

Some examples of operator monotone functions

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1 Introduction and preliminaries

Let f be a real valued continuous function on $(0, \infty)$. We call f n -matrix monotone on $(0, \infty)$ if it holds $f(A) \leq f(B)$, for $n \times n$ self-adjoint matrices A, B with $0 \leq A \leq B$, where $A \leq B$ means

$$(A\xi, \xi) \leq (B\xi, \xi) \quad \forall \xi \in \mathbb{C}^n.$$

When f is n -matrix monotone on $(0, \infty)$ for any positive integer $n \in \mathbb{N}$, f is called operator monotone on $(0, \infty)$. By Löwner's theorem, it is known that f is operator monotone on $(0, \infty)$ if and only if f is a Pick function on $(0, \infty)$, which means the function $f : (0, \infty) \rightarrow \mathbb{R}$ has the analytic continuation $f(z)$ on the upper half plane $\mathbb{H}_+ = \{z \in \mathbb{C} : \Im z > 0\}$ and satisfies the condition $f(\mathbb{H}_+)$ is contained in the closure of $\mathbb{H}_+$ ([1], [3], [7]). For any positive integer $n \in \mathbb{N}$, a real number $\gamma \in \mathbb{R}$, and positive numbers α_i, β_i ($1 \leq i \leq n$) with $\alpha_i \neq \beta_j$ ($1 \leq i, j \leq n$), we define the function $f(t)$ on $(0, \infty)$ as follows:

$$f(t) = t^\gamma \prod_{i=1}^n \frac{\beta_i t^{\alpha_i} - 1}{\alpha_i t^{\beta_i} - 1} \quad (t \neq 1)$$

and $f(1) = 1$. In [9], the author gave the method to investigate the operator monotonicity of functions $f(t)$. Using this result, we consider the operator monotonicity of the function $f(t)$ with some special form.

In section 2, we treat the following functions related to the power difference mean:

$$h(t) = \frac{b t^a - 1}{a t^b - 1}, \quad t \in (0, \infty),$$

for any real number a and b . In section 3, we treat the following functions (extended Petz-Hasegawa's functions) :

$$h(t) = \frac{ab(t-1)^2}{(t^a-1)(t^b-1)}, \quad t \in (0, \infty),$$

for any real number a and b , where we use the notation

$$\frac{t^0 - 1}{0} = \log t \left(= \lim_{a \rightarrow 0} \frac{t^a - 1}{a} \right).$$

We remark that the point-wise limit function $f(t)$ of $\{f_m(t)\}_{m=1}^\infty$ is n -matrix monotone if $f_m(t)$ is n -matrix monotone for all m .

Let, for $\gamma \in \mathbb{R}$ and $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ with $\alpha_i, \beta_i > 0$,

$$f(t) = t^\gamma \prod_{i=1}^n \frac{\beta_i t^{\alpha_i} - 1}{\alpha_i t^{\beta_i} - 1} \quad (t \neq 1)$$

and $f(1) = 1$. We introduce two quantities $F_0(\alpha, \beta)$ and $F(\alpha, \beta)$ for $f(t)$. The following two lemmas related to these quantities are used to determine the operator monotonicity of functions in section 3.

When $0 < \alpha_i, \beta_i \leq 2$, we define

$$\arg \frac{z^{\alpha_i} - 1}{z^{\beta_i} - 1} = 0 \quad \text{for } z \in (0, \infty)$$

and continuously define the argument of $\frac{z^{\alpha_i} - 1}{z^{\beta_i} - 1}$ on $z \in \mathbb{H}_+$. So we can define, for $\alpha_i, \beta_i \leq 2$ ($i = 1, \dots, n$),

$$\arg f(z) = \gamma \arg z + \sum_{i=1}^n \arg \frac{z^{\alpha_i} - 1}{z^{\beta_i} - 1}, \quad \text{for } z \in \mathbb{H}_+,$$

and $\arg f(t) = 0$ for $t \in (0, \infty)$.

If $f(t)$ is non-constant operator monotone, then its analytic continuation $f(z)$ has no zeros and no singular points on $\mathbb{H}_+$ since f is Pick function. It is known (see, [9]:Proposition 3.1) that $f(z)$ has no zeros and no singular points on $\mathbb{H}_+$ if and only if $|\gamma| \leq 2$ and $0 < \alpha_i, \beta_i \leq 2$ ($1 \leq i \leq n$). When $|\gamma| > 2$ or $\max\{\alpha_i, \beta_i : 1 \leq i \leq n\} > 2$, $f(t)$ is not operator monotone.

