

Minimax Theorems for Vector-Valued Multifunctions *

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1 Introduction

We present a Ky Fan type inequality of mixed kind for vector-valued multifunctions. We use it for proving our first type minimax theorem for vector-valued multifunctions. It is a generalization of the classical Sion minimax theorem for scalar functions (in the compact case), as well as, a generalization of a theorem of Tanaka for vector-valued functions.

We use a vector-valued variant for multifunctions of Ky Fan type inequality, described in the another presentation of us in this volume, in order to derive our second type minimax theorems for vector-valued multifunctions, which is stronger than the first one and uses a special notion of convexity for multifunctions.

The theory of vector optimization has been intensively developed in recent years, as currently the interest is focused on vector-valued multifunctions. Important parts of this theory are the minimax problems and saddle point problems, which have their one specific features with respect to the real-valued case. For a development of such vector-valued problems we refer to [T1-T5] and references therein. The vector-valued, set-valued case proposes more possibilities for definitions of saddle points. In this paper we prove also a Nash equilibrium theorem for vector-valued multifunctions using scalarization and Ky Fan's inequality. As a corollary we obtain a loose saddle point theorem for convex-concave multifunctions (with respect to a specified definition). An advantage in our loose saddle point theorems with respect to the existing ones in the literature (see [K-K], [L-V]) is that our conditions are explicit.

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2 Scalar and vector-valued Ky Fan type inequality of mixed kind

Proposition 2.1 (Scalar Ky Fan type inequality of mixed kind). *Assume that the functions $f, g : K \times K \rightarrow \mathbb{R}$, where K is a compact convex nonempty subset of topological vector space, satisfy the properties:*

- (i) $f(\cdots, y), g(x, \cdot)$ are lower semicontinuous for every $x, y \in K$;
- (ii) $f(x, \cdot), g(\cdot, y)$ are quasi-concave for every $x, y \in K$.
- (iii) $\min\{f(x, y), g(x, y)\} \leq 0 \quad \forall x, y \in K$.

Then there exist $x_0, y_0 \in K$ such that

$$\min\left\{\sup_{y \in K} f(x_0, y), \sup_{x \in K} g(x, y_0)\right\} \leq 0.$$

Proof. Define the function

$$h(\tilde{x}, \tilde{y}, x, y) = \min\{f(\tilde{x}, y), g(x, \tilde{y})\}.$$

It is easy to see that $h(\cdot, \cdot, x, y)$ is lower semicontinuous on $K \times K$ and $h(\tilde{x}, \tilde{y}, \cdot, \cdot)$ is quasiconvex on $K \times K$. Applying the classical scalar Ky Fan's inequality (see for instance [A-E]), we obtain the result. ■

Let Y be a Banach space, $C \subset Y$ a closed convex cone with nonempty interior and E a topological vector space.

Definition 2.2 *The multivalued mapping $F : E \rightarrow 2^Y$ is called C -properly quasiconvex if for every two points $x_1, x_2 \in X$ and every $\lambda \in [0, 1]$ we have either*

$$\begin{aligned} F(\lambda x_1 + (1 - \lambda)x_2) &\subset F(x_1) - C & \text{or} \\ F(\lambda x_1 + (1 - \lambda)x_2) &\subset F(x_2) - C. \end{aligned}$$

If $-F$ is C -properly quasiconvex, then F is called C -properly quasiconcave, which is equivalent to $(-C)$ -properly quasiconvex mapping.

Definition 2.3 *We shall say that the multifunction $F : E \rightarrow 2^Y$ is C -lower semicontinuous at x_0 , if for every $y \in F(x_0)$ and every open $V \ni 0$ there exists an open $U \ni x_0$ such that $(y + V + C) \cap F(x) = \emptyset$ for every $x \in U$.*

Definition 2.4 *The multifunction F is called C -upper semicontinuous at x_0 , if for every $y \in C \cup (-C)$ such that $F(x_0) \subset y + \text{int}C$, there exists an open $U \ni x_0$ such that $F(x) \subset y + \text{int}C$ for every $x \in U$.*

