

Characterizations of operators satisfying chaotic order and its applications

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In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H . An operator T is said to be positive (in symbol : $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$. Also an operator T is strictly positive (in symbol : $T > 0$) if T is positive and invertible. We write $A \gg B$ if $\log A \geq \log B$ which is called the chaotic order [FK][FFK2].

Results in this paper will appear in [F9] and [F10].

§1. Generalizations of Kosaki trace inequalities and related trace inequalities on chaotic order

By using an extension of the Furuta inequality and following after Kosaki's nice technique, we shall show generalizations of trace inequalities by Kosaki and related trace inequalities on chaotic order (i.e., $\log A \geq \log B$).

It is well known that $A \geq B \geq 0$ ensures $\text{Tr}(f(A)) \geq \text{Tr}(f(B))$, where Tr denotes the usual trace and f is a continuous increasing function on $\mathbf{R}_+$ with $f(0) = 0$. Kosaki [K] shows the following very interesting trace inequality as a generalization of the above mentioned trace inequality.

Theorem A [K]. Assume $A \geq B \geq 0$ and $p > 1$, $\alpha \geq \max\{-1, \frac{-p}{2}\}$.

(i) Then there exists a partial isometry operator U satisfying

$$A^{\frac{\alpha}{2}} B^p A^{\frac{\alpha}{2}} \leq U^* A^{p+\alpha} U.$$

(ii) For a continuous increasing function f on $\mathbf{R}_+$ with $f(0) = 0$, we have

$$\text{Tr}(f(A^{\frac{\alpha}{2}} B^p A^{\frac{\alpha}{2}})) \leq \text{Tr}(f(A^{p+\alpha}))$$

In the above statements the invertibility of A is assumed when $\alpha < 0$.

Recently Ando-Hiai [AH] established various log-majorization results to ensure excellent and useful inequalities for unitarily invariant norms.

On the other hand, as an extension of [H] and [L], we established the following Furuta inequality ([F1][F2][F3][FUJ] and [KA]).

Theorem B. *If $A \geq B \geq 0$, then for each $r \geq 0$,*

$$(i) \quad (B^r A^p B^r)^{1/q} \geq (B^r B^p B^r)^{1/q}$$

and

$$(ii) \quad (A^r A^p A^r)^{1/q} \geq (A^r B^p A^r)^{1/q}$$

hold for p and q such that $p \geq 0$ and $q \geq 1$ with $(1 + 2r)q \geq p + 2r$.

In Furuta [F8] we established the following extension of Theorem B, which interpolates this log majorization results of Ando-Hiai and the Furuta inequality and moreover extends the results in [FFK1],[FFK2], [FK],[F4] and [F5].

Theorem C [F8]. *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0, 1]$ and $p \geq 1$,*

$$F_{p,t}(A, B, r, s) = A^{\frac{-r}{2}} \{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} A^{\frac{-r}{2}}$$

is a decreasing function of both r and s for any $s \geq 1$ and $r \geq t$ and the following inequality holds $F_{p,t}(A, A, r, s) \geq F_{p,t}(A, B, r, s)$, that is, for each $t \in [0, 1]$ and $p \geq 1$,

$$A^{1-t+r} \geq \{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$$

holds for any $s \geq 1$ and r such that $r \geq t$.

The following Theorem 1.1 is an extension of Theorem A [K].

Theorem 1.1 [F9]. *Let A and B be positive operators such that $A \geq B \geq 0$ with $A > 0$. Assume that $p \geq 1$, $s \geq 1$, $t \in [0, 1]$ and $\beta \geq \max[t - 1, \frac{1}{2}\{t(s + 1) - ps\}]$. Then the following inequalities hold.*

(I) *There exists the a partial isometry operator U satisfying*

$$A^{\frac{\beta}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{\beta}{2}} \leq U^* A^{(p-t)s+\beta} U.$$

(II) *For a continuous increasing function f on $\mathbf{R}_+$ with $f(0) = 0$,*

$$\text{Tr}\{f(A^{\frac{\beta}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{\beta}{2}})\} \leq \text{Tr}\{f(A^{(p-t)s+\beta})\}.$$

The following Theorem 1.2 is a parallel result to Theorem 1.1.

