

AN INTEGRAL REPRESENTATION THEOREM FOR THE HELMHOLTZ EQUATION

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§1. The purpose of this paper is to show the integral representation for positive solutions of the Helmholtz equation $(\Delta - I)f = 0$ on $(0, \infty)^n \times \mathbb{R}^{N-n}$ by a passage to the theory of the heat equation. In the case $n = 0$, it is well-known (see for example [2] and [3]) that every positive solution has an integral representation

$$(1) \quad f(X) = \int_{S^{N-1}} \exp(\langle X, A \rangle) d\mu(A)$$

where μ is a positive measure on the sphere. We give here a new proof of this fact as an illustration of our method. Let $f > 0$ be a solution of $\Delta f = f$ on $\mathbb{R}^N$. Then the function $u(X, t) = e^t f(X)$ satisfies $\Delta u = \frac{\partial u}{\partial t}$ on $\mathbb{R}^N \times \mathbb{R}$. Hence by the integral representation theorem for positive solutions of the heat equation ([1, p.374]) there is a positive measure μ on $\mathbb{R}^N$ such that

$$u(X, t) = \int_{\mathbb{R}^N} \exp(\langle X, A \rangle + t \|A\|^2) d\mu(A).$$

Since $0 = (\Delta - I)^2 f(X) = \int (\|A\|^2 - 1)^2 \exp(\langle X, A \rangle + t(\|A\|^2 - 1)) d\mu(A)$, we have $\text{supp}(\mu) \subset S^{N-1}$, so that (1) is obtained.

In section 2 we describe our main theorem for general $n \geq 1$. After giving the integral representation theorems for the heat equation in

section 3, we prove the theorem in section 4. Finally we make a remark about the minimal Martin boundary at infinity with respect to the Helmholtz equation.

§2. Given integers N and n with $1 \leq n \leq N$, let $D = (0, \infty)^n \times \mathbb{R}^{N-n} = \{X = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N; x_i > 0 \text{ for } i = 1, 2, \dots, n\}$. The Green function of the Helmholtz equation $\Delta - I$ on D is given by

$$G(X, Y) = \int_0^\infty e^{-t} \left[\prod_{i=1}^n \{w(x_i - y_i, t) - w(x_i + y_i, t)\} \prod_{i=n+1}^N w(x_i - y_i, t) \right] dt$$

where $w(x, t) = (4\pi t)^{-1/2} \exp(-x^2/4t)$ if $t > 0$, and $= 0$ if $t \leq 0$.

Now, for each $A = (a_1, \dots, a_N) \in \partial D$, we define

$$H_1(X, A) = \left(\prod_{i \in \tau(A)} \frac{\partial}{\partial y_i} \right) G(X, Y) \Big|_{Y=A}$$

where $\tau(A) = \{i; 1 \leq i \leq n \text{ and } a_i = 0\}$.

For every subset $\Sigma \subset \{1, 2, \dots, n\}$, we put $\Sigma_1 = \{1, 2, \dots, n\} - \Sigma$ and $S_\Sigma = \{A \in S^{N-1}; a_i = 0 \text{ for any } i \in \Sigma \text{ and } a_i > 0 \text{ for any } i \in \Sigma_1\}$. For each $A \in \overline{D} \cap S^{N-1}$, we also define

$$H_2(X, A) = \prod_{i \in \Sigma} x_i \prod_{i \in \Sigma_1} \sinh(a_i x_i) \prod_{i=n+1}^N \exp(a_i x_i) \text{ if } A \in S_\Sigma.$$

Observe that $H_j(\cdot, A)$, $j = 1, 2$, are positive solutions of the Helmholtz equation on D .

We now state the theorem in this paper.

Theorem. For every positive solution f of the Helmholtz equation on $D = (0, \infty) \times \mathbb{R}^{N-n}$, there are unique Borel measures μ_1 on ∂D and μ_2 on $\overline{D} \cap S^{N-1}$ such that

$$(2) \quad f(X) = \int H_1(X, A) d\mu_1(A) + \int H_2(X, A) d\mu_2(A).$$

Furthermore if f is continuous on $\overline{D}$ then $d\mu_1(A) = f(A) d\sigma(A)$, where $d\sigma(\cdot)$ is the surface measure on ∂D .

§3. In this section we give integral representation theorems for the heat equation. Following [1], a solution of the heat equation will be said to be parabolic.

