

Paths and Edge-Connectivity in Graphs

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1. INTRODUCTION

We consider finite undirected graphs passibly with multiple edges but without loops. Let G be a graph and let $V(G)$ and $E(G)$ be the sets of vertices and edges of G respectively. For two distinct vertices x and y , let $\lambda_G(x,y)$ be the maximal number of edge-disjoint paths between x and y , and let $\lambda_G(x,x)=\infty$. For an integer $k \geq 1$, let $\Gamma(G,k)$ be $\{X \subseteq V(G) \mid \text{For each } x,y \in X, \lambda_G(x,y) \geq k\}$.

Let $(s_1, t_1), \dots, (s_k, t_k)$ be pairs of vertices of G . When is the following statement true ?

(1.1) There exist edge-disjoint paths $P_1, \dots, P_k$ such that P_i has ends s_i, t_i ($1 \leq i \leq k$).

Seymour [8] and Thomassen [9] characterised such graphs when $k=2$, and Seymour [8] when $|(s_1, \dots, s_k, t_1, \dots, t_k)|=3$.

For integers $k \geq 1$ and $n \geq 2$, set

$$g(k) = \min\{m \mid \text{If } G \text{ is } m\text{-edge-connected, then (1.1) holds}\},$$

$$\lambda'(k,n) = \min \left\{ m \mid \begin{array}{l} \text{If } |(s_1, \dots, s_k, t_1, \dots, t_k)| \leq n \text{ and} \\ (s_1, \dots, s_k, t_1, \dots, t_k) \in \Gamma(G,m), \text{ then (1.1)} \\ \text{holds} \end{array} \right\},$$

$$\lambda(k,n) = \min \left\{ m \mid \text{If } | \{s_1, \dots, s_k, t_1, \dots, t_k\} | \leq n \text{ and } \lambda_G(s_i, t_i) \geq m \ (1 \leq i \leq k), \text{ then (1.1) holds} \right\},$$

and set

$$\lambda'(k) = \lambda'(k, 2k) = \lambda'(k, m) \ (m > 2k) \text{ and } \lambda(k) = \lambda(k, 2k).$$

Then for each $k \geq 1$,

$$\lambda'(k, 3) = \lambda(k, 3) \text{ and } \lambda(k) \geq \lambda'(k) \geq g(k) \geq k.$$

For $n \geq 4$ and even integer $k \geq 2$,

$$g(k) > k \text{ and } \lambda(k) \geq \lambda(k, n) \geq \lambda'(k, n) > k$$

(see Figure 1 in which $k/2$ represents the number of parallel edges).

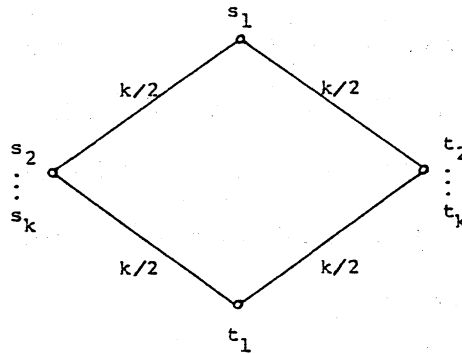


Figure 1.

Thomassen [9] gave following Conjecture 1, and we give following Conjecture 2 slightly stronger than Conjecture 1.

CONJECTURE 1. For each integer $k \geq 1$,

$$g(k) = \begin{cases} k & \text{if } k \text{ is odd} \\ k+1 & \text{if } k \text{ is even} \end{cases}.$$

CONJECTURE 2. For each integer $k \geq 1$,

$$\lambda(k) = \begin{cases} k & \text{if } k \text{ is odd} \\ k+1 & \text{if } k \text{ is even} \end{cases}.$$

It easily follows from Menger's theorem that $\lambda(k) \leq 2k-1$; thus $\lambda(1)=1$ and $\lambda(2)=3$. Cypher [1] proved $\lambda(4) \leq 6$ and $\lambda(5) \leq 7$, and $\lambda(3)=3$ was announced in [5] and proved in [6] by the author. Enomoto and Saito [2] proved $g(4)=5$, and independently Hirata, Kubota and Saito [3] proved $\lambda(k) \leq 2k-3$ ($k \geq 4$).

Our main results are the following.

THEOREM 1. Suppose that $k \geq 2$ is an integer, G is a graph, $\{a_1, a_2\} \subseteq T \subseteq V(G)$, $|T| \leq 3$ and $T \in \mathcal{F}(G, k)$. Then there exists a path P between a_1 and a_2 such that $T \in \mathcal{F}(G-E(P), k-1)$.

THEOREM 2. Suppose that $k \geq 5$ is an odd integer, G is a graph, $\{a_1, a_2, a_3\} \subseteq T \subseteq V(G)$, $a_i \neq a_j$ ($1 \leq i < j \leq 3$), $|T| \leq 5$ and $T \in \mathcal{F}(G, k)$. Then

(1) If $|T| \leq 4$, then there exists a path P between a_1 and a_2 such that $T \in \mathcal{F}(G-E(P), k-1)$.

(2) For $m=2, 3$ if $|T| \leq 4$ and for $m=3$ if $|T|=5$, there exist edge-disjoint paths P_1 between a_1 and a_2 and P_2 between a_1 and a_m such that $T \in \mathcal{F}(G - \bigcup_{i=1}^2 E(P_i), k-2)$.

THEOREM 3. For each integer $k \geq 1$,

$$\lambda(k, 3)=k \text{ and } \lambda(k, 4)=\lambda(k, 5)=\begin{cases} k & \text{if } k \text{ is odd} \\ k+1 & \text{if } k \text{ is even.} \end{cases}$$

In Theorem 2(2) if $m=2$ and $|T|=5$, then the conclusion does not always hold. Figure 2 gives a counterexample with $k=7$.