Theorem 2.5 (Ky Fan type inequality of mixed kind for multifunctions). *Suppose that E_1 and E_2 are topological vector spaces, $X \subset E_1$ is a nonempty convex compact subset, $K \subset E_2$ is a nonempty convex compact subset, C is closed convex strongly pointed cone with nonempty interior in a Banach space Y and $F, G : X \times K \rightarrow 2^Y$ are multifunctions satisfying the following conditions:*

- (i) $G(x, \cdot)$ is C -quasiconvex for every $x \in X$, and $F(\cdot, y)$ is C -properly quasiconvex for every $y \in K$;

(ii) $G(\cdot, y)$ is $-C$ -lower semicontinuous for every $y \in K$, and $F(x, \cdot)$ is $-C$ -upper semicontinuous for every $x \in X$.

(iii) for every $x \in X, y \in K$ we have: either $G(x, y) \cap (-\text{int}C) = \emptyset$ or $F(x, y) \not\subset -\text{int}C$

Then there exist $x_0 \in X, y_0 \in K$ such that for every $x \in X, y \in K$ we have: either $G(x_0, y) \cap (-\text{int}C) = \emptyset$ or $F(x, y_0) \not\subset -\text{int}C$.

Proof. Define

$$\varphi((x, y), (x', y')) := \inf\{f(x, y'), g(x', y)\},$$

where

$$f(x, y) = - \inf_{k \in B} \sup_{z \in F(x, y)} h(k, x, z),$$

$$g(x, y) = - \inf_{k \in B} \inf_{z \in G(x, y)} h(k, x, z)$$

and B is an open base of C . Using Lemmas 3.1, 3.3 of [G-T1] we obtain that $\varphi((\cdot, \cdot), (x', y'))$ is lower semicontinuous for every $x', y' \in K$, and by Lemmas 3.2, 3.4 in [G-T1], $\varphi((x, y), (\cdot, \cdot))$ is quasi-concave for every $x \in X, y \in K$. We have also $\varphi((x, y), (x, y)) \leq 0$ for every $x, y \in K$. Applying Proposition 2.1 we obtain the result. ■

We shall denote by $\sup A$ (resp. $\inf A$), where $A \subset Y$, the set of all efficient points of the set $\overline{A}$ (the norm closure of A) with respect to C (resp. with respect to $-C$), i.e.

$$\sup A = \{a \in \overline{A} : (a + C) \cap A = \{a\}\};$$

$$\inf A = \{a \in \overline{A} : (a - C) \cap A = \{a\}\}.$$

Recall that A is bounded with respect to C , if the set $(a + C) \cap A$ is bounded for every $a \in A$. A classical lemma of R. Phelps [Ph], which is equivalent to Ekeland's variational principle and which we shall use in the sequel, states that $\sup A \neq \emptyset$ (resp. $\inf A \neq \emptyset$), if A is bounded with respect to C (resp. with respect to $-C$).

We shall say that the multivalued mapping $F : X \rightarrow 2^Y$, where X is topological space, is bounded with respect to C , if for every $x \in X$ and every $y \in F(x)$ the set $(y + C) \cap F(x)$ is bounded.

3 Minimax theorems

Theorem 3.1 (Minimax theorem I). Suppose that E_1 and E_2 are topological vector spaces, $X \subset E_1$ is nonempty convex compact subset, $K \subset E_2$ is a nonempty convex compact subset, C is closed convex strongly pointed cone with nonempty interior in a Banach space Y and $F, G : X \times K \rightarrow 2^Y$ are multifunctions, bounded with respect to C and $-C$ respectively, and satisfying the following conditions:

(i) $G(x, \cdot)$ is C -quasiconvex for every $x \in X$, and $-F(\cdot, y)$ is C -properly quasiconvex for every $y \in K$;

(ii) $G(\cdot, y)$ is $-C$ -lower semicontinuous for every $y \in K$, and $F(x, \cdot)$ is C -upper semicontinuous for every $x \in X$.

(iii) for every $x \in X, y \in K$ and every two vectors $z_1, z_2 \in Y$ satisfying $z_1 - z_2 \notin C$, we have

either $[G(x, y) - z_1] \cap (-\text{int}C) = \emptyset$, or $z_2 - F(x, y) \notin -\text{int}C$.