Lemma 1.1 ([9]:Theorem 1.1, Lemma 2.3 and Proposition 3.2). *Let $|\gamma| \leq 2$, $0 < \alpha_i, \beta_i \leq 2$ ($1 \leq i \leq n$).*

(1) *$f(t)$ is operator monotone on $(0, \infty)$ if and only if*

$$\gamma + G_0(\alpha, \beta) \geq 0 \text{ and } \gamma + F_0(\alpha, \beta) \leq 1,$$

where we set

$$g(t) = \prod_{i=1}^n \frac{\beta_i t^{\alpha_i} - 1}{\alpha_i t^{\beta_i} - 1}$$

and define $G_0(\alpha, \beta) = \inf\{\arg g(re^{\pi i}) : r \in (0, \infty)\}/\pi$ and $F_0(\alpha, \beta) = \sup\{\arg g(re^{\pi i}) : r \in (0, \infty)\}/\pi$.

(2) $G_0(\alpha, \beta) = -F_0(\beta, \alpha)$ and $F_0(\alpha, \beta) \geq 0$.

(3) $F_0(\alpha, \beta) + G_0(\alpha, \beta) = \sum_{i=1}^n (\alpha_i - \beta_i)$.

(4) When $0 < b < a \leq 2$,

$$0 < b \leq 1 \Leftrightarrow G_0(a, b) \geq 0 \Leftrightarrow F_0(a, b) \leq a - b,$$

where we use the notation a (resp. b) instead of $\alpha = (a)$ (resp. $\beta = (b)$).

For $0 \leq a, b \leq 2$, we define

$$F(a, b) = \begin{cases} a - b & \text{if } a \geq b, 0 \leq b \leq 1 \\ a - 1 & \text{if } 1 < a, b \leq 2 \\ 0 & \text{if } a < b, 0 \leq a \leq 1 \end{cases}.$$

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ ($0 < \alpha_i, \beta_i \leq 2$) and σ and τ permutations on $\{1, \dots, n\}$ satisfying with $\alpha_{\sigma(1)} \leq \alpha_{\sigma(2)} \leq \dots \leq \alpha_{\sigma(n)}$ and $\beta_{\tau(1)} \leq \beta_{\tau(2)} \leq \dots \leq \beta_{\tau(n)}$. Then we define

$$F(\alpha, \beta) = \sum_{i=1}^n F(\alpha_{\sigma(i)}, \beta_{\tau(i)}).$$

Lemma 1.2 ([9]:Theorem 1.2). For $|\gamma| \leq 2$, $0 < \alpha_i, \beta_i \leq 2$, the function

$$f(t) = t^\gamma \prod_{i=1}^n \frac{\beta_i t^{\alpha_i} - 1}{\alpha_i t^{\beta_i} - 1}$$

becomes operator monotone on $(0, \infty)$ if

$$\gamma - F(\beta, \alpha) \geq 0 \text{ and } \gamma + F(\alpha, \beta) \leq 1.$$

2 Functions related the power difference mean

The following characterization is well-known:

Lemma 2.1 ([6]:Theorem 2.4.3). The function $h : (0, \infty) \rightarrow \mathbb{R}$ is 2-matrix monotone if and only if h is in $C^1(0, \infty)$ and

$$\begin{pmatrix} h^{[1]}(\lambda_1, \lambda_1) & h^{[1]}(\lambda_1, \lambda_2) \\ h^{[1]}(\lambda_2, \lambda_1) & h^{[1]}(\lambda_2, \lambda_2) \end{pmatrix} \geq 0$$

for any $\lambda_1, \lambda_2 \in (0, \infty)$, where

$$h^{[1]}(\lambda, \mu) = \begin{cases} \frac{h(\lambda) - h(\mu)}{\lambda - \mu} & \lambda \neq \mu \\ h'(\lambda) & \lambda = \mu \end{cases}.$$