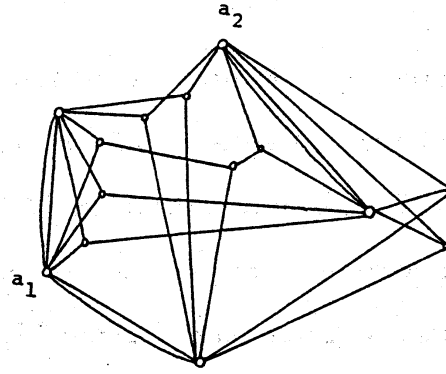


Figure 2.

When k is odd and $|s_1, \dots, s_k, t_1, \dots, t_k| \geq 4$, if for some $1 \leq i \leq k$,

$$\lambda_G(s_i, t_i) < k,$$

then (1.1) does not always hold. Figure 3 gives a counterexample.

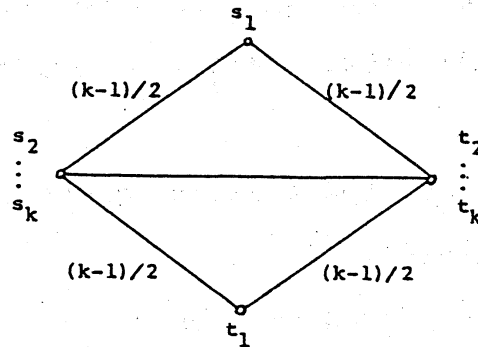


Figure 3.

Notations and Definitions. Let $X, Y \subseteq V(G)$, $F \subseteq E(G)$, $\{x, y\} \subseteq V(G)$ and $e \in E(G)$. We often denote $\{x\}$ by x and $\{e\}$ by e . The subgraph of G induced by X is denoted by $\langle X \rangle_G$ and the subgraph obtained from G by deleting X (F) is denoted by $G - X$ ($G - F$). $\partial_G(X, Y)$ denotes the set of edges with one end in X and the other in Y , and $\partial_G(X)$ denotes $\partial_G(X, V(G) - X)$. $\lambda_G(X, Y)$ denotes the maximal number of edge-disjoint paths with one end in X and the other in Y . $\partial_G(X)$

is called an n -cut if $|\partial_G(X)|=n$ and $\langle X \rangle_G$ and $\langle V(G)-X \rangle_G$ are both connected. An n -cut $\partial_G(X)$ is called nontrivial if $|X| \geq 2$ and $|V(G)-X| \geq 2$, trivial otherwise. $d_G(x)$ denotes the degree of x and $N_G(x)$ denotes the set of vertices adjacent to x . We regard a path and a cycle as subgraphs of G . A path $P=P[x,y]$ denotes a path between x and y , and for $x', y' \in V(P)$, $P(x', y')$ denotes a subpath of P between x' and y' .

2. PROOF OF THEOREM 1

For a vertex $w \in V(G)$ and $b, c \in N_G(w)$, we let $G_w^{b,c}$ be the graph $(V(G), E(G) \cup e - \{f, g\})$, where e is a new edge with ends b and c , $f \in \partial_G(w, b)$ and $g \in \partial_G(w, c)$. We require the following lemmas.

LEMMA 2.1 (Mader [4]). Suppose that w is a non-separating vertex of a graph G with $d_G(w) \geq 4$ and with $|N_G(w)| \geq 2$. Then there exist $b, c \in N_G(w)$ such that for each $x, y \in V(G)-w$,

$$\lambda_{G_w^{b,c}}(x, y) = \lambda_G(x, y).$$

Now we prove Theorem 1 by induction on $|E(G)|$. We may assume that $a_1 \neq a_2$ and $|T|=3$. If G has a nontrivial k -cut $\partial_G(X)$ ($X \subseteq V(G)$) separating T , then let H (K) be the graph obtained from G by contracting $V(G)-X$ (X) to a new vertex u (v). Set $T_H = (X \cap T) \cup u$ and $T_K = (T-X) \cup v$. We may

let $|T \cap X| = 2$. By induction for H and $(T \cap X) \cup u$ instead of for G and T , the result holds. Thus the result follows. Hence we may assume that each edge is incident to a vertex of T .

Case 1. There exists $x \in V(G) - T$.

If $d_G(x) \geq 4$, then by Lemma 2.1 there exists $b, c \in N_G(x)$ such that for each $y, z \in V(G) - x$,

$$\lambda_{G_x^{b,c}}(y, z) = \lambda_G(y, z).$$

By induction the result holds in G_x . Thus we may assume that $d_G(x) = 3$ and clearly that $N_G(x) = T$. Now the path $P[a_1, a_2]$ with $E(P) \subseteq \partial_G(x)$ is a required path.

Case 2. $V(G) = T$.

The result easily follows.

3. PROOF OF THEOREM 2.

We call a graph G is elemental for $V_1 \subseteq V(G)$ if $V(G) = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ and for each $x \in V_2$, $d_G(x) = 3$, $|N_G(x)| = 3$ and $N_G(x) \subseteq V_1$. We call a graph G is elemental for $V_1 \subseteq V(G)$ and an integer $k \geq 1$ if G is elemental for V_1 and for each $x \in V_1$, $d_G(x) = k$. For integers $p \geq 0$ and $q \geq 0$, we call a graph G is $G(p, q)$ if G is elemental for some $V_1 = \{x_1, x_2, x_3\} \subseteq V(G)$, $|V(G) - V_1| = q$ and $|\partial_G(x_i, x_j)| = p$ ($1 \leq i < j \leq 3$). Let G be an elemental graph for $V_1 \subseteq V(G)$. We call a subgraph S an elemental star if $V(S) \subseteq V_1$, $|V(S)| = 2$ and $|E(S)| = 1$, or if for some $x \in V(G) - V_1$, $V(S) = N_G(x) \cup x$ and $E(S) = \partial_G(x)$.