Theorem 1.2 [F9]. *Let A and B be positive invertible operators such that $A \gg B$ (i.e., $\log A \geq \log B$). Assume that $p \geq u > 0$, $s \geq 1$, $\alpha \in [0, 1]$ and $\beta \geq -u\alpha$. Then the following inequalities hold.*

(I) *There exists the a partial isometry operator U satisfying*

$$A^{\frac{\beta}{2}} (A^{\frac{u\alpha}{2}} B^p A^{\frac{u\alpha}{2}})^s A^{\frac{\beta}{2}} \leq U^* A^{(u\alpha+p)s+\beta} U.$$

(II) *For a continuous increasing function f on $\mathbf{R}_+$ with $f(0) = 0$,*

$$\text{Tr}\{f(A^{\frac{\beta}{2}} (A^{\frac{u\alpha}{2}} B^p A^{\frac{u\alpha}{2}})^s A^{\frac{\beta}{2}})\} \leq \text{Tr}\{f(A^{(u\alpha+p)s+\beta})\}.$$

Corollary 1.3 [F9]. *Let A and B be positive invertible operators such that $A \gg B$ (i.e., $\log A \geq \log B$). Assume that $p > 0$, and $\beta \geq 0$. Then the following inequalities hold.*

(I) *There exists the a partial isometry operator U satisfying*

$$A^{\frac{\beta}{2}} B^p A^{\frac{\beta}{2}} \leq U^* A^{p+\beta} U.$$

(II) *For a continuous increasing function f on $\mathbf{R}_+$ with $f(0) = 0$,*

$$\text{Tr}\{f(A^{\frac{\beta}{2}} B^p A^{\frac{\beta}{2}})\} \leq \text{Tr}\{f(A^{p+\beta})\}.$$

§2. Extensions of Ando's characterization of operators satisfying $\log A \geq \log B$ and its applications

It is shown in Ando [A] that $A \gg B$ holds if and only if $A^p \geq (A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{1}{2}}$ holds for all $p \geq 0$.

In this section , we state an extension of this result and its application.

Theorem 2.1 [F10]. *Let A and B be positive invertible operators. Then the following (I) and (II) holds.*

(I) *If $A \gg B$ (i.e., $\log A \geq \log B$) , then for each $\alpha \in [0, 1]$, and all $p \geq 0$ and $u \geq 0$,*

$$G_{p,u,\alpha}(A, B, r, s) = A^{-\frac{ur}{2}} \{A^{\frac{ur}{2}} (A^{\frac{u\alpha}{2}} B^p A^{\frac{u\alpha}{2}})^s A^{\frac{ur}{2}}\}^{\frac{u(\alpha+r)}{(u\alpha+p)s+ur}} A^{-\frac{ur}{2}}$$

is a decreasing function of both s and r such that $s \geq 1$ and $r + \alpha \geq 1$.

(II) $A \gg B$ holds if and only if $g(s, r) = A^{-\frac{r}{2}} (A^{\frac{r}{2}} B^s A^{\frac{r}{2}})^{\frac{r}{s+r}} A^{-\frac{r}{2}}$ is a decreasing function of both $r \geq 0$ and $s \geq 0$.

Theorem 2.2 [F10]. *Let A and B be positive invertible operators. Then the following assertions are mutually equivalent.*

(I) $A \gg B$ (i.e., $\log A \geq \log B$).

(II) For each $\alpha \in [0, 1]$, and all $p \geq 0$ and $u \geq 0$,

$$A^{u(\alpha+r)} \geq \{A^{\frac{ur}{2}} (A^{\frac{u\alpha}{2}} B^p A^{\frac{u\alpha}{2}})^s A^{\frac{ur}{2}}\}^{\frac{u(\alpha+r)}{(u\alpha+p)s+ur}}$$

holds for any $s \geq 1$ and r such that $r + \alpha \geq 1$.

(III) For each $\alpha \in [0, 1]$, and all $p \geq 0$,

$$A^{p(\alpha+r)} \geq \{A^{\frac{pr}{2}} (A^{\frac{p\alpha}{2}} B^p A^{\frac{p\alpha}{2}})^s A^{\frac{pr}{2}}\}^{\frac{\alpha+r}{(\alpha+1)s+r}}$$

holds for any $s \geq 1$ and r such that $r + \alpha \geq 1$.

