

Asymptotic Stability for the Linear Integro-Differential Equation $\dot{x}(t) = ax(t) - b \int_{t-h}^t x(s) ds$

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1 Introduction

In this paper, we will discuss the uniform asymptotic stability of the zero solution of the linear integro-differential equation

$$\dot{x}(t) = ax(t) - b \int_{t-h}^t x(s) ds, \quad (\text{E})$$

where a and b are real and $h > 0$. As a special case, for $a = 0$, (E) becomes

$$\dot{x}(t) = -b \int_{t-h}^t x(s) ds \quad (\text{I})$$

and in [4] it is shown that the zero solution of (I) is uniformly asymptotically stable if and only if

$$0 < bh^2 < \frac{\pi^2}{2}.$$

There are also some stability results for (I) with a generalized continuously distributed delay which is expressed in Stieltjes integral [3]. In case $a < 0$, some sufficient stability conditions for $a < 0$ are obtained by using Liapunov functionals in [1].

But, there exist no results on the stability of (E) for the case $a > 0$ as far as the authors know. So, we will study (E) for $a > 0$ and give results on the uniform asymptotic stability of (E).

2 Main results

We obtain the following theorems on the uniform asymptotic stability of

$$\dot{x}(t) = ax(t) - b \int_{t-h}^t x(s) ds, \quad (\text{E})$$

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where $a > 0$, b is real and $h > 0$.

Theorem 2.1. *Let $a^2 < 2b$. Then, the zero solution of (E) is uniformly asymptotically stable if and only if*

$$\frac{a}{b} < h < \frac{1}{\sqrt{2b - a^2}} \operatorname{Cos}^{-1} \frac{a^2 - b}{b} \quad (2.1)$$

is satisfied.

Theorem 2.2. *Let $a^2 > 2b$. Then, the zero solution of (E) is not uniformly asymptotically stable for all $h > 0$.*

To give the proofs of Theorems 2.1 and 2.2 with the root analysis, we need to consider the characteristic equation of (E) which is expressed in the form

$$\lambda = a - b \int_{-h}^0 e^{\lambda s} ds \quad (C)$$

and we introduce the following results which are used without proofs.

Theorem A. [2] *The zero solution of (E) is uniformly asymptotically stable if and only if any root of (C) has a negative real part.*

Now, let $\nu(h)$ be the number of roots of (C) including multiplicity whose real parts are positive at h . Then, the following property holds.

Theorem B. [5] *Let $h_i (i = 0, 1, 2, \dots)$ be constants at which (C) has a root on the imaginary axis of the complex plane. Then, the number $\nu(h)$ is constant on each interval $h_i < h < h_{i+1}$.*

Lemma C. [6] *If $a - bh \geq 0$, then (C) has a nonnegative real root.*

Lemma C implies that if $0 < h \leq a/b$ then the zero solution of (E) is not uniformly asymptotically stable by Theorem A. In case $b \leq 0$, (C) has a nonnegative real root for all $h > 0$ because the condition $a - bh > 0$ is satisfied. Thus, in case $b \leq 0$, we also see easily that the zero solution of (E) is not uniformly asymptotically stable. Hereafter, we assume $b > 0$.

3 The proof of main results

In this section, we will prove Theorems 2.1 and 2.2 regarding a and b are fixed constants and h is a variable. At first, we give a proof of Theorem 2.1.

Proof of Theorem 2.1. (Sufficiency.) Our purpose is to show $\nu(h) = 0$ when h satisfies the condition (2.1). However, it is difficult to get $\nu(h) = 0$ directly only by using the condition (2.1). So at first we will discuss the case $h \in (0, a/b)$.