We require the following lemmas.

LEMMA 3.1 (Okamura [7]). Suppose that $k \geq 4$ is an integer, G is a graph, $\{s, t\} \subseteq T \subseteq V(G)$ and $T \in \mathcal{F}(G, k)$. Then

(1) For each non-separating edge e incident to s , there exists a path P between s and t passing through e such that

$$T \in \mathcal{F}(G - E(P), k-2) \text{ and } \{s, t\} \in \mathcal{F}(G - E(P), k-1).$$

(2) For each vertex a of $T - \{s, t\}$ with fewer degree than $2k$ and for each edge f incident to a , there exists a path P between s and t not passing through a such that

$$T \in \mathcal{F}(G - E(P), k-2), \{s, t, a\} \in \mathcal{F}(G - E(P), k-1),$$

and

$$\{s, a\} \text{ or } \{t, a\} \in \mathcal{F}(G - E(P) - f, k-1).$$

(3) For each non-separating edges e and e' incident to s , there exists a cycle C passing through e and e' such that

$$T \in \mathcal{F}(G - E(C), k-2).$$

LEMMA 3.2 (Okamura [7]). Suppose that $n \geq 4$ is an integer and $k \geq 3$ is an odd integer. If for each odd integer $1 \leq m \leq k$,

$$\lambda'(m, n) = m,$$

then

$$\lambda(k, n) = k \quad \text{and} \quad \lambda(k+1, n) = k+2.$$

LEMMA 3.3. Suppose that $k \geq 3$ is an integer, G is an elemental graph for $T \subseteq V(G)$ and k , $T \in \mathcal{F}(G, k)$, G has no nontrivial k -cut separating T , and that S_1, S_2, S_3 are elemental stars of G . If $V(S_1) \cap V(S_2) \cap V(S_3) = \emptyset$, then

$$T \in \Gamma(G - \bigcup_{i=1}^3 E(S_i), k-2).$$

Proof. Assume that $X \subseteq V(G)$, $|X| \leq |V(G) - X|$ and X separates T . Set $G' = G - \bigcup_{i=1}^3 E(S_i)$. If $|X| = 1$, then let $X = \{x\}$. Since $d_{G'}(x) \geq d_G(x) - 2 = k - 2$, we have $|\partial_{G'}(X)| \geq k - 2$. If $|X| \geq 2$, then $|\partial_G(X)| \geq k + 1$, and so $|\partial_{G'}(X)| \geq k - 2$. Now Lemma 3.3 is proved.

LEMMA 3.4. Suppose that $k \geq 2$ is an integer, G is an elemental graph for $T = \{x_1, x_2, x_3, x_4\} \subseteq V(G)$ and k , $|T| = 4$ and $T \in \Gamma(G, k)$. Then

(1) One of the following holds.

(i) $\partial_G(x_1, x_2) \neq \emptyset$, $\partial_G(x_1, x_3) \neq \emptyset$, or for some $y \in V(G) - T$, $N_G(y) = \{x_1, x_2, x_3\}$.

(ii) k is even, $|\partial_G(x_2, x_3)| = k/2$, and

$$|\{y \in V(G) - T \mid N_G(y) = \{x_i, x_1, x_4\}\}| = k/2 \quad (i=2,3).$$

(2) One of the following holds.

(i) For each $1 \leq i < j \leq k$, G has an elemental star S containing x_i and x_j .

(ii) k is even and G is the graph obtained from four cycle by replacing each edge by $k/2$ parallel edges.

(3) If G has no nontrivial k -cut separating T , then

(i) $\partial_G(x_1, x_2) \neq \emptyset$ or G has two elemental stars containing x_1 and x_2 .

(ii) One of the following holds.

(a) G has edge-disjoint paths $P_1[x_1, x_2]$ and $P_2[x_1, x_3]$ such that for $i=2$ or 4 ,

$(x_i, x_3) \in \Gamma(G - \bigcup_{i=1}^2 E(P_i), k-1)$ and $T \in \Gamma(G - \bigcup_{i=1}^2 E(P_i), k-2)$.

(b) For each $e \in \partial_G(x_3) - \partial_G(x_3, x_2)$, G has edge-disjoint paths $P_1[x_1, x_2]$ and $P_2[x_1, x_3]$ such that $e \in E(P_2)$ and $T \in \Gamma(G - \bigcup_{i=1}^2 E(P_i), k-2)$.

Proof. For $1 \leq i, j \leq 4$, set

$$p_{i,j} = |\partial_G(x_i, x_j)|,$$

$$R_i = \{y \in V(G) - T \mid N_G(y) = T - x_i\},$$

$$r_i = |R_i|.$$

(1) Assume $p_{1,2} = p_{1,3} = r_4 = 0$. Then

$$d_G(x_1) = k = p_{1,4} + r_2 + r_3,$$

$$d_G(x_4) = k = p_{1,4} + p_{2,4} + p_{3,4} + r_1 + r_2 + r_3$$

Thus

$$p_{2,4} = p_{3,4} = r_1 = 0.$$

Since $T \in \Gamma(G, k)$, we have

$$|\partial_G(\{x_2, x_3\})| = r_2 + r_3 \geq k.$$

Thus

$$p_{1,4} = 0.$$

By comparing $d_G(x_i)$ with $d_G(x_j)$ for $1 \leq i < j \leq 3$, we have

$$r_2 = r_3 = p_{2,3}.$$

Now (ii) follows.