Then for every z_1 such that

$$(a) \quad z_1 - \text{int}C \supset \sup_{x \in X} \inf_{y \in K} G(x, y),$$

and for every z_2 such that

$$(b) \quad z_2 + \text{int}C \supset \inf_{y \in K} \sup_{x \in X} F(x, y),$$

we have $z_1 - z_2 \in C$.

Proof. Assume the contrary. By (ii) it follows that $G(\cdot, y) - z_1$ is $-C$ -lower semicontinuous and $z_2 - F(x, \cdot)$ is $-C$ -upper semicontinuous. By (i) it follows that $G(x, \cdot) - z_1$ is C -quasiconvex and $z_2 - F(\cdot, y)$ is C -properly quasiconvex. So, using (iii) we apply Theorem 2.5 and obtain that there exist points x_0, y_0 such that for every $x \in X, y \in K$ we have:

$$\text{either } (G(x_0, y) - z_1) \cap (-\text{int}C) = \emptyset$$

$$\text{or } z_2 - F(x, y_0) \notin -\text{int}C.$$

Assume that there exists $x \in X$ such that

$$z_2 - F(x, y_0) \subset -\text{int}C.$$

Then

$$(G(x_0, y) - z_1) \cap (-\text{int}C) = \emptyset \quad \forall y \in K.$$

This implies

$$\left(\inf_{y \in K} G(x_0, y) \right) \cap (z_1 - \text{int}C) = \emptyset. \quad (1)$$

It is easy to see, using Phelps lemma (see [Ph]) that for any set S which is bounded with respect to C , we have

$$S \subset \sup S - C \quad (2)$$

So, for $S = \inf_{y \in K} G(x_0, y)$, by (2) we have (using (a))

$$\begin{aligned} \inf_{y \in K} G(x_0, y) &\subset \sup_{x \in X} \inf_{y \in K} G(x, y) - C \\ &\subset z_1 - \text{int}C - C \\ &= z_1 - \text{int}C, \end{aligned}$$

which is a contradiction with (1). Therefore

$$z_2 - F(x, y_0) \not\subset -\text{int}C \quad \forall x \in X.$$

This implies

$$\sup_{x \in X} F(x, y_0) \not\subset z_2 + \text{int}C \quad (3)$$

By (b) and (2) we obtain

$$\begin{aligned} z_2 + \text{int}C &= z_2 + \text{int}C + C \\ &\supset \inf_{y \in K} \sup_{x \in X} F(x, y) + C \\ &\supset \sup_{x \in X} F(x, y_0), \end{aligned}$$

which is a contradiction with (3). ■

Definition 3.2 A multifunction $F : E \rightarrow 2^Y$ is called (in the sense of [K-T-H, Definition 3.6])

- (a) type-(v) C -properly quasiconvex if for every two points $x_1, x_2 \in X$ and every $\lambda \in [0, 1]$ we have either $F(\lambda x_1 + (1 - \lambda)x_2) \subset F(x_1) - C$ or $F(\lambda x_1 + (1 - \lambda)x_2) \subset F(x_2) - C$;
- (b) type-(iii) C -properly quasiconvex if for every two points $x_1, x_2 \in X$ and every $\lambda \in [0, 1]$ we have either $F(x_1) \subset F(\lambda x_1 + (1 - \lambda)x_2) + C$ or $F(x_2) \subset F(\lambda x_1 + (1 - \lambda)x_2) + C$.

If $-F$ is type-(v) [resp. type-(iii)] C -properly quasiconvex, then F is said to be type-(v) [resp. type-(iii)] C -properly quasiconcave, which is equivalent to type-(v) [resp. type-(iii)] $(-C)$ -properly quasiconvex mapping.

The following theorem is a generalization (in the compact case) of a scalar two-function result of Simon [S, Theorem 1.4], which in turn is a generalization of Sion's minimax theorem [Si].