(IV) For each $\alpha \in [0, 1]$, and all $p \geq 0$,

$$A^{\alpha+r} \geq \{A^{\frac{r}{2}} (A^{\frac{\alpha}{2}} B^p A^{\frac{\alpha}{2}})^s A^{\frac{r}{2}}\}^{\frac{\alpha+r}{(\alpha+p)s+r}}$$

holds for any $s \geq 1$ and r such that $r + \alpha \geq 1$.

Theorem 2.3 [F10]. *Let A and B be positive invertible operators. If $A \geq (A^{\frac{1}{2}} B A^{\frac{1}{2}})^{\frac{1}{2}}$, then for each $\alpha \in [0, 1]$, and all $p \geq 1$ and $u \geq 1$,*

$$H_{p,u,\alpha}(A, B, r, s) = A^{-\frac{ur}{2}} \{A^{\frac{ur}{2}} (A^{\frac{u\alpha}{2}} B^p A^{\frac{u\alpha}{2}})^s A^{\frac{ur}{2}}\}^{\frac{u(\alpha+r)}{(u\alpha+p)s+ur}} A^{-\frac{ur}{2}}$$

is a decreasing function of both s and r such that $s \geq 1$ and $r + \alpha \geq 1$.

Theorem 2.4 [F10]. *Let A and B be positive invertible operators. Then the following assertions are mutually equivalent.*

(I) $A \geq (A^{\frac{1}{2}} B A^{\frac{1}{2}})^{\frac{1}{2}}$.

(II) For each $\alpha \in [0, 1]$, and all $p \geq 1$ and $u \geq 1$,

$$A^{u(\alpha+r)} \geq \{A^{\frac{ur}{2}} (A^{\frac{u\alpha}{2}} B^p A^{\frac{u\alpha}{2}})^s A^{\frac{ur}{2}}\}^{\frac{u(\alpha+r)}{(u\alpha+p)s+ur}}$$

holds for any $s \geq 1$ and r such that $r + \alpha \geq 1$.

(III) For each $\alpha \in [0, 1]$, and all $p \geq 1$,

$$A^{p(\alpha+r)} \geq \{A^{\frac{pr}{2}} (A^{\frac{p\alpha}{2}} B^p A^{\frac{p\alpha}{2}})^s A^{\frac{pr}{2}}\}^{\frac{\alpha+r}{(\alpha+1)s+r}}$$

holds for any $s \geq 1$ and r such that $r + \alpha \geq 1$.

(IV) For each $\alpha \in [0, 1]$, and all $p \geq 1$,

$$A^{\alpha+r} \geq \{A^{\frac{r}{2}} (A^{\frac{\alpha}{2}} B^p A^{\frac{\alpha}{2}})^s A^{\frac{r}{2}}\}^{\frac{\alpha+r}{(\alpha+p)s+r}}$$

holds for any $s \geq 1$ and r such that $r + \alpha \geq 1$.

Remark 2.1. When we replace the hypothesis $A \gg B$ in Theorem 2.2 by $A \geq (A^{\frac{1}{2}} B A^{\frac{1}{2}})^{\frac{1}{2}}$ in Theorem 2.4, then all the results in Theorem 2.2 remain valid if we replace $p \geq 0$ and $u \geq 0$ in Theorem 2.2 by $p \geq 1$ and $u \geq 1$ in Theorem 2.4.

Also we remark that when we replace the hypothesis $A \gg B$ in (I) of Theorem 2.1 by $A \geq (A^{\frac{1}{2}} B A^{\frac{1}{2}})^{\frac{1}{2}}$ in Theorem 2.3, but the result in (I) of Theorem 2.1 remain valid if we replace $p \geq 0$ and $u \geq 0$ in Theorem 2.1 by $p \geq 1$ and $u \geq 1$ in Theorem 2.3.

Remark 2.2. Put $r + \alpha = 1$ and $s = 1$ in (IV) of Theorem 2.4, then $A^u \geq (A^{\frac{u}{2}} B^p A^{\frac{u}{2}})^{\frac{u}{p+u}}$ holds for any $p \geq 1$ and $u \geq 1$, which has been shown in [Theorem 2, FFW]. Also put $r + \alpha = 1$ and $s = 1$ in (IV) of Theorem 2.2, then we have $A^u \geq (A^{\frac{u}{2}} B^p A^{\frac{u}{2}})^{\frac{u}{p+u}}$ holds for any $p \geq 0$ and $u \geq 0$, and this result is already obtained in [Theorem 1, FFK2] and [Theorem 1, F5].