(2) Assume $p_{1,2} = r_3 = r_4 = 0$. Then by comparing $d_G(x_1) + d_G(x_2)$ with $d_G(x_3) + d_G(x_4)$, we have

$$r_1 = r_2 = p_{3,4} = 0.$$

Now by comparing $d_G(x_3) = k = p_{1,3} + p_{2,3}$ with $d_G(x_i)$ for $i=1, 2$, we have

$$p_{1,4}=p_{2,3} \text{ and } p_{2,4}=p_{1,3}.$$

Moreover

$$|\partial_G(\{x_1, x_4\})| = p_{1,3} + p_{2,4} = 2p_{1,3} \geq k,$$

$$|\partial_G(\{x_1, x_3\})| = p_{1,4} + p_{2,3} = 2p_{1,4} \geq k.$$

Thus

$$p_{1,3}=p_{2,3}=p_{2,4}=p_{1,4},$$

and so (ii) follows.

(3) (i) We assume $p_{1,2}=r_4=0$, and then prove $r_3 \geq 2$.

Since any cut separating $\{x_1, x_3\}$ and $\{x_2, x_4\}$ or separating $\{x_1, x_4\}$ and $\{x_2, x_3\}$ has more than k edges we have

$$(3.1) \quad p_{1,4} + p_{2,3} + p_{3,4} + r_1 + r_2 + r_3 \geq k+1,$$

and

$$(3.2) \quad p_{1,3} + p_{2,4} + p_{3,4} + r_1 + r_2 + r_3 \geq k+1.$$

By comparing $d_G(x_3) + d_G(x_4)$ with (3.1)+(3.2), we have

$$r_3 \geq 2.$$

(ii) If there exists $f \in \partial_G(\{x_1, x_3\})$, then by Lemma 2.1 G has a path $P[x_3, x_2]$ such that $f \in E(P)$, $\{x_3, x_2\} \in \Gamma(G-E(P), k-1)$ and $T \in \Gamma(G-E(P), k-2)$, and so (a) follows. Thus we may let

$$p_{1,3}=p_{1,2}=0,$$

then by (1)

$$r_4 > 0.$$

If $r_3 > 0$, then for $y_1 \in R_4$ and $y_2 \in R_3$,

$$\{x_3, x_4\} \in \Gamma(G - \bigcup_{i=1}^2 \partial_G(y_i), k-1) \text{ and } T \in \Gamma(G - \bigcup_{i=1}^2 \partial_G(y_i), k-2),$$

and so (a) follows. Thus we may let

$$r_3 = 0.$$

Then by (1) and (3)

$$p_{1,4} > 0 \text{ and } r_4 \geq 2.$$

Let y be another end of e , then $y = x_4$ or $y \in R_i$ ($i=1,2$ or 4).

In each case (b) easily follows.

LEMMA 3.5. Suppose that $k \geq 3$ is an odd integer, G is a graph, $\{x_1, x_2, x_3\} \subseteq T \subseteq V(G)$, $x_i \neq x_j$ ($1 \leq i < j \leq 3$), $T \in \mathcal{F}(G, k)$ and $e \in E(G)$. If following (i) or (ii) holds, then for $m=2,3$, G has edge-disjoint paths $P_1[x_1, x_2]$ and $P_2[x_1, x_m]$ such that $e \in E(P_1) \cup E(P_2)$ and $T \in \mathcal{F}(G - \bigcup_{i=1}^2 E(P_i), k-2)$.

(i) $e \in \partial_G(x_1, x_2)$,

(ii) $e \in \partial_G(x_1, y)$ for some $y \in V(G) - T$ with $d_G(y) = 3$ and with $N_G(y) = \{x_1, x_2, x_3\}$.

Proof. Assume that (i) holds. By Theorem 1 if $m=2$, then G has a cycle C such that $e \in E(C)$ and $T \in \mathcal{F}(G - E(C), k-2)$, and if $m=3$, then G has a path $P[x_2, x_3]$ such that $e \in E(P)$ and $T \in \mathcal{F}(G - E(P), k-2)$.

Assume that (ii) holds. We may assume that G is 2-connected. If $d_G(x_3) = d > k$, then we replace x_3 by d vertices of degree k (Figure 4 gives an example with $d=8$ and $k=5$), producing a new graph G' . In G' we assign x_3 on $N_{G'}(y) - \{x_1, x_2\}$. If the result holds in G' , then clearly the result holds in G , and so we may assume that $d_G(x_3) = k$. Let $f \in \partial_G(x_3) - \partial_G(y, x_3)$. By Lemma 3.1

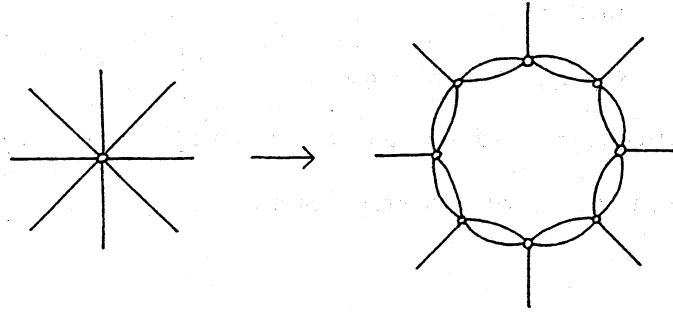


Figure 4.