Theorem 3.3 (Minimax theorem II). Suppose that E_1 and E_2 are topological vector spaces, $X \subset E_1$ is a nonempty convex compact subset, $K \subset E_2$ is a nonempty convex compact subset, C is closed convex strongly pointed cone with nonempty interior in a Banach space Y and $F, G : X \times K \rightarrow 2^Y$ are multifunctions, bounded with respect to C and $-C$ respectively, such that the set $\bigcup_{y \in K} \sup \bigcup_{x \in X} F(x, y)$ is bounded with respect to $-C$ and the set $\bigcup_{x \in X} \inf \bigcup_{y \in K} G(x, y)$ is bounded with respect to C . Suppose that F and G satisfy the following conditions:

- (i) $G(x, \cdot)$ is type-(iii) C -properly quasiconvex on K for every $x \in X$;
and $F(\cdot, y)$ is type-(iii) C -properly quasiconcave on K for every $y \in K$;
 - (ii) $G(\cdot, y)$ is $-C$ -lower semicontinuous for every $y \in K$, and $F(x, \cdot)$ is C -lower semicontinuous for every $x \in X$.
 - (iii) $F(x, y) - G(x, y) \subset -C$ for every $x \in X, y \in K$.
- Then there exist two points

$$z_1 \in \sup \bigcup_{x \in X} \inf \bigcup_{y \in K} G(x, y)$$

and

$$z_2 \in \inf \bigcup_{y \in K} \sup \bigcup_{x \in X} F(x, y)$$

such that $z_1 - z_2 \in C$.

For the proof of this theorem we need the following result.

Theorem 3.4 ([G-T] Theorem 4.4). Let K be a nonempty convex subset of a topological vector space E , Y a Banach space, and $F : K \times K \rightarrow 2^Y$ a multifunction. Assume that

1. $C : K \rightarrow 2^Y$ is a multifunction with a closed graph such that $C(x)$ is closed convex cone with compact base $B(x) = (2\overline{B}_Y \setminus B_Y) \cap C(x)$ for every x ;
2. for every $x, y \in K$, $F(\cdot, y)$ is $C(x)$ -lower semicontinuous and locally bounded;
3. there exists a multifunction $G : K \times K \rightarrow 2^Y$ such that

- (a) for every $x \in K$, $G(x, x) \subset -C(x)$,
- (b) $F(x, y) \not\subset -C(x)$ implies $G(x, y) \not\subset -C(x)$,

- (c) $G(x, \cdot)$ is type-(iii) $C(x)$ -properly quasiconcave on K for every $x \in K$;
4. there exists a nonempty compact convex subset D of K such that for every $x \in K \setminus D$, there exists $y \in D$ with $F(x, y) \not\subset -C(x)$.

Then, the solutions set

$$S = \{x \in K : F(x, y) \subset -C(x), \text{ for all } y \in K\}$$

is a nonempty and compact subset of D .

Proof of Theorem 3.3. Define the mapping $H : X \times K \times X \times K \rightarrow 2^Y$ by

$$H(\tilde{x}, \tilde{y}, x, y) = F(x, \tilde{y}) - G(\tilde{x}, y).$$

Applying Theorem 3.4 for H we obtain that there exists x_0, y_0 such that

$$H(x_0, y_0, x, y) \subset -C \quad \forall x \in X, \forall y \in K,$$

whence

$$\sup_{x \in X} \cup_{y \in K} F(x, y_0) - \inf_{y \in K} \cup_{x \in X} G(x_0, y) \subset -C. \quad (4)$$

By (2) we obtain

$$\sup_{x \in X} \cup_{y \in K} F(x, y_0) \subset \inf_{y \in K} \sup_{x \in X} \cup_{y \in K} F(x, y) + C$$

and

$$\inf_{y \in K} \cup_{x \in X} G(x_0, y) \subset \sup_{x \in X} \inf_{y \in K} \cup_{x \in X} G(x, y) - C.$$

Therefore, by (4) there exist

$$z_1 \in \sup_{x \in X} \inf_{y \in K} \cup_{y \in K} G(x, y), c_1 \in C$$

and

$$z_2 \in \inf_{y \in K} \sup_{x \in X} \cup_{x \in X} F(x, y), c_2 \in C$$

such that

$$z_2 + c_2 - (z_1 - c_1) \in -C,$$

which implies

$$z_1 - z_2 \in C + c_1 + c_2 \subset C. \quad \blacksquare$$

4 Nash equilibrium and loose saddle point theorems

Definition 4.1 The multifunction $F : E \supset X \rightarrow 2^Z$, where X is a convex nonempty subset, is called C -convex, if for every $x, y \in X, \lambda \in [0, 1], u \in \lambda F(x) + (1 - \lambda)F(y)$ there exists $v \in F(\lambda x + (1 - \lambda)y)$ such that $u - v \in C$. If F is $-C$ -convex, then F is called C -concave.