Theorem 2.2. *Let a, b be real numbers and*

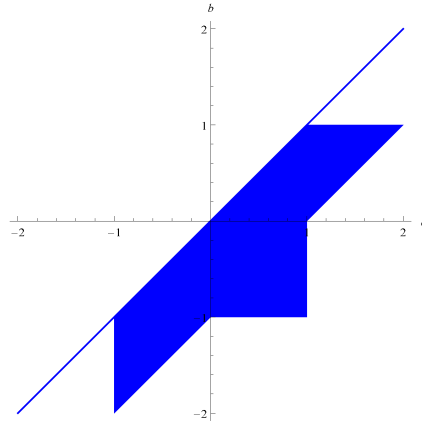
$$h(t) = \frac{b t^a - 1}{a t^b - 1}, \quad t \in (0, \infty).$$

Then we have

- (1) *h is increasing on $(0, \infty)$ if $a > b$ and decreasing if $a < b$.*
- (2) *h becomes operator monotone on $(0, \infty)$ if and only if the point (a, b) belongs to the set*

$$\Omega = \{a = b\} \cup \{(a, b) : 0 \leq a - b \leq 1, a \geq -1, b \leq 1\} \cup ([0, 1] \times [-1, 0])$$

in the (a, b) -plane:



- (3) *h is 2-matrix monotone on $(0, \infty)$ if and only if h is operator monotone on $(0, \infty)$.*

Proof. (1) We set

$$\frac{dh(t)}{dt} = \frac{t^{b-1}}{(t^b - 1)^2} k(t),$$

where $k(t) = \frac{b}{a}((a - b)t^a - at^{a-b} + b)$. Since

$$k(1) = 0, \quad \frac{dk(t)}{dt} = b(a - b)t^{a-1}(1 - t^{-b}),$$

we have $k(t) \geq 0$ for $t \in (0, \infty)$ if $a > b$. This means $h(t)$ is positive and increasing on $(0, \infty)$ if $a > b$.

Remarking the fact

$$h(t) = \frac{b t^a - 1}{a t^b - 1} = \frac{1}{\frac{a}{b} \frac{t^b - 1}{t^a - 1}},$$

$h(t)$ is decreasing on $(0, \infty)$ if $a < b$.

(2) This has been proved in [9]:Example 3.4(1).

(3) It suffices to show that $(a, b) \in \Omega$ if $h(t)$ is 2-matrix monotone on $(0, \infty)$.

We assume that h is 2-matrix monotone and not constant. By (1) we have $a > b$ and

$$h'(t) = \frac{b}{a} \frac{k_1(t)}{(t^b - 1)^2} \geq 0 \quad \text{for all } t \in (0, \infty),$$

where $k_1(t) = (a - b)t^{a+b-1} - at^{a-1} + bt^{b-1}$. So we have, for any $s, t > 0$,

$$\begin{pmatrix} h'(s) & h^{[1]}(s, t) \\ h^{[1]}(s, t) & h'(t) \end{pmatrix} \geq 0,$$

equivalently, $h'(s)h'(t) - (h^{[1]}(s, t))^2 \geq 0$. Then we set

$$\begin{aligned} D(s, t) &= h'(s)h'(t) - (h^{[1]}(s, t))^2 \\ &= \frac{b^2}{a^2} \frac{1}{(s^b - 1)^2(t^b - 1)^2(s - t)^2} \times (k_1(s)k_1(t)(s - t)^2 - k_2(s, t)), \end{aligned}$$

where

$$\begin{aligned} k_2(s, t) &= ((s^a - 1)(t^b - 1) - (s^b - 1)(t^a - 1))^2 \\ &= ((s^a - 1)t^b - (s^b - 1)t^a - s^a + s^b)^2, \end{aligned}$$

and we remark $k_1(s)k_1(t)(s - t)^2, k_2(s, t) \geq 0$ for all $s, t \in (0, \infty)$.