G has a path $P[x_1, x_2]$ such that $x_3 \notin V(P)$, $T \in \mathcal{P}(G-E(P), k-2)$, $\{x_1, x_2, x_3\} \in \mathcal{P}(G-E(P), k-1)$ and $\{x_i, x_3\} \in \mathcal{P}(G-E(P)-f, k-1)$ ($i=1$ or 2). Then $y \notin V(P)$, because $d_G(x_3)=k$ and $d_G(y)=3$. Moreover $T \in \mathcal{P}(G-E(P)-y, k-2)$. Thus the result follows.

Now we prove Theorem 2. We may assume that G is 2-connected, $d_G(x)=k$ for each $x \in T$ (see the proof of Lemma 3.5 and Figure 4, in this case we can assign x on any vertex of new $d_G(x)$ vertices of degree k) and that $d_G(y)=3$ for each $y \in V(G)-T$ (see Case 1 in the proof of Theorem 1). We proceed by induction on $|E(G)|$. If $|T| \leq 3$, then the results follows from Theorem 1. Thus let $|T| \geq 4$.

Case 1. G has a nontrivial k -cut $\partial_G(X) = \{e_1, \dots, e_k\}$ ($X \subseteq V(G)$) separating T .

We define H, K, u, v, T_H and T_K similarly as in the proof of Theorem 1. If $|X \cap T| = 1$, then the results hold in K , and so in G . Thus let $|X \cap T| \geq 2$ and $|T-X| \geq 2$.

We require the following.

(3.3) If G has a nontrivial k -cut $\partial_G(Y) = \{f_1, \dots, f_k\}$

$(Y \subseteq X)$ separating T , then we may assume that $(X-Y) \cap T \neq \emptyset$.

Proof. Assume $(X-Y) \cap T = \emptyset$. Let $b_i, (c_i)$ be the end of $e_i, (f_i)$ in $Y \cap V(G) - X \cap (Y) (1 \leq i \leq k)$. We may assume that the graph obtained from $\langle X-Y \rangle_G$ by adding $b_1, \dots, b_k, c_1, \dots, c_k, e_1, \dots, e_k, f_1, \dots, f_k$ has edge-disjoint paths $P_1[b_1, c_1], \dots, P_k[b_k, c_k]$. Let G' be the graph obtained from $G - (X-Y)$ by adding new edges $g_1, \dots, g_k$, where g_i has ends b_i and $c_i (1 \leq i \leq k)$. Then $|E(G')| < |E(G)|$, and the results of Theorem 2 hold in G' , and so in G . Now (3.3) is proved.

(3.4) If $|X-T|=2$ ($|T-X|=2$), then we may assume that $H(K)$ is $G(p, q)$ ($G(p', q')$) for some integers p and q (p' and q').

Proof. Assume $|X \cap T|=2$. If H has a nontrivial k -cut $\partial_H(Y) (Y \subseteq V(H) - u)$ separating T_H , then by (3.3) $(X-Y) \cap T \neq \emptyset$, and so $|T \cap Y|=1$. Then by taking Y instead of X the results of Theorem 2 hold. Thus we may assume that an end of each edge of H is in T_H . Hence the result easily follows.

We return to the proof of Theorem 2. By Lemma 3.5 we may assume the following.

(3.5) $\partial_G(a_1, a_i) = \emptyset$ ($i=2, m$) and for each $y \in V(G) - T$, $\{a_1, a_2, a_m\} \not\subseteq N_G(y)$.

Let $a_1 \in X$.

(1) Now $|X-T|=|T-X|=2$. If $a_2 \in X$, then by (3.4) the result easily follows. Thus let $a_2 \in V(G)-X$. Since

$$p+q \geq (k+1)/2 \quad \text{and} \quad p'+q' \geq (k+1)/2,$$

for some $1 \leq i \leq k$, H has an elemental star S_1 containing a_1 and e_i and K has an elemental star S_2 containing a_2 and e_i . Then $T \in \Gamma(G - \bigcup_{i=1}^2 E(S_i), k-1)$.

(2) Subcase 1-1. $\{a_2, a_m\} \subseteq X$.

H has required paths. If one of them passes through u , then we can deduce the result by using Lemma 3.1(3) on K .

Subcase 1-2. $\{a_2, a_m\} \subseteq V(G)-X$ and $|X \cap T|=2$.

Set $X \cap T = \{a_1, a_5\}$. By (3.4) H is $G(p, q)$. Thus if following (3.6) or (3.7) holds, then the result follows.

(3.6) For some $e_i \in \partial_H(u, a_1)$, K has edge-disjoint paths $P_1[v, a_2]$ and $P_2[v, a_m]$ such that $e_i \in E(P_1) \cup E(P_2)$ and $T_K \in \Gamma(K - \bigcup_{i=1}^2 E(P_i), k-2)$.

(3.7) For some $e_i, e_j \in \partial_H(u) - \partial_H(u, a_5)$, K has edge-disjoint paths $P_1[v, a_2]$ and $P_2[v, a_m]$ such that $\{e_i, e_j\} \subseteq E(P_1) \cup E(P_2)$ and $T_K \in \Gamma(K - \bigcup_{i=1}^2 E(P_i), k-2)$.

If $p=0$, then $\partial_H(u, a_5) = \emptyset$, and so (3.7) follows. Thus let $p > 0$. If $|T-X|=2$, then by (3.4) K is $G(p', q')$, and so (3.6) follows. Thus let $|T-X|=3$ and $m=3$. Set $T-X = \{a_2, a_3, a_4\}$.

Subcase 1-2-1. K has nontrivial k -cut $\partial_K(Y)$ ($Y \subseteq V(K)-v$) separating T_K .