Before proving sufficiency of Theorem 2.1, we give four lemmas. Now, we consider the case where h increases minutely from zero. Then, we have

Lemma 3.1. *For a sufficiently small $h > 0$, (C) has no pairs of imaginary roots $\lambda = x \pm iy$ such that $x > 0, y > 0$.*

Proof. Suppose that there exists a pair of imaginary roots $\lambda = x \pm iy$ ($x > 0, y > 0$). Here, we note that (C) has a root of complex conjugate. Thus, it is sufficient to discuss $\lambda = x + iy$ only. Then, substituting $\lambda = x + iy$ for (C), we have

$$x = a - b \int_{-h}^0 e^{xs} \cos ys \, ds, \quad (3.1)$$

$$y = -b \int_{-h}^0 e^{xs} \sin ys \, ds. \quad (3.2)$$

From (3.2),

$$y \leq b \left| \int_{-h}^0 e^{xs} \sin ys \, ds \right| \leq b \int_{-h}^0 e^{xs} |\sin ys| \, ds.$$

Also, from $|\sin ys| \leq y|s|$,

$$y \leq by \int_{-h}^0 e^{xs} |s| \, ds < bhy \int_{-h}^0 e^{xs} \, ds = bhy \times \frac{1}{x} (1 - e^{-xh}) < \frac{bhy}{x}.$$

Hence,

$$0 < x < bh,$$

so that $x \rightarrow +0$ as $h \rightarrow +0$. However, from (3.1), we have $x \rightarrow a - 0$ as $h \rightarrow +0$, which is a contradiction. This completes the proof of Lemma 3.1. $\square$

Lemma 3.2. *Suppose $a^2 < 2b$ and $0 < h < a/b$. Then, (C) has a positive real root. Moreover, the positive real root is simple.*

Proof. Suppose $\lambda = x$ is a root of (C). Now, we define the characteristic function of (E) expressed as

$$p(\lambda, a, b, h) := \lambda - a + b \int_{-h}^0 e^{\lambda s} \, ds. \quad (3.3)$$

Then from (3.3),

$$p(x, a, b, h) = x - a + b \int_{-h}^0 e^{xs} \, ds = 0. \quad (3.4)$$

(3.4) is reduced to

$$\frac{1}{x}\{x^2 - ax + b - be^{-xh}\} = 0.$$

Here, we define the function $q(x)$ as

$$q(x) := x^2 - ax + b - be^{-xh}. \quad (3.5)$$

Thus, it is sufficient to show that there exists a root $x > 0$ which satisfies $q(x) = 0$. From (3.5),

$$q'(x) = 2x - a + bhe^{-xh}, \quad (3.6)$$

$$q''(x) = 2 - bh^2e^{-xh}. \quad (3.7)$$

From (3.6) and (3.7), we obtain

$$q(0) = 0, \quad q'(0) = -a + bh < 0, \quad q(a) = b(1 - e^{-ah}) > 0.$$

Then, there exists the root $x^* \in (0, a)$ which satisfies $q(x^*) = 0$. Moreover, we note that $q''(x)$ is monotone increasing for all $x \geq 0$ and

$$q''(0) = 2 - bh^2 > 2 - b\left(\frac{a}{b}\right)^2 = 2 - \frac{a^2}{b} > 0,$$

so we have $q''(x) > 0$ on $[0, \infty)$. Therefore, $q'(x)$ is monotone increasing on $[0, \infty)$ and x^* is determined uniquely.

Here, by Rolle's theorem we also see that there exists the root $\tilde{x} \in (0, x^*)$ which satisfies $q'(\tilde{x}) = 0$. Since $q'(x)$ is monotone increasing on $[0, \infty)$, we have $q'(x^*) > 0$, which implies that $x = x^* > 0$ is a simple root of $q(x) = 0$. $\square$