When $b > 0$ and $a + b + 1 < 2a$, we have

$$\lim_{t \rightarrow \infty} D(s, t) < 0 \quad \text{for some } s$$

because

$$\begin{aligned} & \lim_{t \rightarrow \infty} t^{-2a} (k_1(s)k_1(t)(s - t)^2 - k_2(s, t)) \\ &= \lim_{t \rightarrow \infty} t^{-2a+(a+b+1)} (((a - b) - at^{-b} + bt^{-a})k_1(s)(st^{-1} - 1)^2 \\ & \quad - ((s^a - 1)t^{b-a} - (s^b - 1) - (s^a - s^b)t^{-a})^2) \\ &= -(s^b - 1)^2 < 0. \end{aligned}$$

This contradicts to the assumption of h . The highest degree d_1 (resp. d_2) of t in $k_1(s)k_1(t)(s - t)^2$ (resp. $k_2(t)$) is

$$(d_1, d_2) = \begin{cases} (a + b + 1, 2a) & (0 < b < a) \\ (a + 1, 2a) & (b < 0 < a) \\ (a + 1, 0) & (b < a < 0) \end{cases},$$

and the lowest degree d'_1 (resp. d'_2) of t in $k_1(s)k_1(t)(s-t)^2$ (resp. $k_2(t)$) is

$$(d'_1, d'_2) = \begin{cases} (b-1, 0) & (0 < b < a) \\ (b-1, 2b) & (b < 0 < a) \\ (a+b-1, 2b) & (b < a < 0) \end{cases}.$$

By using the similar argument as above, when $d_1 < d_2$ or $d'_1 > d'_2$, we have

$$D(s, t) < 0 \text{ for a sufficiently large } t(> 0) \text{ and some fixed } s,$$

or

$$D(s, t) < 0 \text{ for a sufficiently small } t(> 0) \text{ and some fixed } s.$$

So 2-matrix monotonicity of h implies

$$d_1 \geq d_2 \text{ and } d'_1 \leq d'_2.$$

This means that $(a, b) \in \Omega$ if h is 2-matrix monotone. □

3 Extended Petz-Hasegawa's functions

We consider the operator monotonicity of the function

$$h(t) = \frac{ab(t-1)^2}{(t^a-1)(t^b-1)}, \quad t \in (0, \infty),$$

for any real number a and b . When $b = 1-a$ and $-1 \leq a \leq 2$, this function is called Petz-Hasegawa's function and becomes operator monotone on $(0, \infty)$ (see [4], [5], [7]).

Theorem 3.1. *Let a, b be real numbers and*

$$h(t) = \frac{ab(t-1)^2}{(t^a-1)(t^b-1)}.$$

Then h becomes operator monotone on $(0, \infty)$ if and only if the point (a, b) belongs to the following set:

$$\Omega = \{(a, b) : a \in [-1, 2], g_1(a) \leq b \leq g_2(a)\},$$

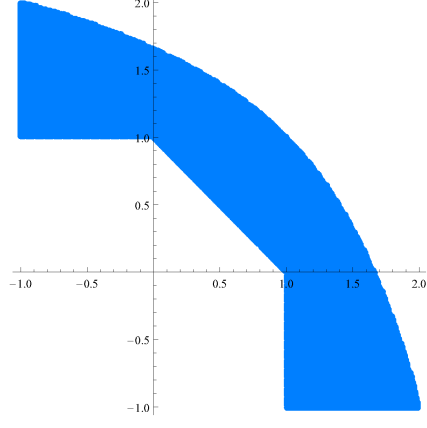
where

$$g_1(a) = \begin{cases} 1, & a \in [-1, 0] \\ 1-a, & a \in [0, 1] \\ -1, & a \in [1, 2] \end{cases}$$

and $g_2(a)$ is satisfying $1 - a \leq g_2(a) \leq 2 - a$ and the following equation:

$$\left(\frac{a - g_2(a)}{a} \frac{\sin a\pi}{\sin(a + g_2(a))\pi}\right)^a = \left(\frac{g_2(a) - a}{g_2(a)} \frac{\sin g_2(a)\pi}{\sin(g_2(a) + a)\pi}\right)^{g_2(a)}.$$

This set Ω in the (a, b) -plane is as follows:



where the boundary curve $g_2(a)$ is given by computations of approximate values.

Proof. The function h is symmetric for a and b . So we may assume that $a \geq b$. We can rewrite $h(t)$ as follows:

$$\begin{aligned} h(t) &= ab \cdot \frac{(t-1)^2}{(t^a-1)(t^b-1)} & a \geq b \geq 0 \\ &= a(-b)t^{-b} \cdot \frac{(t-1)^2}{(t^a-1)(t^{-b}-1)} & b < 0 \leq a \\ &= (-a)(-b)t^{-a-b} \cdot \frac{(t-1)^2}{(t^{-a}-1)(t^{-b}-1)} & b \leq a < 0 \end{aligned}$$

By the remark before Lemma 1.1, we have $|a|, |b| \leq 2$ if h is operator monotone.