For $x, t \in \mathbb{R}$ and $a \geq 0$, we put

$$k(x, t, a) = \begin{cases} w(x-a, t) - w(x+a, t) & \text{if } a > 0 \\ \frac{x}{t} w(x, t) & (= 0 \text{ if } t = 0) \text{ if } a = 0 \end{cases}$$

and

$$k^*(x, t, a) = \begin{cases} \sinh(ax) \exp(ta^2) & \text{if } a > 0 \\ x & \text{if } a = 0. \end{cases}$$

Let $D = (0, \infty) \times \mathbb{R}^{N-n}$ as before. For each $(X, t) \in D \times (-\infty, \infty)$ and $(A, s) \in \overline{D} \times [-\infty, \infty)$ we define

$$K((X,t),(A,s)) = \begin{cases} \prod_{i=1}^n k(x_i, t-s, a_i) \prod_{i=n+1}^N w(x_i - a_i, t-s) & \text{if } s \in \mathbb{R}. \\ \prod_{i=1}^n k^*(x_i, t, a_i) \prod_{i=n+1}^N \exp(a_i x_i + t a_i^2) & \text{if } s = -\infty. \end{cases}$$

The following was proved in [4, Theorems 2.2 and 3.4] in the case $n = 1$ (see also [5]), and a similar proof can be carried out for arbitrary $n \geq 1$ so that we have

Proposition 1. For every positive parabolic function u on $D \times (0, \infty)$, there is a unique Borel measure μ on $\partial(D \times (0, \infty))$ such that

$$u(X, t) = \int K((X, t), (A, s)) d\mu(A, s).$$

In particular if u is continuous on $\overline{D} \times [0, \infty)$ then $d\mu(A, s) = u(A, s) d\sigma(A, s)$, where $d\sigma(.,.)$ is the surface measure of $\partial(D \times (0, \infty))$, and if u is continuous on $\overline{D} \times (0, \infty)$ then $d\mu(A, s) = u(A, s) d\sigma(A) ds$ on $\partial D \times (0, \infty)$, where $d\sigma(.)$ is the surface measure on ∂D .

By the Appell transform the integral representation on $D \times (-\infty, 0)$ was given in [4, Theorem 4.1] (in the case $n = 1$). Since this method is available for arbitrary $n \geq 1$, we also see

Proposition 2. For every positive parabolic function u on $D \times (-\infty, 0)$ there is a unique Borel measure μ on $\partial D \times (-\infty, 0) \cup \overline{D} \times \{-\infty\}$ such that

$$(3) \quad u(X,t) = \int K((X,t),(A,s))d\mu(A,s).$$

In particular if u is continuous on $\overline{D} \times (-\infty, 0)$ then $d\mu(A,s) = u(A,s)d\sigma(A)ds$ on $\partial D \times (-\infty, 0)$.

We remark here that the second assertion is deduced from the last assertion in Proposition 1 by applying the Appell transform.

Before returning to the Helmholtz equation, we make an observation on the Martin boundary of $D \times (-\infty, 0)$ with respect to the heat equation. (For details, we refer to [1, p.262-383]). Let $A_1 = (a_1, a_2, \dots, a_N)$ with $a_i = 1, 1 \leq i \leq n$ and $= 0, n+1 \leq i \leq N$. Then $((A_1, 0), D \times (-\infty, 0))$ is a Martin point set pair ([1, p.359]). By the same manner as in [1, p.374-375, in the case $N = n = 1$] we see that the Martin boundary $\partial^M(D \times (-\infty, 0))$ for this pair is $\partial D \times (-\infty, 0) \cup \overline{D} \times \{-\infty\} \cup \{0_\infty\}$ and the Martin kernel is given by

$$K^*((X,t),(A,s)) = \frac{K((X,t),(A,s))}{K((A_1,0),(A,s))}$$

for $(A,s) \in \partial D \times (-\infty, 0) \cup \overline{D} \times \{-\infty\}$ and $K^*((X,t), 0_\infty) = 0$. In the Martin topology, $(Y,r) \in D \times (-\infty, 0)$ tends to $(A,s) \in \partial D \times (-\infty, 0)$ if and only if $(Y,r) \rightarrow (A,s)$, (Y,r) tends to $(A, -\infty) \in \overline{D} \times \{-\infty\}$ if and only if $r \rightarrow -\infty$ and $Y/-r \rightarrow A$, and (Y,r) tends to 0_∞ if and only if $r \rightarrow 0$ or $\|Y\|/(1-r) \rightarrow \infty$. Thus, on $\partial D \times (-\infty, 0)$ the Martin topology coincides with the Euclidean topology. Similarly to [1, p.367], we also see that 0_∞ is the only non-minimal Martin boundary point. If u is positive parabolic on $D \times (-\infty, 0)$ and $P^f u(A_1, 0) < \infty$ (the parabolic fine limit at