Let $k^0 \in \text{int}C$ be fixed. Define the functions

$$h(x) = \inf \{t \in \mathbf{R} : x \in tk^0 - C\},$$

$$\varphi(x) = \inf h(F(x)),$$

$$\psi(x) = \sup h(F(x)).$$

It is easy to see that h is continuous and sublinear (see [Tam1], [Tam2]).

Lemma 4.2 *Let the multifunction $F : E \supset X \rightarrow 2^Z$ be C -convex. Then the function φ is convex.*

Proof. Let $x_1, x_2 \in X$. By definition of φ and h , for every $\varepsilon > 0$ there exist $z_i \in F(x_i), t_i \in \mathbf{R}, i = 1, 2$ such that

$$z_i - t_i k^0 \in -C \quad (5)$$

and

$$t_i < \varphi(x_i) + \varepsilon.$$

By definition of C -convex multifunction,

$$\exists v \in F(\lambda x_1 + (1 - \lambda)x_2) : \lambda z_1 + (1 - \lambda)z_2 \in v + C. \quad (6)$$

By (5) we have

$$-C \ni \lambda(z_1 - t_1 k^0) + (1 - \lambda)(z_2 - t_2 k^0) = \lambda z_1 + (1 - \lambda)z_2 - (\lambda t_1 + (1 - \lambda)t_2)k^0. \quad (7)$$

By (6) and (7) we have

$$\begin{aligned} v &\in \lambda z_1 + (1 - \lambda)z_2 - C \\ &\subset (\lambda t_1 + (1 - \lambda)t_2)k^0 - C - C \\ &= (\lambda t_1 + (1 - \lambda)t_2)k^0 - C. \end{aligned}$$

Hence

$$\begin{aligned} h(v) &\leq \lambda t_1 + (1 - \lambda)t_2 \\ &< \lambda \varphi(x_1) + (1 - \lambda)\varphi(x_2) + 2\varepsilon. \end{aligned}$$

Therefore

$$\varphi(\lambda x_1 + (1 - \lambda)x_2) := \inf_{z \in F(\lambda x_1 + (1 - \lambda)x_2)} h(z) \leq \lambda \varphi(x_1) + (1 - \lambda)\varphi(x_2) + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrarily small, we obtain

$$\varphi(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda \varphi(x_1) + (1 - \lambda)\varphi(x_2). \quad \blacksquare$$

Definition 4.3 *The multifunction $F : E \rightarrow 2^Z$ will be called (C, k^0) -upper semicontinuous at x_0 , if for every $\varepsilon > 0$ there exists an open $U \ni x_0$ such that*

$$[(\varphi(x_0) - \varepsilon)k^0 - C] \cap F(x) = \emptyset \quad \forall x \in U.$$

Lemma 4.4 *If F is $-C$ -lower semicontinuous, then φ is upper semicontinuous.*

Proof. Let $x_0 \in E, \varepsilon > 0$ be fixed and $y_0 \in F(x_0)$ be such that

$$h(y_0) < \inf h(F(x_0)) + \varepsilon.$$

By continuity of h , there exists an open $V \ni 0$ such that

$$h(v) < \varepsilon \quad \forall v \in V.$$

By definition of $-C$ -lower semicontinuity, there exists an open $U \ni x_0$ such that

$$F(x) \cap (y_0 + V - C) \neq \emptyset \quad \forall x \in U.$$

Let $y \in F(x) \cap (y_0 + V - C)$. Then $y = y_0 + v - c$ for some $v \in V, c \in C$ and we can write

$$\begin{aligned} \varphi(x) &= \inf_{y' \in F(x)} h(y') \\ &\leq h(y) \\ &\leq h(y_0) + h(v) + h(-c) \quad (\text{by sublinearity of } h) \\ &\leq \varphi(x_0) + 2\varepsilon. \end{aligned}$$