§3. Characterizations of operators satisfying $\log A \geq \log B$ associated with some operator equations and parallel results

Theorem 3.1 [F10]. *Let A and B be positive invertible operators. Then the following assertions are mutually equivalent.*

(I) $A \gg B$ (i.e., $\log A \geq \log B$) .

(II) For each $\alpha \in [0, 1]$ and for any $p \geq u > 0$ and $s \geq 1$, there exists a unique invertible positive contraction $T_{p,u,\alpha}(s)$ such that

$$T_{p,u,\alpha}(s)A^{ps+u\alpha(s-1)}T_{p,u,\alpha}(s) = A^{-\frac{u\alpha}{2}}(A^{\frac{u\alpha}{2}}B^pA^{\frac{u\alpha}{2}})^sA^{-\frac{u\alpha}{2}}.$$

(III) For any $p \geq 0$, there exists a unique invertible positive contraction T_p such that $T_p A^p T_p = B^p$.

Next we state the following parallel result to Theorem 3.1.

Theorem 3.2 [F10]. *Let A and B be positive invertible operators. Then the following assertions are mutually equivalent.*

(I) $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}.$

(II) For each $\alpha \in [0, 1]$ and for any $p \geq u \geq 1$ and $s \geq 1$, there exists a unique invertible positive contraction $T_{p,u,\alpha}(s)$ such that

$$T_{p,u,\alpha}(s)A^{ps+u\alpha(s-1)}T_{p,u,\alpha}(s) = A^{-\frac{u\alpha}{2}}(A^{\frac{u\alpha}{2}}B^pA^{\frac{u\alpha}{2}})^sA^{-\frac{u\alpha}{2}}.$$

(III) For any $p \geq 1$, there exists a unique invertible positive contraction T_p such that $T_p A^p T_p = B^p$.

Related to Theorem 3.1 and Theorem 3.2, we have the following parallel result by the same way as proof of Theorem 3.1.

Theorem 3.3 [F10]. *Let A and B be positive invertible operators. Then the following assertions are mutually equivalent.*

(I) $A \geq B \geq 0.$

(II) For each $t \in [0, 1]$ and for any $p \geq 2$ and $s \geq 1$, there exists a unique invertible positive contraction $T_{p,t}(s)$ such that

$$T_{p,t}(s)A^{(p-t)s+t-1}T_{p,t}(s) = A^{\frac{t-1}{2}}(A^{\frac{t}{2}}B^pA^{\frac{t}{2}})^sA^{\frac{t-1}{2}}.$$

(III) For any $p \geq 1$, there exists a unique invertible positive contraction T_p such that $T_p A^p T_p = A^{-\frac{1}{2}}B^{p+1}A^{-\frac{1}{2}}.$

(IV) $A^{1+p} \geq (A^{\frac{p}{2}}B^{p+2}A^{\frac{p}{2}})^{\frac{1}{2}}$ for all $p \geq 0$.

Results in this section are extensions of [Theorem 2, F6] and [Theorem 2.1, F7].

§4. Addendum . An extension of Theorem B

Recently we have the following result as an extension form of Theorem B.

Theorem D [FFK3]. *If $A \geq B > 0$, then $(B^\gamma A^\alpha B^\gamma)^\beta \geq (B^\gamma B^\alpha B^\gamma)^\beta$ holds under any one of the following conditions ;*

- (i) $\frac{1}{\beta} \leq \alpha$, $0 < \beta < 1$, and $\gamma = \frac{\alpha\beta - 1}{2(1 - \beta)}$
- (ii) $\frac{1}{\beta} \leq \alpha \leq 1$, $1 < \beta \leq 2$, and $\gamma = \frac{\alpha\beta - 1}{2(1 - \beta)}$
- (iii) $\frac{1}{2} \leq \alpha \leq 1$, $2 \leq \beta$, and $\gamma = \frac{\alpha\beta - 1}{2(1 - \beta)}$.