$(A_1, 0)$, cf. [1, p.359]), then there is a unique Borel measure μ^* on $\partial^M(D \times (-\infty, 0))$ with $\int d\mu^* = P_u^f(A_1, 0)$ such that

$$(4) \quad u(X, t) = \int K((X, t), (A, s)) d\mu^*(A, s).$$

§4. In this section we give a proof of the theorem. Now, let $f > 0$ be a solution of $\Delta f = f$ on $D = (0, \infty)^n \times \mathbb{R}^{N-n}$.

We first assume that f is continuous on $\bar{D}$. Then the function $u(X, t) = e^t f(X)$ is continuous on $\bar{D} \times (-\infty, 0)$ and parabolic on $D \times (-\infty, 0)$. By Proposition 2, there is a Borel measure μ_2 on $\bar{D}$ (from now on we identify $\bar{D} \times \{-\infty\}$ with $\bar{D}$) such that

$$(5) \quad u(X, t) = \iint_{-\infty}^0 K((X, t), (A, s)) e^s f(A) ds d\sigma(A) + \int K((X, t), (A, -\infty)) d\mu_2(A).$$

An elementary calculation shows that for each $A \in \partial D$

$$(6) \quad e^{-t} \int_{-\infty}^0 K((X, t), (A, s)) e^s ds = \int_0^\infty e^{-t} K((X, t), (A, 0)) dt = H_1(X, A),$$

which also implies that $e^{-t} \int K((X, t), (A, -\infty)) d\mu_2(A)$ is independent of t and is a solution of $\Delta f = f$. It follows that $\text{supp}(\mu_2) \subset \bar{D} \cap S^{N-1}$, for

$$(7) \quad \begin{aligned} 0 &= (\Delta - I)^2 \int_{\bar{D}} K((X, t), (A, -\infty)) d\mu_2(A) \\ &= \int_{\bar{D}} (\|A\|^2 - 1)^2 K((X, t), (A, -\infty)) d\mu_2(A). \end{aligned}$$

Since for each $A \in \overline{D} \cap S^{N-1}$

$$(8) \quad \lim_{t \uparrow 0} K((X, t), (A, -\infty)) = H_2(X, A) \text{ (increasingly),}$$

we have the second part of the Theorem by letting $t \uparrow 0$ in (5).

In the general case, we put

$$f_m(X) = f(x_1 + 1/m, x_2 + 1/m, \dots, x_n + 1/m, x_{n+1}, \dots, x_N)$$

and $u_m(X, t) = e^t f_m(X)$ for each $m \geq 1$. Then f_m is continuous on $\overline{D}$ and satisfies the Helmholtz equation on D . Hence by (4) and the above proof, there exists a Borel measure $\mu_{2,m}$ on $\overline{D} \cap S^{N-1}$ such that

$$(9) \quad \begin{aligned} e^t f_m(X) &= \int K^*((X, t), (A, s)) d\mu_{1,m}^*(A, s) + \int K^*((X, t), (A, -\infty)) d\mu_{2,m}^*(A) \\ &= \iint K((X, t), (A, s)) e^s f_m(A) ds d\sigma(A) + \int K((X, t), (A, -\infty)) d\mu_{2,m}(A), \end{aligned}$$

where $\mu_{1,m}^* = K((A_1, 0), (A, s)) e^s f_m(A) ds d\sigma(A)$ and $\mu_{2,m}^* = K((A_1, 0), (A, -\infty)) d\mu_{2,m}(A)$. Since $P_u^f(A_1, 0) = \lim_{t \uparrow 0} e^t f_m(A_1) = f_m(A_1)$ is bounded in m , $(\mu_{i,m}^*)_{m=1}^\infty$ ($i = 1, 2$) is a vaguely bounded sequence of positive measures on the Martin boundary $\partial^M(D \times (-\infty, 0))$, so that we may assume that this has a vague limit μ_i^* ($i = 1, 2$). Then we see that $\text{supp}(\mu_2^*) \subset \overline{D} \cap S^{N-1}$ and

$$(10) \quad \lim_{m \rightarrow \infty} \int K((X, t), (A, -\infty)) d\mu_{2,m}(A) = \int K^*((X, t), (A, -\infty)) d\mu_2(A).$$