By (3.3) We may let $|Y \cap T_K| = |T_K - Y| = 2$. Let K_1 and K_2 be the graphs obtained from K by contracting Y and $V(K)-Y$ to a vertex respectively. Then similarly as (3.4)

K_i is $G(p_i, q_i)$ for some integers p_i and q_i ($i=1,2$)

Let M be

$$\{ \{x_1, x_2\} \subseteq V(K) - T_K \mid \partial_K(x_1, x_2) \neq \emptyset \},$$

and let M' be

$$\{ x \mid \text{For some } N \in M, x \in N \}.$$

For each $N \in M$, $N \cap V(K_i) \neq \emptyset$ ($i=1,2$),

$d_{K-N}(a_j) = d_{K-N}(v) = k-1$ ($j=2,3,4$) and $T_K \in \Gamma(K-N, k-1)$.

If $k=|M|$, then $p_1=p_2=0$ and the result easily follows,

and so let $k > |M|$. $K-M'$ is elemental for T_K and $k-|M|$.

Assume that $k-|M|$ is even and $K-M'$ is the graph obtained from four cycle by replacing each edge by $(k-|M|)/2$ parallel edges. For each cycle C of $K-M'$ such that $|V(C)|=|E(C)|=4$, we have $T_K \in \Gamma(G-E(C), k-2)$. If $\partial_G(a_1, a_4) \neq \emptyset$, then (3.6) follows, and if not, then by (3.5) a_1 is adjacent to p vertices of M' . If $|M| \geq 2$, then (3.6) follows. Thus assume $1 \geq |M| \geq p \geq 1$. Since $(k-|M|)/2 \geq (5-1)/2=2$, for some $1 \leq i < j \leq k$,

$$\{e_i, e_j\} \subseteq \partial_H(u) - \partial_H(u, a_5),$$

and K has a four cycle C such that $|V(C)|=|E(C)|=4$ and

$\{e_i, e_j\} \subseteq E(C)$. Hence (3.7) follows.

By Lemma 3.4(2) we may assume that for each two vertices of T_K , $K-M'$ has an elemental star containing them. Set $a_0=v$, and for $i, j=0,2,3,4$, set

$$p_{i,j} = |\partial_K(a_i, a_j)|,$$

$$r_i = |\{x \in V(K) - T_K \mid N_K(x) = T_K - a_i\}|.$$

For $i, j=0,2,3,4$, since $|\partial_K(\{a_i, a_j\})| \geq k$,

$$p_{i,j} \leq (k-1)/2.$$

If a_1 is adjacent to a vertex of M' in G , then (3.6) follows. If for some $x \in V(G) - T$, $N_G(x) = \{a_1, a_i, a_4\}$ ($i=2$ or 3), then (3.6) follows. Thus and by (3.5) we may assume that

$$|\partial_G(a_1, a_4)| = p.$$

If $a_4 \in Y$, then (3.6) easily follows, and thus let $T_H - Y = \{a_0, a_4\}$. Since $p_{0,4} \geq |\partial_G(a_1, a_4)| = p > 0$, by Lemma 3.4(1) we have

$$p_{4,2} > 0, p_{4,3} > 0, \text{ or } r_0 > 0,$$

and

$$p_{0,2} > 0, p_{0,3} > 0, \text{ or } r_4 > 0.$$

If $r_0 > 0$, $r_4 > 0$, $p_{0,2} \cdot p_{3,4} > 0$, or $p_{0,3} \cdot p_{2,4} > 0$, then (3.6) follows (note that K_i is $G(p_i, q_i)$ for $i=1,2$). Thus we may assume that

$$(3.8) \quad p_{0,2} > 0, p_{2,4} > 0 \text{ and } r_0 = r_4 = p_{0,3} = p_{3,4} = 0.$$

Assume $|M| = 0$. Then

$$d_G(a_3) = p_{2,3} + r_2 \text{ and } p_{2,3} \leq (k-1)/2,$$

and so

$$(3.9) \quad r_2 \geq (k+1)/2 \geq p+1.$$

By comparing $d_G(a_2)$ with $d_G(a_4)$ we have

$$p_{0,2} + p_{2,3} = p_{0,4} + r_2.$$

Thus

$$(3.10) \quad p_{0,2} > p_{0,4} \geq p.$$

From (3.9) and (3.10), (3.7) follows.

Now we may let $|M| > 0$. Since $\{a_2, a_3\} \subseteq Y$, we have

$$|\partial_K(Y)| = k = d_K(a_2) + d_K(a_3) - 2p_{2,3} - |M| \\ = 2k - 2p_{2,3} - |M|,$$

and so

$$2p_{2,3} + |M| = k.$$

Since $dg(a_3) = k = p_{2,3} + r_2 + |M|$,

$$r_2 = p_{2,3}.$$

Since $dg(a_3) = 2r_2 + |M|$, $dg(a_4) = p_{0,4} + p_{2,4} + r_2 + r_3 + |M|$,

and $p_{2,4} > 0$ (by (3.8)), we have

$$(3.11) \quad r_2 \geq a_{0,4} + 1 \geq p + 1.$$

By comparing $dg(a_2)$ with $dg(a_4)$, we have

$$p_{0,2} = p_{0,4}.$$

Thus

$$(3.12) \quad p_{0,2} + |M| \geq p + 1.$$

From (3.11) and (3.12), (3.7) follows.

Subcase 1-2-2. K has no nontrivial k -cut separating T_K .

We may assume that an end of each edge of K in T_K and K is elemental for T_K . The proof is similar as the case $|M| = 0$ in the proof of Subcase 1-2-1.

Subcase 1-3. $\{a_2, a_m\} \subseteq V(G) - X$ and $|X \cap T| = 3$.

