

Perturbative Expansion of the Chern-Simons Integral

Itaru Mitoma

Department of Mathematics, Saga University

1 Introduction

The necessity of the asymptotic expansion in infinite dimensions arises from the pioneering work of the Chern-Simons theory by Witten [10] in 1989.

However, the exponential 3rd term of the Chern-Simons theory is less likely to be handled by techniques known at present, so that we will challenge a new method called the Fujiwara-Kumano-go method [5, 8] as a possibility.

Let M be a compact oriented smooth 3-manifold, G a simply connected, connected compact simple Lie group, and $P \rightarrow M$ a principal G -bundle over M . Let denote by $\Omega^r(M, \mathfrak{g})$, the space of $\mathfrak{g}$ -valued smooth r -forms on M .

Let $\mathcal{A}$ denote the space of connections on P and $\mathcal{G}$ the group of gauge transformations on P . Note that, by fixing a reference connection on P as the origin, we may identify $\mathcal{A}$ with the (infinite-dimensional) vector space $\Omega^1(M, \mathfrak{g})$, and $\mathcal{G}$ with the space $C^\infty(M, G)$ of smooth maps from M to G , respectively. Then the *Chern-Simons integral* of an integrand $F(A)$ is given by

$$(1.1) \quad \int_{\mathcal{A}/\mathcal{G}} F(A) e^{L(A)} \mathcal{D}(A),$$

where the Chern-Simons Lagrangian L is defined by

$$(1.2) \quad L(A) = -\frac{ik}{4\pi} \int_M \text{Tr} \left\{ A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right\},$$

$\mathcal{D}(A)$ is the *Feynman measure* and the parameter k is a positive integer called the *level of charges*.

Among various integrands, the most typical example of gauge invariant observables is the *Wilson line* defined by

$$(1.3) \quad F(A) = \prod_{j=1}^s \text{Tr}_{R_j} \mathcal{P} \exp \int_{\gamma_j} A,$$

where $\mathcal{P}$ denotes the product integral (see [4]), γ_j , $j = 1, 2, \dots, s$, are closed oriented loops, and the trace Tr is taken with respect to some irreducible representation R_j of G assigned to each γ_j .

In Section 2, we give several relevant basic notions, after which we state our results precisely. In Section 3, we prove theorems stated in Section 2 by using the Fujiwara-Kumano-go method.

Throughout this paper, $\sqrt{z}$ is understood to denote the branch of the root of $z \in \mathbb{C}$ where $-\frac{\pi}{2} < \arg \sqrt{z} < \frac{\pi}{2}$.

2 Definitions and Results

From the method of superfields of the perturbative formulation of the Chern-Simons integral [2, 3], we have the Lorentz gauge fixed form of the Chern-Simons integral written as

$$(2.1) \quad \int_{\mathcal{A}} \int_{\Phi} \int_{\widehat{C}'} \int_{C'} \mathcal{D}(A) \mathcal{D}(\phi) \mathcal{D}(\tilde{c}') \mathcal{D}(c') F(A_0 + A) \\ \times \exp \left[ik((A, \phi), Q_{A_0}(A, \phi))_+ - \frac{ik}{4\pi} \int_M \text{Tr} \frac{2}{3} A \wedge A \wedge A + \int_M \text{Tr} \tilde{c}' d_{A_0} * D_A c' \right].$$

Here A_0 is a background connection, Q_{A_0} is a twisted Dirac operator and $(\cdot, \cdot)_+$ is the inner product of the Hilbert space $L^2(\Omega_+) = L^2(\Omega^1(M, \mathfrak{g}) \oplus \Omega^3(M, \mathfrak{g}))$ given by

$$((A, \phi), (B, \varphi))_+ = (A, B) + (\phi, \varphi),$$

where the inner product and the norm on $\Omega^r(M, \mathfrak{g})$ are defined by

$$(2.2) \quad (\omega, \eta) = - \int_M \text{Tr} \omega \wedge * \eta, \quad |\cdot| = \sqrt{(\cdot, \cdot)}.$$

By heuristically considering

$$\int_{\widehat{C}'} \int_{C'} \mathcal{D}(\tilde{c}') \mathcal{D}(c') \exp \left[\int_M \text{Tr} \tilde{c}' d_{A_0} * D_A c' \right] \\ = \det * d_{A_0} * D_A = \det \Delta_0 \det_R * d_{A_0} * D_A,$$

and balancing out $\det \Delta_0$ by a normalizaion of (2.1), we arrive at the perturbative heuristic formulation of the normalized Chern-Simons integral such that

$$(2.3) \quad \frac{1}{Z} \int_{\mathcal{A}} \int_{\Phi} \mathcal{D}(A) \mathcal{D}(\phi) F(A_0 + A) \\ \times \det_R * d_{A_0} * D_A \exp \left[ik((A, \phi), Q_{A_0}(A, \phi))_+ - \frac{ik}{4\pi} \int_M \text{Tr} \frac{2}{3} A \wedge A \wedge A \right],$$

where

$$(2.4) \quad Z = \int_{\mathcal{A}} \int_{\Phi} \mathcal{D}(A) \mathcal{D}(\phi) \exp \left[ik((A, \phi), Q_{A_0}(A, \phi))_+ \right].$$

To provide mathematical meaning to this, we have first of all to regularize the formal determinant $\det_R * d_{A_0} * D_A$. Briefly we recall Albeverio-Mitoma[1].