Remark 4.1. (i) and (ii) are announced in [Y, p 61], but we remark that (i) is nothing but exchange of parameters p , q and r in Theorem B , that is, put $(1 + 2r)q = p + 2r$ for $r \geq 0$, $p \geq 1$ and $q \geq 1$ in Theorem B, then we easily obtain $p \geq q \geq 1$ and we have only to replace p by α , r by γ and $\frac{1}{q}$ by β , then we have (i) which is nothing but another expression of (i) in Theorem B.

Moreover a simple proof of (ii) is obtained in [FFK3] along a method of [F2] by using polar decomposition. Also in [FFK3] we obtained (iii) along a method of [F2]. We have to assume invertibility of A and B in the cases (ii) and (iii) since $\gamma \leq 0$.

Remark 4.2. We should mention that Tanahashi [TA] has obtained several interesting results closely related to Theorem D.

REFERENCES

- [A] T.Ando, On some operator inequality, Math. Ann., **279**(1987),157-159.
- [AH] T.Ando and F.Hiai, Log majorization and complementary Golden-Thompson type inequalities, Linear Alg. and Its Appl., **197,198** (1994),113-131.
- [FUJ] M.Fujii, Furuta's inequality and its mean theoretic approach, J. Operator Theory, **23** (1990),67-72.
- [FFK1] M.Fujii, T.Furuta and E.Kamei, Operator functions associated with Furuta's inequality, Linear Alg. and Its Appl., **149**(1991),91-96.
- [FFK2] M.Fujii, T.Furuta and E.Kamei, Furuta's inequality and its application to Ando's theorem, Linear Alg. and Its Appl., **179**(1993),161-169.

- [FFK3] M.Fujii, T.Furuta and E.Kamei, Complements to the Furuta inequality, Proc. Japan Acad., **70** (1994),239-242.
- [FFW] M.Fujii, T.Furuta and D.Wang, An application of the Furuta inequality to operator inequalities on chaotic order,**40** (1994), 317-321.
- [FK] M.Fujii, and E.Kamei, Furuta's inequality and its application to Ando's Theorem, Proc. Amer. Math. Soc., **115** (1992),409-413.
- [F1] T.Furuta, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1+2r)q \geq p+2r$, Proc. Amer. Math. Soc., **101** (1987),85-88.
- [F2] T.Furuta, Elementary proof of an order preserving inequality, Proc. Japan Acad., **65** (1989),126.
- [F3] T.Furuta, A proof via operator means of an order preserving inequality, Linear Alg. and Its Appl., **113**(1989),129-130.
- [F4] T.Furuta, Two operator functions with monotone property, Proc. Amer. Math. Soc. **111**(1991),511-516.
- [F5] T.Furuta, Applications of order preserving operator inequalities, Operator Theory: Advances and Applications, **59**(1992),180-190.
- [F6] T.Furuta, Some norm inequalities and operator inequalities via the Furuta inequality Acta Sci. Math. (Szeged) **57** (1993),139-145.
- [F7] T.Furuta, Applications of the Furuta inequality to operator inequalities and norm inequalities preserving some orders, Operator Theory: Advances and Applications, **61** (1993),115-122.
- [F8] T.Furuta, Extension of the Furuta inequality and log-majorization by Ando-Hiai, to appear in Linear Alg. and Its Appl.,(1994).
- [F9] T.Furuta, Generalizations of Kosaki trace inequalities and related trace inequalities on chaotic order, to appear in Linear Alg. and Its Appl..
- [F10] T.Furuta, Characterizations of operators satisfying $\log A \geq \log B$ and its applications, preprint.
- [H] E. Heinz, Beiträge zur Störungstheorie der Spektralzerlegung, Math. Ann., **123**(1951),415-438.
- [KA] E. Kamei, A satellite to Furuta's inequality, Math. Japon, **33** (1988),883-886.

- [K] H. Kosaki, On some trace inequality, Proc. of the Center for Mathematical Analysis , Australian National University, (1991),129-134.
- [L] K.Löwner, Über monotone Matrixfunktionen, Math. Z., **38** (1934),177-216.
- [TA] K. Tanahashi, 巾が負の古田不等式 , preprint.
- [Y] T.Toshino, Introduction to Operator Theory, Pitman Research Notes in Mathematics Series 300, Longman Scientific and Technical (1993).

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