Lemma 4.5 *If F is (C, k^0) -upper semicontinuous, then φ is lower semicontinuous.*

Proof. Let $x_0 \in E, y \in F(x_0)$ and $x \in U$, where U is given by the definition of (C, k^0) -upper semicontinuity of F at x_0 . Let $z \in F(x)$. Then by definition we have:

$$\begin{aligned} 0 &\leq \inf\{t : z - tk^0 \in (\varphi(x_0) - \varepsilon)k^0 - C\} \\ &= \inf\{t : z - (t + \varphi(x_0) - \varepsilon)k^0 \in -C\} \\ &= \varepsilon - \varphi(x_0) + \inf\{t : z - tk^0 \in -C\} \\ &= \varepsilon - \varphi(x_0) + h(z). \end{aligned}$$

Hence $\varphi(x_0) \leq h(z) + \varepsilon$, and $z \in F(x)$ is arbitrary, this implies $\varphi(x_0) \leq \varphi(x) + \varepsilon$. ■

Below we prove a Nash equilibrium type theorem and a loose saddle point theorem. The proofs are based on scalarization via the previous lemmas and on the Ky Fan inequality.

Let E_1, E_2 be topological vector spaces, Z be a Banach space, $X \subset E_1, Y \subset E_2$ be convex compact nonempty subsets and $C_i \subset Z$ be closed convex cones with nonempty interiors, $k_i^0 \in \text{int}C_i, i = 1, 2$.

Theorem 4.6 (Nash equilibrium). *Let the multifunctions $F_i : X \times Y \rightarrow 2^Z$ be (C_i, k_i^0) -upper semicontinuous. Assume that $F_1(\cdot, y)$ is C_1 -convex for every $y \in Y$, $F_1(x, \cdot)$ is $-C_1$ -lower semicontinuous for every $x \in X$, $F_2(x, \cdot)$ is C_2 -convex for every $x \in X$ and $F_2(\cdot, y)$ is $-C_2$ -lower semicontinuous for every $y \in Y$. Then there exists a Nash equilibrium, $(x_0, y_0) \in X \times Y$, which means*

$$\begin{aligned} F_1(x, y_0) \cap [\inf h(F_1(x_0, y_0))k_1^0 - \text{int}C_1] &= \emptyset \quad \forall x \in X, \\ F_2(x_0, y) \cap [\inf h(F_2(x_0, y_0))k_2^0 - \text{int}C_2] &= \emptyset \quad \forall y \in Y. \end{aligned}$$

Proof. Define

$$f(x, y, \bar{x}, \bar{y}) = \inf h(F_1(x, y)) - \inf h(F_1(\bar{x}, y)) + \inf h(F_2(x, y)) - \inf h(F_2(x, \bar{y}))$$

By Lemma 4.2, $f(x, y, \cdot, \cdot)$ is concave for every $x \in X, y \in Y$ and by Lemmas 4.4, 4.5, $f(\cdot, \cdot, \bar{x}, \bar{y})$ is lower semicontinuous for every $\bar{x} \in X, \bar{y} \in Y$. By Ky Fan's inequality (see [A-E, Theorem 6.3.5]) there exists $(x_0, y_0) \in X \times Y$ such that

$$\sup_{(\bar{x}, \bar{y}) \in X \times Y} f(x_0, y_0, \bar{x}, \bar{y}) \leq 0$$

Putting $\bar{y} = y_0$ we obtain

$$\inf h(F_1(x_0, y_0)) \leq \inf h(F_1(x, y_0)) \quad \forall x \in X, \quad (8)$$

and putting $\bar{x} = x_0$ we obtain

$$\inf h(F_2(x_0, y_0)) \leq \inf h(F_2(x_0, y)) \quad \forall y \in Y. \quad (9)$$

But (8) implies

$$F_1(x, y_0) \cap [\inf h(F_1(x_0, y_0))k_1^0 - \text{int}C_1] = \emptyset$$

and (9) implies

$$F_2(x_0, y) \cap [\inf h(F_2(x_0, y_0))k_2^0 - \text{int}C_2] = \emptyset,$$

which finishes the proof. ■

In the special case when $F_1 = -F_2$ and $C_1 = C_2 = C$, $k_1^0 = k_2^0 = k^0$, we obtain the following loose saddle point theorem.