Case (1) $0 < b \leq a \leq 2$: We can consider $\gamma = 0$, $\alpha = (1, 1)$, and $\beta = (a, b)$. Since

$$\lim_{r \rightarrow \infty} \arg h(re^{\pi i}) = \lim_{r \rightarrow \infty} \arg abr^{2-a-b} \cdot \frac{(e^{\pi i} - 1/r)^2}{(e^{a\pi i} - 1/r^a)(e^{b\pi i} - 1/r^b)} = (2 - a - b)\pi$$

and $G_0(\alpha, \beta) \leq 2 - a - b \leq F_0(\alpha, \beta)$, it follows that, by Lemma 1.1,

$$1 \leq a + b \leq 2$$

if h is operator monotone.

When $0 < b \leq a \leq 1$, we have

$$\begin{aligned} F(\alpha, \beta) &= F(1, a) + F(1, b) = (1 - a) + (1 - b) = 2 - a - b, \\ -F(\beta, \alpha) &= -(F(a, 1) + F(b, 1)) = -(0 + 0) = 0. \end{aligned}$$

By Lemma 1.2, $a + b \geq 1$ and $0 < b \leq a \leq 1$ implies that h is operator monotone.

Case (2) $-2 \leq b \leq 0 < a \leq 2$: We can consider $\gamma = -b$, $\alpha = (1, 1)$, and $\beta = (a, -b)$. Since

$$\begin{aligned} \lim_{r \rightarrow \infty} \arg h(re^{\pi i}) &= \lim_{r \rightarrow \infty} \arg a(-b)r^{2-a}e^{-b\pi i} \cdot \frac{(e^{\pi i} - 1/r)^2}{(e^{a\pi i} - 1/r^a)(e^{-b\pi i} - 1/r^{-b})} \\ &= (-b + 2 - a - (-b))\pi = (2 - a)\pi \end{aligned}$$

and

$$\begin{aligned} \lim_{r \rightarrow 0+} \arg h(re^{\pi i}) &= \lim_{r \rightarrow 0+} \arg a(-b)r^{-b}e^{-b\pi i} \cdot \frac{(re^{\pi i} - 1)^2}{(r^ae^{a\pi i} - 1)(r^{-b}e^{-b\pi i} - 1)} \\ &= -b\pi, \end{aligned}$$

we have $1 \leq a \leq 2$ and $-1 \leq b \leq 0$ if h is operator monotone.

When $1 \leq a \leq 2$ and $-1 \leq b \leq 0$, we have

$$\begin{aligned} -b + F(\alpha, \beta) &= -b + F(1, a) + F(1, -b) = -b + 0 + (1 - (-b)) = 1, \\ -b + G(\alpha, \beta) &= -b - F(a, 1) - F(-b, 1) = -b - (a - 1) - 0 = 1 - (a + b). \end{aligned}$$

By Lemma 1.2, $a + b \geq 1$, $1 \leq a \leq 2$, and $-1 \leq b \leq 0$ implies that h is operator monotone.

Case (3) $-2 \leq a, b < 0$: We can consider $\gamma = -a - b$, $\alpha = (1, 1)$, and $\beta = (-a, -b)$. Since

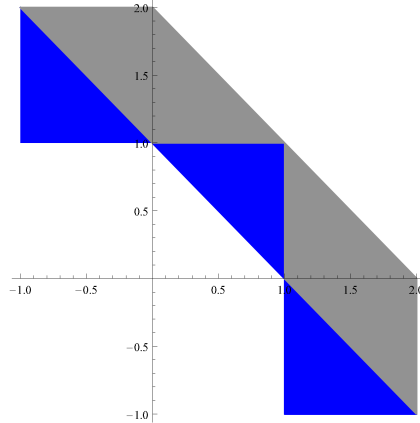
$$\lim_{r \rightarrow \infty} \arg h(re^{\pi i}) = \lim_{r \rightarrow \infty} \arg abr^2e^{(-a-b)\pi i} \cdot \frac{(e^{\pi i} - 1/r)^2}{(e^{-a\pi i} - 1/r^{-a})(e^{-b\pi i} - 1/r^{-b})} = 2\pi,$$

h is not operator monotone.