Now, we consider a case where (C) has a root on the imaginary axis of the complex plane. At first, we assume that (C) has a pair of purely imaginary roots $\lambda = \pm i\omega$ ($\omega > 0$; constant) at the first time $h = h^*$ when h is increases from zero. Then for $\lambda \neq 0$, we rewrite (C) as follows:

$$\lambda^2 = a\lambda - b(1 - e^{-\lambda h}), \quad (C^*)$$

where $a > 0$, $b > 0$ and $h > 0$. Substituting $\lambda = i\omega$ and $h = h^*$ for (C*),

$$-\omega^2 = ia\omega - b(1 - e^{-i\omega h^*}).$$

From the above,

$$-\omega^2 = -b + b \cos \omega h^*, \quad (3.8)$$

$$a\omega = b \sin \omega h^*. \quad (3.9)$$

From (3.8) and (3.9),

$$\left(\frac{b-\omega^2}{b}\right)^2 + \left(\frac{a\omega^2}{b}\right)^2 = 1.$$

Because of $\omega > 0$, we have $a^2 < 2b$ and

$$\omega = \sqrt{2b - a^2}. \quad (3.10)$$

Substituting (3.10) for (3.8) and (3.9),

$$\cos \sqrt{2b - a^2} h^* = \frac{a^2 - b}{b}, \quad (3.11)$$

$$\sin \sqrt{2b - a^2} h^* = \frac{a\sqrt{2b - a^2}}{b} > 0. \quad (3.12)$$

From (3.11) and (3.12),

$$h^* = \frac{1}{\sqrt{2b - a^2}} \operatorname{Cos}^{-1} \frac{a^2 - b}{b}.$$

We see that (C) has a pair of purely imaginary roots $\lambda = \pm i\omega$ at $h = h^*$. However, we have not yet made reference to the case where (C) has the root $\lambda = 0$. Next, we prove the following lemma on the root $\lambda = 0$.

Lemma 3.3. (C) has the root $\lambda = 0$ at $h = a/b$. Moreover, there exist no positive real roots at $h = a/b$.

Proof. We will give a proof with the characteristic function $p(x, a, b, h)$ expressed by (3.3), where $x \in \mathbf{R}$. We have

$$p(0, a, b, \frac{a}{b}) = -a + b \left\{ 0 - \left(-\frac{a}{b} \right) \right\} = 0,$$

then (C) has the root $\lambda = 0$ at $h = a/b$.

Now, we assume that (C) has a positive real root $x = x^*$ at $h = a/b$, that is, $p(x^*, a, b, a/b) = 0$. Then,

$$\frac{\partial}{\partial x} p(x, a, b, \frac{a}{b}) = \frac{1}{x^2} \{ x^2 + axe^{-\frac{a}{b}x} - b + be^{-\frac{a}{b}x} \}.$$

Here, we define the function $f(x)$ as

$$f(x) := x^2 + axe^{-\frac{a}{b}x} - b + be^{-\frac{a}{b}x}.$$

Then,

$$f'(x) = 2x \left(1 - \frac{a^2}{2b} e^{-\frac{a}{b}x} \right).$$

Because of $a^2 < 2b$, $f'(x) > 0$ holds for all $x > 0$. Hence, $f(x)$ is monotone increasing for all $x > 0$. Moreover, from $f(0) = 0$, we have $f(x) > 0$ for all $x > 0$. This implies that $\frac{\partial}{\partial x}p(x, a, b, a/b) > 0$ for all $x > 0$. Since $p(0, a, b, a/b) = 0$ and $\frac{\partial}{\partial x}p(x, a, b, a/b) > 0$ is satisfied for all $x > 0$, so we obtain $p(x^*, a, b, a/b) > 0$, which contradicts the initial assumption. Therefore, (C) has no positive real root at $h = a/b$. It is clear that (C) has the root $\lambda = 0$ at $h = a/b$. Thus, the proof of Lemma 3.3 is complete. $\square$

Now, we show that $h^* > a/b$ if $0 < a^2 < 2b$. Let b be fixed and define the function $g(a)$ as

$$g(a) := \sqrt{2b - a^2} \left(\frac{1}{\sqrt{2b - a^2}} \text{Cos}^{-1} \frac{a^2 - b}{b} - \frac{a}{b} \right).$$

Then,

$$g(a) = \text{Cos}^{-1} \frac{a^2 - b}{b} - \frac{a\sqrt{2b - a^2}}{b}.$$

It is easily seen that $g(0) = \pi$, $g(\sqrt{2b}) = 0$ and $g'(a) = -2\sqrt{2b - a^2}/b < 0$ for all $a \in (0, \sqrt{2b})$. Therefore, we have that $g(a) > 0$ for all $a \in (0, \sqrt{2b})$, so $h^* > a/b$ holds.