Now $m = 3$. By (3.4) K is $G(p', q')$. Set $X \cap T = \{a_1, a_4, a_5\}$

If H has nontrivial k -cut $\partial_H(Y)$ ($Y \subseteq V(H) - u$) separating T_H , then we may let $|Y \cap T_H| = 2$. Then for Y or $V(G) - Y$

instead of X Subcase 1-1 or Subcase 1-2 occurs. Thus we may assume that this is not the case and H is elemental for T_H .

If following (3.13) or (3.14) holds, then the result follows.

(3.13) For some $e_i \in \partial_K(v) - \bigcup_{i=2}^3 \partial_K(v, a_i)$, H has edge-disjoint paths $P_1[a_1, u]$ and $P_2[a_1, u]$ such that

$e_i \in E(P_1) \cup E(P_2)$ and $T_H \in \Gamma(H - \bigcup_{i=1}^2 E(P_i), k-2)$.

(3.14) For $i=2$ or 3 and for some $e_i \in \partial_K(v, x_1)$ and $e_j \in \partial_K(v) - \partial_K(v, x_1)$, H has edge-disjoint paths $P_1[a_1, u]$ and $P_2[a_1, u]$ such that

$\{e_i, e_j\} \subseteq E(P_1) \cup E(P_2)$ and $T_H \in \Gamma(H - \bigcup_{i=1}^2 E(P_i), k-2)$.

Set $a_0 = u$ and for $i, j=0, 1, 4, 5$ set

$$P_{i,j} = |\partial_H(a_i, a_j)|,$$

$$R_i = \{x \in V(H) - T_H \mid N_H(x) = T_H - a_i\},$$

$$r_i = |R_i|.$$

By (3.5) $p_{0,1} = 0$.

Assume $p_{1,4} = p_{1,5} = 0$. If $r_0 \leq (k-1)/2$, then

$$r_4 + r_5 = d_G(a_1) - r_0 \geq (k+1)/2 \geq p' + 1,$$

and so (3.13) or (3.14) follows. Thus let $r_0 \geq (k+1)/2$.

Since $d_G(a_0) = p_{0,4} + p_{0,5} + r_1 + r_4 + r_5$ and

$d_G(a_5) = p_{0,5} + p_{4,5} + r_0 + r_1 + r_4$, we have

$$p_{0,4} + r_5 = p_{4,5} + r_0.$$

Hence

$$d_G(a_4) = k \geq p_{0,4} + r_0 + r_5 \geq 2r_0 > k,$$

a contradiction.

Now we may let $p_{1,i} > 0$ for $i=4$ or 5 , say $i=4$.

Since $p_{0,1} = 0$ and by Lemma 3.4(3), we have

$$r_4 + r_5 \geq 2.$$

For each $x \in R_4 \cup R_5$, if x is adjacent to a vertex of $V(K) - T_K$ in G , then (3.13) follows, thus assume that

$\partial_G(x, a_i) \neq \emptyset$ ($i=2$ or 3). For each $x, y \in R_4 \cup R_5$, if

$\partial_G(x, a_2) \neq \emptyset$ and $\partial_G(y, a_3) \neq \emptyset$, then (3.14) follows,

thus assume that for $i=2$ or 3 , $\partial_G(x, a_i) = \partial_G(y, a_i) = \emptyset$, say $i=3$,

and that $r_4 + r_5 \leq p'$.

Assume $r_4 > 0$. For some $e_i \in \partial_K(v) - \partial_K(v, a_2)$, e_i is incident to a_4 or a vertex of R_1 in G , because

$$p' + q' \geq (k+1)/2 > p_{0,5}.$$

Thus (3.14) follows.

Now we may assume that $r_4 = 0$, $r_5 > 0$ and $p_{1,5} = 0$.

Thus $p_{0,1} = p_{1,5} = r_4 = 0$, contrary to Lemma 3.4(1).

Subcase 1-4. $a_2 \in X$ and $a_m \in V(G) - X$.

now $m=3$.

Subcase 1-4-1. $|X \cap T| = 2$.

By (3.4) $H = G(p, q)$, and by (3.5) $p = 0$. Since $|T_K| \leq 4$, by induction K has a path $p[v, a_3]$ such that $T_K \in \mathcal{P}(K - E(P), k-1)$. Let $e_1 \in E(P)$. H has an elemental star S_1 containing a_1 and e_1 . Let S_2 be another elemental star of H . Then $T_H \in \mathcal{P}(H - \bigcup_{i=1}^2 E(S_i), k-2)$, and so the result follows.

Subcase 1-4-2. $|X \cap T| = 3$ and $|T - X| = 2$.

Assume that H has a nontrivial k -cut $\partial_H(Y) = \{f_1, \dots, f_k\}$ ($Y \subseteq V(H) - u$) separating T_H . Then we may assume that $|Y \cap T_H| = 2$, $a_2 \in Y$ and $a_1 \in X - Y$. Let H_1 (H_2) be the graph obtained from H by contracting $V(H) - Y$ (Y) to a new vertex u_1 (u_2). Then similarly as (3.4) H_i is $G(p_i, q_i)$ for some integers p_i and q_i ($i=1, 2$). If $p_2 = 0$, then the result easily follows. If $p_2 > 0$, then we may let $\{f_1, e_1\} \subseteq \partial_G(a_1)$ and we can easily deduce the result.