Now, we denote by $\mu_{1,1}^{**}$ and $\mu_{1,2}^{**}$ the restrictions of the measure μ_1^*

to $\partial D \times (-\infty, 0)$ and to $\bar{D}$, respectively. We shall show that there is a measure μ_1 on D such that $\mu_{1,1}^{**}(A,s) = K((A_1,0),(A,s))e^s d\mu_1(A)ds$. Let ψ be an arbitrary continuous function on $\partial D \times (-\infty, 0)$ with compact support and fix $-\infty < s_0 < 0$. We can easily check that the function $\psi(A,s)e^{sK((A_1,0),(A,s))}/K((A_1,0),(A,s_0))$ in (A,s) is continuous and has compact support on $\partial D \times (-\infty, 0)$ and that there is a constant $C = C(\psi, s_0) > 0$ such that $e^{sK((A_1,0),(A,s))} \geq CK((A_1,0),(A,s_0))$ on $\text{supp}(\psi)$. Since

$$\begin{aligned} f_m(A_1) &\geq \int \int_{\partial D}^0 K((A_1,0),(A,s))e^s f_m(A) d\sigma(A) ds \\ &\geq C \int \int_{\text{supp}(\psi)} K((A_1,0),(A,s_0)) f_m(A) d\sigma(A) ds, \end{aligned}$$

we may assume that $K((A_1,0),(A,s_0))f_m(A)d\sigma(A)$ converges vaguely to a Borel measure $\tilde{\mu}$ on ∂D as $m \rightarrow \infty$. Then

$$\begin{aligned} \int \psi d\mu_{1,1}^{**} &= \lim_{m \rightarrow \infty} \int \psi d\mu_{1,m}^* \\ &= \lim_{m \rightarrow \infty} \iint \frac{\psi(A,s)e^{sK((A_1,0),(A,s))}}{K((A_1,0),(A,s_0))} K((A_1,0),(A,s_0)) f_m(A) d\sigma(A) ds \\ &= \iint \psi(A,s)e^{sK((A_1,0),(A,s))} \frac{1}{K((A_1,0),(A,s_0))} d\tilde{\mu}(A) ds. \end{aligned}$$

Therefore $d\mu_{1,1}^{**} = K((A_1,0),(A,s))e^s d\mu_1(A)ds$, where $d\mu_1(A) = (K((A_1,0),(A,s_0)))^{-1} d\tilde{\mu}(A)$.

Consequently, letting $m \rightarrow \infty$ in (9) and remarking (6) and (10), we

have

$$\begin{aligned} e^t f(X) &= \iint K((X,t),(A,s)) e^s ds d\mu_1(A) + \int K^*((X,t),(A,-\infty)) d(\mu_{1,2}^{**} + \mu_2^*)(A) \\ &= \int e^t H_1(X,A) d\mu_1(A) + \int K((X,t),(A,-\infty)) d\mu_2(A,s), \end{aligned}$$

where $d\mu_2(A) = (K((A_1,0),(A,-\infty)))^{-1} d(\mu_{1,2}^{**} + \mu_2^*)(A)$. By the same manner as in (7) we see $\text{supp}(\mu_2) \subset \overline{D} \cap S^{N-1}$. Hence, as a consequence of (8), the desired integral representation (2) follows by letting $t \uparrow 0$. Since the uniqueness of the representation measures follows from Proposition 2, we obtain our theorem.

§5. It is easily seen that our method is also available for the operator $\Delta - cI$ (c : real constant) on D . Remark that if $c < 0$ there is no positive solution. In the Martin boundary theoretic view point, our result explains that the minimal Martin boundary of D at infinity with respect to $\Delta - cI$ (i.e., the set of normalized minimal solutions which vanish at all finite boundary points) is homeomorphic to $c(S^{N-1} \cap \overline{D}) = \{cA, A \in S^{N-1} \cap \overline{D}\}$.

On the other hand, Landis & Nadirashvili [6] tells us that

$$\{f; \Delta f = 0 \text{ and } f > 0 \text{ in } D_E, f = 0 \text{ on } \partial D_E\}$$

is one dimensional, where $E \subset S^{N-1}$ is a domain with Lipschitz boundary and $D_E = \{X \in \mathbb{R}^N; X \neq 0, X/\|X\| \in E\}$. By these observations it can be conjectured that the minimal Martin boundary of D_E at infinity with

respect to $\Delta - cI$ would be homeomorphic to $\overline{cE}$, but we know no other example which reinforces this conjecture.

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