By Lemmas C and 3.2, (C) has a nonnegative and simple real root for $h \in (0, a/b]$. We also see that (C) has no imaginary roots for a sufficiently small $h > 0$ by Lemma 3.1. Moreover, (C) has the zero root but no positive real roots by Lemma 3.3. Here, noting that (C) has no pair of purely imaginary roots during h moves from zero to h^* , we see that a root of whose real part is nonnegative for $h \in (0, a/b]$ is unique and the real root determined by Lemma C. Thus, we have $\nu(h) = 1$ for $h \in (0, a/b]$.

Finally, we investigate $\nu(h)$ for the case $h > a/b$. Now, we will investigate the behavior of the root $\lambda(h)$ with $\lambda(a/b) = 0$. Then, we show the following lemma for the behavior of $\lambda(h)$.

Lemma 3.4. *For the root $\lambda(h)$ with $\lambda(a/b) = 0$, $\text{Re}(d\lambda/dh)|_{\lambda=0, h=a/b} < 0$ holds.*

Proof. We again use the characteristic function $p(\lambda, a, b, h)$ given by (3.3). Then, we obtain

$$\frac{\partial}{\partial \lambda}p(\lambda, a, b, h) = 1 + b \int_{-h}^0 se^{\lambda s} ds, \quad \frac{\partial}{\partial h}p(\lambda, a, b, h) = be^{-\lambda h}.$$

By the theorem on implicit function, we obtain

$$\frac{d\lambda}{dh} = -\frac{be^{-\lambda h}}{1 + b \int_{-h}^0 se^{\lambda s} ds}. \quad (3.13)$$

Here, we investigate the behavior of the root $\lambda(a/b)$ when h increases minutely from $h = a/b$. Then, from (3.13),

$$\text{Re} \frac{d\lambda}{dh} \Big|_{\lambda=0, h=\frac{a}{b}} = -\frac{b}{1 - \frac{a^2}{2b}} < 0.$$

This completes the proof of Lemma 3.4. $\square$

Lemma 3.4 shows that the root $\lambda(h)$ moves into the left half-plane of the complex plane when h increases minutely from a/b . Since $\nu(h) = 1$ for $h \in (0, a/b)$, we have $\nu(h) = 0$ for $h \in (a/b, h^*)$. Thus, the zero solution of (E) is uniformly asymptotically stable on the interval $a/b < h < h^*$ by Theorem A. From the above, the proof of sufficiency of Theorem 2.1 is complete.

(Necessity.) We will show that the uniform asymptotic stability of (E) implies the condition (2.1). We consider the contraposition of this statement, that is,

Proposition 3.1. *Suppose $a^2 < 2b$. If $h \leq a/b$ or $h \geq h^*$. Then the zero solution of (E) is not uniformly asymptotically stable.*

Proof. First, we consider a case of $h \leq a/b$. Then by Lemma C, (C) has a nonnegative real root, which implies $\nu(h) > 0$. Hence, the zero solution of (E) is not uniformly asymptotically stable.

Next, we consider a case of $h \geq h^*$. We proved that the (C) had a pair of purely imaginary roots $\lambda = \pm i\omega$ at $h = h^*$ in the proof of sufficiency. Here, we investigate the behavior of the root $\lambda(h)$ with $\lambda(h^*) = \pm i\omega$ when h increases minutely from h^* .