So we have that Ω is contained in

$$\{(a, b) : -1 \leq a \leq 0, g_1(a) \leq b \leq 2\} \cup \{(a, b) : 0 \leq a \leq 2, g_1(a) \leq b \leq 2 - a\}$$

and h is operator monotone if the point (a, b) is contained in the following three triangles:



By the numerical computation of $F_0(\alpha, \beta)$ and $G_0(\alpha, \beta)$, we can replace the above figure to the figure in the statement of Theorem 3.1.

We consider the function $g_2(a)$. By the symmetry of a and b , we only consider the case $1 \leq a \leq 2$ and $1 - a \leq b \leq 2 - a$. We remark that $f(t)$ is operator monotone on $(0, \infty)$ if and only if $\Im f(re^{\pi i}) \geq 0$ for all $r \geq 0$ by Lemma 1.1(1). Since

$$\begin{aligned} & \Im f(re^{\pi i}) \\ &= \frac{(r+1)^2}{|(r^a e^{a\pi i} - 1)(r^b r^{b\pi i} - 1)|^2} \Im ab(r^a e^{-a\pi i} - 1)(r^{-b\pi i} - 1) \\ &= \frac{(r+1)^2 r^b}{|(r^a e^{a\pi i} - 1)(r^b r^{b\pi i} - 1)|^2} ab(-r^a \sin(a+b)\pi + \sin b\pi + r^{a-b} \sin a\pi), \end{aligned}$$

the signature of $\Im f(re^{\pi i})$ is equal to that of

$$k(r) = ab(-r^a \sin(a+b)\pi + \sin b\pi + r^{a-b} \sin a\pi)$$

for all $r > 0$. We can see that the solution r_0 of $k'(r) = 0$ is

$$\left(\frac{(a-b) \sin a\pi}{a \sin(a+b)\pi} \right)^{1/b},$$

and $k(r)$ is decreasing on $(0, r_0)$ and increasing on (r_0, ∞) . So we have $f(t)$ is operator monotone on $(0, \infty)$ if and only if $k(r_0) \geq 0$. As a relation of a and b satisfying with $k(r_0) = 0$, we can get the following:

$$\left(\frac{a-b}{a} \frac{\sin a\pi}{\sin(a+b)\pi} \right)^a = \left(\frac{b-a}{b} \frac{\sin b\pi}{\sin(b+a)\pi} \right)^b.$$

So we can get the desired relation. □

The following is a program drawing a part of this figure in $[1, 2] \times [-1, 1]$ by Mathematica.

```

pick1={};

f10[a_,b_,z_]:= -Arg[z^a-1]-Arg[z^b-1];

minmaxf10[a_,b_]:=Module[{zval,zmin,zmax},
  zval=Table[f10[a,b,-0.001*i],{i,1,1000}];
  zmin=Min[zval];
  zmax=Max[zval];
  {Min[{zmin,(2-a-b)*Pi-zmax}],Max[{zmax,(2-a-b)*Pi-zmin}]}}

Do[ { c = minmaxf10[a,b];
  If[ ( c[[1]]>=0 ) && ( c[[2]] <= Pi) ,
    pick1 = Append[ pick1, {a,b} ] ; ] } ,
  {a,1.01, 2.0, 0.01}, {b, 0.01, 1.0, 0.01}]

pick2={};

f00[a_,b_,z_]:= -Arg[z^a-1]-Arg[z^(-b)-1];

minmaxf00[a_,b_]:=Module[{zval,zmin,zmax},
  zval=Table[f00[a,b,-0.001*i],{i,1,1000}];
  zmin=Min[zval];
  zmax=Max[zval];
  {Min[{zmin,(2-a+b)*Pi-zmax}] + (-b)*Pi,
  Max[{zmax,(2-a+b)*Pi-zmin}]+ (-b)*Pi}]

Do[ { c = minmaxf00[a,b];
  If[ ( c[[1]]>=0 ) && ( c[[2]] <= Pi) ,
    pick2 = Append[ pick2, {a,b} ] ; ] } ,
  {a,1.01, 2.0, 0.01}, {b, -0.01, -1.0, -0.01}]

ListPlot[ {pick1, pick2}, AspectRatio->Automatic,
  AxesOrigin->{0,0},PlotStyle->PointSize[0.01]]

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