Now we may assume that H has no nontrivial k -cut

separating T_H and H is elemental for T_H . Set $X \cap T = \{a_1, a_2, u, a_4\}$ and $T - X = \{a_3, a_5\}$. For a_1, a_2, u, a_4 instead of x_1, x_2, x_3, x_4 , (a) or (b) of Lemma 3.4(3) holds. If (a) holds, then the result easily follows, thus assume that (b) holds. Since $|\partial_H(u) - \partial_H(u, a_2)| \geq (k+1)/2$ and $p' + q' \geq (k+1)/2$, for some $1 \leq i \leq k$,

$$e_i \in \partial_H(u) - \partial_H(u, a_2) \text{ and } e_i \in \partial_K(v) - \partial_K(v, a_5),$$

and so the result follows.

Case 2. G has no nontrivial k -cut separating T .

We may assume that G is elemental for T . If $|T|=4$, then by Lemma 3.3 the result follows. Thus let $|T|=5$ and $m=3$. Set $T = \{a_1, a_2, a_3, a_4, a_5\}$ and for $1 \leq i, j, l \leq 5$, set

$$p_{i,j} = |\partial_G(a_i, a_j)|,$$

$$R(i, j, l) = \{x \in V(G) - T \mid N_G(x) = \{a_i, a_j, a_l\}\},$$

$$r(i, j, l) = |R(i, j, l)|.$$

We require the following.

(3.15) For each distinct $1 \leq i, j, l \leq 5$, G has an elemental star containig $\{a_i, a_j\}$ or $\{a_i, a_l\}$.

Proof. Assume that each elemental star of G does not contain $\{a_1, a_2\}$ nor $\{a_1, a_3\}$. Then

$$d_G(a_1) = p_{1,4} + p_{1,5} + r(1,4,5).$$

Since $p_{i,j} \leq (k-1)/2$ for each i, j , we have $r(1,4,5) > 0$.

Let F be a cut of G separating $\{a_1, a_4, a_5\}$ and $\{a_2, a_3\}$, then

$|F| = d_G(a_4) + d_G(a_5) - (p_{1,4} + p_{1,5} + 2r(1,4,5)) < k$,
a contradiction. Now (3.15) is proved.

We return to the proof of Theorem 2. By (3.5)

$$p_{1,2} = p_{1,3} = r(1,2,3) = 0.$$

If $r(1,2,i) > 0$ and $r(1,3,j) > 0$ ($i, j = 4$ or 5), then the result follows. Thus and by (3.15) we may assume that

$$r(1,2,4) > 0 \text{ and } r(1,3,i) = 0 \quad (i = 4, 5).$$

By (3.15)

$$p_{i,5} + r(i,5,2) + r(i,5,4) > 0 \quad (i = 1, 3).$$

If $p_{1,5} > 0$, $p_{3,5} > 0$, $r(1,5,2) \cdot r(3,5,4) > 0$, or $r(1,5,4) \cdot r(3,5,2) > 0$, then by Lemma 3.3 the result follows.

Thus we may assume that for $(i, j) = (2, 4)$ or $(4, 2)$,

$$p_{1,5} = p_{3,5} = 0, \quad r(1,5,i) = r(3,5,i) = 0,$$

and

$$r(1,5,j) \cdot r(3,5,j) > 0.$$

Assume $r(1,5,2) = r(3,5,2) = 0$. Then

$$d_G(x_1) = p_{1,4} + r(1,2,4) + r(1,4,5),$$

and

$$d_G(x_4) \geq p_{1,4} + r(1,2,4) + r(1,4,5) + r(3,4,5) > k,$$

a contradiction. Thus

$$r(1,5,4) = r(3,5,4) = 0.$$

Since $r(1,2,5) > 0$, by the same argument we have

$$p_{1,4} = p_{3,4} = 0.$$

Thus

$$d_G(x_1) = r(1,2,4) + r(1,2,5)$$

and

$$d_G(x_2) \geq r(1,2,4) + r(1,2,5) + r(2,3,5) > k,$$

a contradiction.

4. PROOF OF THEOREM 3.

Suppose that $k \geq 1$ is an integer, G is a graph, $T = (s_1, \dots, s_k, t_1, \dots, t_k) \subseteq V(G)$ and $T \in \mathcal{P}(G, k)$. We prove that if $|T|=3$, or if k is odd and $|T|=4$ or 5 , then (1.1) holds by induction on k .

Assume $|T|=3$. By Theorem 1 G has a path $p[s_k, s_k]$ such that $T \in \mathcal{P}(G - E(P), k-1)$. By induction for $k-1$, (1.1) holds in $G - E(P)$, and so for k , (1.1) holds.

Assume that $k \geq 5$ is odd and $|T|=4$ or 5 . For some $1 \leq i < j \leq k$, if $|T|=4$, then

$$s_i = s_j \text{ or } t_j,$$

and if $|T|=5$, then

$$s_i = s_j \text{ or } t_j \text{ and } (s_i, t_i) \neq (s_j, t_j),$$

say for $i=k-1$ and $j=k$. By Theorem 2 G has edge-disjoint paths $P_1[s_{k-1}, t_{k-1}]$ and $P_2[s_k, t_k]$ such that $T \in \mathcal{P}(G - \bigcup_{i=1}^2 E(P_i), k-2)$. By induction for $k-2$, (1.1) holds in $G - \bigcup_{i=1}^2 E(P_i)$, and so for k , (1.1) holds in G .

Thus for integer $k \geq 1$,

$$\lambda'(k, 3) = \lambda(k, 3) = k,$$

and for odd integer $k \geq 1$,

$$\lambda'(k, 4) = \lambda'(k, 5) = k.$$

By Lemma 3.2 for odd integer $k \geq 1$,

$$\lambda(k, 4) = \lambda(k, 5) = k \text{ and } \lambda(k+1, 4) = \lambda(k+1, 5) = k+2.$$

Now Theorem 3 is proved.

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