From (C*), the characteristic function of (E) is defined as follows:

$$p^*(\lambda, a, b, h) := \lambda^2 - a\lambda + b(1 - e^{-\lambda h}). \quad (3.14)$$

From (3.14),

$$\frac{\partial}{\partial \lambda} p^*(\lambda, a, b, h) = 2\lambda - a + bhe^{-\lambda h}, \quad \frac{\partial}{\partial h} p^*(\lambda, a, b, h) = b\lambda e^{-\lambda h}.$$

Then,

$$\left. \frac{d\lambda}{dh} \right|_{\substack{\lambda=i\omega \\ h=h^*}} = \frac{-ib\omega e^{-i\omega h^*}}{2i\omega - a + bh^* e^{-i\omega h^*}} = \frac{-ib\omega e^{-i\omega h^*}(-2i\omega - a + bh^* e^{i\omega h^*})}{|2i\omega - a + bh^* e^{-i\omega h^*}|^2} \quad (3.15)$$

by the theorem on implicit function. Now, we define the function $h_1(\omega)$ as follows:

$$h_1(\omega) := -ib\omega e^{-i\omega h^*}(-2i\omega - a + bh^* e^{i\omega h^*}).$$

Then, $h_1(\omega)$ is reduced to

$$h_1(\omega) = b\omega\{(-2\omega \cos \omega h^* + a \sin \omega h^*) + i(2\omega \sin \omega h^* + a \cos \omega h^* - bh^*)\}.$$

Considering the real part of $h_1(\omega)$, we have

$$\begin{aligned} \operatorname{Re} h_1(\omega) &= b\omega(-2\omega \cos \omega h^* + a \sin \omega h^*) \\ &= b\omega(-2\omega \frac{b - \omega^2}{b} + a \frac{a\omega}{b}) \\ &= \omega^4 > 0 \end{aligned}$$

from (3.8)–(3.10). This implies that the real part of the numerator of (3.15) is positive. We also define the function $h_2(\omega)$ as follows:

$$h_2(\omega) := 2i\omega - a + bh^*e^{-i\omega h^*}.$$

Then, $h_2(\omega)$ becomes

$$h_2(\omega) = (-a + bh^* \cos \omega h^*) + i(2\omega - bh^* \sin \omega h^*) \quad (3.16)$$

Here, we consider a case where both the real and imaginary part of $h_2(\omega)$ are zero. Then from (3.16),

$$-a + bh^* \cos \omega h^* = 0, \quad (3.17)$$

$$2\omega - bh^* \sin \omega h^* = 0. \quad (3.18)$$

By (3.17) $\times \sin \omega h^* + (3.18) \times \cos \omega h^*$, we obtain

$$-a \sin \omega h^* + 2\omega \cos \omega h^* = 0.$$

Therefore, from (3.8)–(3.10), we have $\omega = 0$ only, which contradicts $\omega > 0$. Hence, we showed $\frac{\partial}{\partial \lambda} p^*(i\omega, a, b, h^*) \neq 0$.

Thus, $\text{Re}(d\lambda/dh)|_{\substack{\lambda=i\omega \\ h=h^*}} > 0$ holds. This means that a pair of purely imaginary roots $\lambda(h)$ move into the right half-plane of the complex plane when h increases minutely from h^* . Therefore, since we have $\nu(h) > 0$, the zero solution of (E) is not uniformly asymptotically stable by Theorem A. This completes the proof of Proposition 3.1. $\square$

From Proposition 3.1, we can show the necessity of Theorem 2.1. Thus, the proof of Theorem 2.1 is finished completely. $\square$

Proof of Theorem 2.2. Next, we give a proof of Theorem 2.2. we note that $\nu(h) = 1$ for $h \in (0, a/b)$ and (C) has the root $\lambda = 0$ at $h = a/b$ in the same way as Theorem 2.1. Thus, it is sufficient to investigate the behavior of the root $\lambda(h)$ with $\lambda(a/b) = 0$ and $\nu(h)$ for all $h > a/b$. Here, for all $h > a/b$, the following lemmas holds.