Theorem 4.7 (Loose saddle point theorem). *Suppose that the multifunction $F : X \times Y \rightarrow 2^Z$ have compact images and is (C, k^0) -lower semicontinuous and $(-C, -k^0)$ -lower semicontinuous, $F(\cdot, y)$, $y \in Y$ is C -convex and C -lower semicontinuous, $F(x, \cdot)$, $x \in X$ is C -concave and $-C$ -lower semicontinuous. Then there exists a loose saddle point $(x_0, y_0) \in X \times Y$, namely there exist $z_1, z_2 \in F(x_0, y_0)$, such that*

$$(z_1 - \text{int}C) \cap F(x, y_0) = \emptyset \quad \forall x \in X,$$

$$(z_2 + \text{int}C) \cap F(x_0, y) = \emptyset \quad \forall y \in Y.$$

REFERENCES

- [A-E] J.-P. Aubin, I. Ekeland, *Applied Nonlinear Analysis*, John Wiley & Sons, 1984.
- [G-T1] P. Gr. Georgiev, T. Tanaka, *Vector-valued, set-valued variants of Ky Fan's inequality*, *Journal of Nonlinear and Convex Analysis*, 1(3) (2000), 245–254.
- [G-T12] P. Gr. Georgiev, T. Tanaka, *Fan's inequality for set-valued maps*, to appear in *Proceedings of the third World Congress of Nonlinear Analysts*, Catania, Italy, 2000.
- [T1] T. Tanaka, *Generalized semicontinuity and existence theorems for cone saddle points*, *Appl. Math. Optim.*, 36 (1997), 313–322.
- [T2] T. Tanaka, *Generalized quasiconvexities, cone saddle points and minimax theorem for vector-valued functions*, *J. Optim. Theory Appl.* 81(2) (1994), 355–377.
- [T3] T. Tanaka, *Two types of minimax theorems for vector-valued functions*, *J. Optim. Theory Appl.*, 68(2) (1991), 321–334.
- [T4] T. Tanaka, *Existence theorems for cone saddle points of vector-valued functions in infinite-dimensional spaces*, *J. Optim. Theory Appl.*, 62 (1989), 127–138.
- [T5] T. Tanaka, *Some minimax problems of vector-valued functions*, *J. Optim. Theory Appl.*, 59 (1988), 505–524.
- [K-K] I.-S. Kim and Y.-T. Kim, *Loose saddle points of set-valued maps in topological vector spaces*, *Appl. Math. Letters* 12 (1999), 21–26.
- [K-T-H] D. Kuroiwa, T. Tanaka, and T.X.D. Ha, *On cone convexity of set-valued maps*, *Nonlinear Anal. TMA*, 30(3) (1997), 1487–1496.

[L-V] D. T. Luc and C. Vargas, *A saddle point theorem for set-valued maps*, Nonlinear Anal. 18, 1–7 (1992).

[Ph] R.R.Phelps, *Convex functions, monotone operators and differentiability*. Lecture Notes in Mathematics. Springer Verlag, Vol.1364.

[S] S. Simons, *Two-function minimax theorems and variational inequalities for functions on compact and noncompact sets, with some comments on fixed-point theorems*, Proc. Symp. Pure Math., 45 (1986), Part 2, 377–392.

[Si] M. Sion, *On a general minimax theorems*, Pacific J. Math. 8 (1958), 171–176.

[Tam1] Chr. Tammer, *A generalization of Ekeland's variational principle*, Optimization 25, 1992, 129–141.

[Tam2] Chr. Tammer, *A variational principle and applications for vectorial control approximation problems*, Reports of the Inst. of Optimization and Stohastics, Martin-Luther Universität Halle-Wittenberg, 1996.