

A note on equalities in group algebras and character values

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1 Introduction

In this report I reproduce my 20 minutes talk given at Kyoto RIMS on March 9, 1998.

Let G and H be finite groups of order n . A mapping f from G into H is called a planar function of degree n if, for each element $\tau \in H$ and $u \in G^* = G - \{1\}$, there exists exactly one $x \in G$ such that $f(ux)f(x)^{-1} = \tau$. In [2] Hiramine has shown that if both G and H are abelian groups of order $3p$ with $p(\geq 5)$ a prime, then there exists no planar function from G into H . To prove this he has established two results on character values. Their proofs are slightly complicated. In this note we shall give short proofs.

We follow the notation and terminology of [2].

2 Planar Functions and Equations in Group Algebras

Let G and H be finite groups of order n . Throughout this article elements of G will be denoted by small Roman letters and elements of H by small Greek letters.

Let f be a mapping from G into H and $S_\alpha = \{x \in G | f(x) = \alpha\}$, $\alpha \in H$. If $S_\alpha \neq \emptyset$, we set $\hat{S}_\alpha = \sum_{x \in S_\alpha} x \in C[G]$ and $\hat{S}_\alpha^{-1} = \sum_{x \in S_\alpha} x^{-1} \in C[G]$,

otherwise $\hat{S}_\alpha = \hat{S}_\alpha^{-1} = 0$, where $C[G]$ is the group algebra of G over the complex number field C . Let $G_0 = G \times H$ be the direct product of groups G, H .

To prove the results we need two propositions.

The following is Proposition 2.1 [2].

Proposition 1 *The following are equivalent.*

(i) *The function f is planar.*

(ii) *In the group algebra $C[G]$ of G ,*

$$\sum_{\alpha \in H} \hat{S}_{\tau\alpha} \hat{S}_\alpha^{-1} = \sum_{\alpha \in H} \hat{S}_{\alpha\tau}^{-1} \hat{S}_\alpha = \begin{cases} \hat{G} + n - 1 & \text{if } \tau = 1, \\ \hat{G} - 1 & \text{otherwise.} \end{cases}$$

REMARK 1. If $\tau \neq 1$, then it follows from the equation in (ii) of the proposition above that in the group algebra $C[G_0]$ of G_0 ,

$$\sum_{\alpha \in H} \hat{S}_{\tau\alpha} \tau \alpha \hat{S}_\alpha^{-1} \alpha^{-1} = (\hat{G} - 1)\tau.$$

We prove the following

Proposition 2 *We have in $C[G_0]$,*

$$\left(\sum_{\alpha \in H} \hat{S}_\alpha \alpha \right) \left(\sum_{\beta \in H} \hat{S}_\beta^{-1} \beta^{-1} \right) = \hat{G} + n - 1 + \sum_{\tau \in H, \tau \neq 1} (\hat{G} - 1)\tau.$$

In order to prove proposition 2 we need the Remark 1 above and an easy

Lemma 1

$$\left(\sum_{\alpha \in H} \alpha \right) \left(\sum_{\beta \in H} \beta^{-1} \right) = \sum_{\tau \in H} \sum_{\beta \in H} (\beta\tau) \beta^{-1}.$$

3 Proofs of Hiramine' Results

We start with the following well-known facts about character theory. These facts play important parts in the proofs of his results.

Fact 1 Let G be an abelian group and χ an arbitrary (linear) character of G . Then χ is a homomorphism from G into $C^* = C - \{0\}$. So we can extend this homomorphism χ to an algebra homomorphism $\bar{\chi}$ from $C[G]$ into C .

Fact 2 Let H_1, H_2 be finite groups and G_1 the direct product of H_1, H_2 . Then all irreducible characters of G_1 are obtained as follows. Let $\chi_0, \dots, \chi_{s-1}$ be the irreducible characters of H_1 , $\rho_0, \dots, \rho_{t-1}$ the irreducible characters of H_2 . Then G_1 has exactly st irreducible characters $\Psi_{ij} (0 \leq i \leq s-1, 0 \leq j \leq t-1)$, satisfying $\Psi_{ij}(h_1 h_2) = \chi_i(h_1) \rho_j(h_2)$, where $h_1 \in H_1, h_2 \in H_2$.

PROOF. See [1, p.54].

REMARK 2. In Fact 2 if both χ_i and ρ_j are linear characters, then Ψ_{ij} is a homomorphism from G_1 to C^* . As in Fact 1, we have an algebra homomorphism $\bar{\Psi}_{ij}$ from $C[G_1]$ into C which is an extension of Ψ_{ij} .

In the remainder of this section we assume that f is a planar function and that G is an abelian group of order n . Let $\chi_0 (= 1_G), \dots, \chi_{n-1}$ be the irreducible (linear) characters of G , where 1_G denote the principal character of G . We set

$$d_i^{(\alpha)} = \begin{cases} \sum_{x \in S_\alpha} \chi_i(x) & \text{if } S_\alpha \neq \emptyset, \\ 0 & \text{if } S_\alpha = \emptyset \end{cases}$$

for each $0 \leq i \leq n-1$ and for each $\alpha \in H$. Now we state Hiramine' results [2] and give our proof to Result 2.

Result 1 The following hold

(i) $d_0^{(\alpha)} = |S_\alpha|$ and

$$\sum_{\alpha \in H} d_0^{(\tau\alpha)} d_0^{(\alpha)} = \sum_{\alpha \in H} d_0^{(\alpha\tau)} d_0^{(\alpha)} = \begin{cases} 2n-1 & \text{if } \tau = 1, \\ n-1 & \text{otherwise.} \end{cases}$$

(ii) For $i \neq 0$,

$$\sum_{\alpha \in H} d_i^{(\tau\alpha)} \overline{d_i^{(\alpha)}} = \sum_{\alpha \in H} \overline{d_i^{(\alpha\tau)}} d_i^{(\alpha)} = \begin{cases} n-1 & \text{if } \tau = 1, \\ -1 & \text{otherwise.} \end{cases}$$

(Here $\bar{d}$ denotes the complex conjugate of $d \in C$.)

PROOF. We omit our proof of this result.

Result 2 *With the same notation and assumption as in Result 1, suppose that H is abelian and let $\rho_0(= 1_H), \dots, \rho_{n-1}$ be the irreducible characters of H . Set $z_{ij} = \sum_{\alpha \in H} d_i^{(\alpha)} \rho_j(\alpha)$. Then,*

(i) $z_{0,0} = n$, and $z_{i,0} = 0 (i \neq 0)$

(ii) For $j \neq 0$, $z_{ij} \overline{z_{ij}} = n$.

PROOF. Since χ_i and ρ_j are linear, from Remark 2 we see that $\overline{\Psi}_{ij}$ is an algebra homomorphism from $C[G_0]$ into C . First we shall prove (ii). We apply $\overline{\Psi}_{ij}$ ($j \neq 0$) to the equation in Proposition 2. Then we get (ii). We have proved (ii). Next we shall prove (i). Similarly by using $\overline{\Psi}_{i0}$ we can prove (i). This completes the proof of Result 2.

REMARK 3. Nakagawa[3] has proved Result 2 by using Gaussian sums.

References

- [1] L. Dornhoff: *Group representation theory, Part A*, Marcel Dekker, New York, 1971.
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- [3] N. Nakagawa: *Left cyclic planar functions of degree p^n* , Utilitas Math. in press.