Lemma 3.5. *Suppose $a^2 \geq 2b$. Then, (C) has no roots on the imaginary axis for all $h > a/b$.*

Lemma 3.5 implies that $\nu(h)$ is constant for all $h > a/b$. Since (C) has no pairs of purely imaginary roots and no the zero root for all $h > a/b$ by using the characteristic function $p(\lambda, a, b, h)$, it is easy to give a proof of Lemma 3.5. In this paper, we omit the details of this proof.

Now, we will investigate the behavior of $\lambda(h)$ with $\lambda(a/b) = 0$, which it is quiet important on $\nu(h)$ for all $h > a/b$. Then, we must prove the following property.

Lemma 3.6. *Suppose $a^2 > 2b$. Then, we have $\nu(h) > 0$ for all $h > a/b$.*

Proof. In case $a^2 > 2b$, we consider a sign of $\operatorname{Re}(d\lambda/dh)|_{\lambda=0, h=a/b}$ in the same way as Lemma 3.4. Then,

$$\operatorname{Re} \frac{d\lambda}{dh} \Big|_{\lambda=0, h=\frac{a}{b}} = -\frac{b}{1 - \frac{a^2}{2b}} > 0.$$

It implies that $\lambda(h)$ moves to the right half-plane when h is increased from a/b minutely and the root which exists in the right half-plane for $h \in (0, a/b)$ remains in the right half-plane. Hence, we have $\nu(h) = 2 > 0$ for all $h > a/b$ from Lemma 3.5. $\square$

Thus, by Lemmas 3.5 and 3.6 we show that the zero solution of (E) is not uniformly asymptotically stable for all $h > 0$, so the proof of Theorem 2.2 is complete. $\square$

4 Critical case $a^2 = 2b$ and conjectures

In this section, we consider the case $a^2 = 2b$. Here, in the same way as case $a^2 > 2b$, we can see easily that $\nu(h) = 1$ for $h \in (0, a/b)$ and (C) has the root $\lambda = 0$ at $h = a/b$. However, if we introduce the characteristic function $p(\lambda, a, b, h)$. Then,

$$p(0, a, b, \frac{a}{b}) = 0, \quad \frac{\partial}{\partial \lambda} p(0, a, b, \frac{a}{b}) = 0, \quad \frac{\partial^2}{\partial \lambda^2} p(0, a, b, \frac{a}{b}) = \frac{a^3}{3b^2} \neq 0.$$

Therefore, we see that the root $\lambda = 0$ is a double root of (C), and so we cannot analyze the behavior of $\lambda(h)$ with $\lambda(a/b) = 0$ by using the derivative

$$\operatorname{Re} \frac{d\lambda}{dh} \Big|_{\lambda=0, h=\frac{a}{b}} = -\frac{b}{1 - \frac{a^2}{2b}}.$$

Thus, we need to discuss the case $a^2 = 2b$ by another method. But, to our regret we cannot find the new method now. By the numerical examples (Figures 1 through 3), we are convinced that the zero solution of (E) is not uniformly asymptotically stable in case $a^2 = 2b$. Then, we have the following conjecture.

Conjecture 4.1. *Let $a^2 = 2b$. Then, the zero solution of (E) is not uniformly asymptotically stable for all $h > 0$.*

If we can prove Conjecture 4.1, then we can show immediately the following statement by Theorems 2.1 and 2.2.

Conjecture 4.2. *The zero solution of (E) is uniformly asymptotically stable if and only if conditions $a^2 < 2b$ and*

$$\frac{a}{b} < h < \frac{1}{\sqrt{2b - a^2}} \operatorname{Cos}^{-1} \frac{a^2 - b}{b}$$

are satisfied.

