C. Li, and C. Tomei. Matrix factorizations and integrable systems. *Comm. Pure Appl. Math.*, **42** (1989), 443-521.

7. N. Ercolani, H. Flaschka and S. Singer. The Geometry of the Full Kostant-Toda. Preprint 1994.
8. H. Flaschka. On the Toda Lattice II. *Prog. Theor. Phys.*, **51** (1974), 703-716.
9. M.I. Gekhtman and M.Z. Shapiro. Completeness of Real Toda Flows and Totally Positive Matrices. Preprint 1994.
10. J. Humphreys. *Introduction to Lie Algebras and their Representations*. New York: Springer 1972.
11. Y. Kodama, and K. McLaughlin. Explicit Integration of the Full Symmetric Toda hierarchy and the Sorting property. to appear in *Lett. Math. Phys.* (solv-int/9502006)
12. Y. Kodama, and J. Ye. Toda Hierarchy with Indefinite metric. Preprint. (solv-int/9505010)
13. Y. Kodama, and J. Ye. Iso-spectral Deformations of General Matrix and Their Reductions on Lie Algebras. Preprint.
14. B. Kostant. The solution to a generalized Toda lattice and representation theory. *Adv. in Math.*, **34** (1979), 195-338.
15. S.V. Manakov. Complete Integrability and Stochastization of Discrete Dynamical Systems. *Sov. Phys. JETP*, **40** (1975), 703-716. (*Zh. Exp. Teor. Fiz.*, **67** (1974), 543-555.)
16. J. Moser. Finitely Many Mass Points on the Line Under the Influence of an Exponential Potential - An Integrable System. in: *Dynamical Systems, Theory and Applications*, J. Moser ed. *Lec. Notes in Phys.*, Vol.38, Springer-Verlag, Berlin and New York, 1975, pp. 467-497.
17. T. R. Nanda. Isospectral Flows on Banded Matrices. *Ph.D. Dissertation, New York University*, 1982.
18. W.W. Symes. The QR Algorithm and Scattering for the Finite Non-periodic Toda Lattice. *Physica D*, **4** (1982), 275-278.
19. G. Szegő. *Orthogonal Polynomials*. AMS Colloquium Publications, vol.23, 1939.
20. D.S. Watkins. Isospectral flows, *SIAM Rev.*, **26** No.3 (1984), 379-392.