Finally, we will show the behavior of solutions numerically for the case $a^2 = 2b$ which illustrate Conjecture 4.2. Then, we fix $a = 4$ and $b = 8$ and take the initial function as $\phi(t) = 100 + 20 \sin t$. We put the parameter h as follows and illustrate Conjecture 4.2 with drawing the solution curves.

Figure 1 : $h = 0.45$ ($0 < h < a/b$)

Figure 2 : $h = 0.5$ ($h = a/b$)

Figure 3 : $h = 0.55$ ($h > a/b$)

Figures 1 through 3 suggest that the zero solution of (E) is not uniformly asymptotically stable for all $h > 0$ in case $a^2 = 2b$.

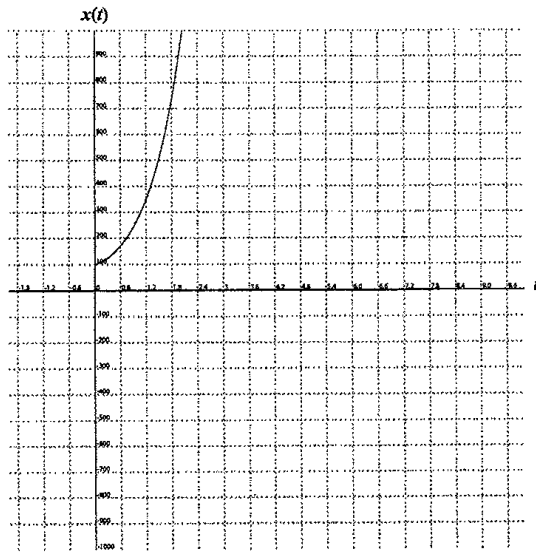


Figure 1: $h = 0.45$ ($h < a/b$)

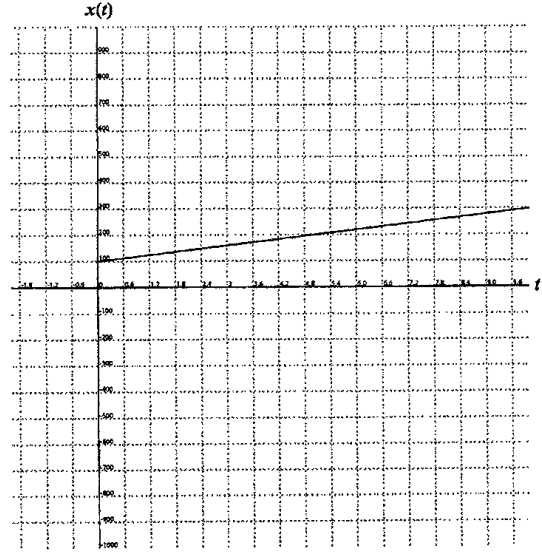


Figure 2: $h = 0.5$ ($h = a/b$)

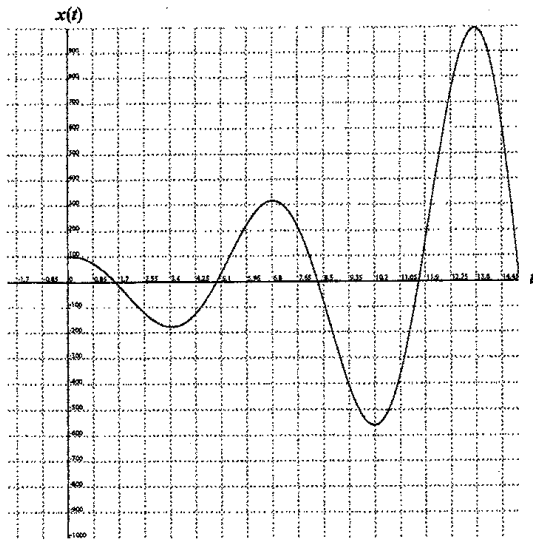


Figure 3: $h = 0.55$ ($h > a/b$)

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