Since Q_{A_0} is self-adjoint and elliptic, Q_{A_0} has pure point spectrum [8]. Let $\lambda_j, e_j = (e_j^A, e_j^\phi), j = 1, 2, \dots$ be the eigenvalues and vectors of Q_{A_0} in $L^2(\Omega_+)$.

Let $\{\nu_j, \xi_j, j = 1, 2, \dots\}$ be the eigensystem of Δ_0 in $L^2(\Omega^0)$,

$$\tilde{c}' = \sum_{j=1}^{\infty} \xi_j \tilde{c}'_j \quad \text{and} \quad c' = \sum_{j=1}^{\infty} \xi_j c'_j.$$

Define

$$a_{R,ij}^\ell = -\frac{1}{\nu_i} \int_M \text{Tr } r_i d_{A_0} \xi_i \wedge * [e_\ell^A, r_j \xi_j].$$

Since M is compact, we can choose appropriate real numbers $r_j > 0, j = 1, 2, \dots$ such that

$$(2.5) \quad \sum_{i,j} |a_{R,ij}^\ell| < 1.$$

For an integer p , we define the Hilbert subspace $H_p(\Omega_+)$ of $L^2(\Omega_+)$ with new inner product $(\cdot, \cdot)_p$ defined by

$$((A, \phi), (B, \varphi))_p = (A, (I + Q_{A_0}^2)^p B) + (\phi, (I + Q_{A_0}^2)^p \varphi),$$

where I is the identity operator on $L^2(\Omega_+)$, and the p -norm on $H_p(\Omega_+)$ is defined as usual by $\|\cdot\|_p = \sqrt{(\cdot, \cdot)_p}$. Henceforth we denote $H_p(\Omega_+)$ briefly by H_p whenever no confusion may occur. Let $h_i = (1 + \lambda_i^2)^{-p/2} e_i$ be the CONS of H_p . Choose a sufficiently large p satisfying the condition

$$\sum_{i=1}^{\infty} (1 + \lambda_i^2)^{-p} |\lambda_i| < \infty,$$

and guaranteeing the regularizations in what follows.

From now on, we use the brief notations such that

$$\beta_j = (1 + \lambda_j^2)^{-p/2}, \quad \text{and} \quad a_j = \beta_j^2 \lambda_j.$$

Then for $(A, \phi) \in H_p$, we consider instead of $\det_R * d_{A_0} * D_A$, the regularized determinant defined by

$$\det_{Reg}(A) = \lim_{m \rightarrow \infty} \det_{Reg}^m(A),$$

where

$$\det_{Reg}^m(A) = \begin{vmatrix} a_{11}^R(A) & a_{12}^R(A) & \cdots & a_{1m}^R(A) \\ a_{21}^R(A) & a_{22}^R(A) & \cdots & a_{2m}^R(A) \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}^R(A) & a_{m2}^R(A) & \cdots & a_{mm}^R(A) \end{vmatrix},$$

$$a_{ii}^R(A) = 1 + \sum_{\ell=1}^{\infty} \beta_{\ell}((A, \phi), h_{\ell})_p a_{R,ii}^{\ell}$$

and

$$a_{ij}^R(A) = \sum_{\ell=1}^{\infty} \beta_{\ell}((A, \phi), h_{\ell})_p a_{R,ij}^{\ell}.$$

The regularized determinant is well defined for sufficiently large p , which is guaranteed by the increasing rates of eigenvalues of Q_{A_0} (c) of Lemma 1.6.3 in [6]).

Next we proceed to a regularizing the holonomy. From Mitoma-Nishikawa [9], for a given closed smooth curve $\gamma : [0, 1] \rightarrow M$ in M , for each $t \in [0, 1]$ and sufficiently small $\epsilon > 0$ there exists a Poincare dual $C_{\gamma}^{\epsilon}(t)$ such that

$$\lim_{\epsilon \rightarrow 0} \sup_{0 \leq t \leq 1} \left| (A, C_{\gamma}^{\epsilon}(t)) - \sum_{i=1}^3 \int_0^t A_i^{\alpha}(\gamma(\tau)) \dot{\gamma}^i(\tau) d\tau \right| = 0.$$

Since

$$(A, C_{\gamma}^{\epsilon}(t)) = \left((A, \phi), (I + Q_{A_0}^2)^{-p} (C_{\gamma}^{\epsilon}(t), 0) \right)_p,$$

by setting

$$(2.6) \quad \tilde{C}_{\gamma}^{\epsilon}(t) = (I + Q_{A_0}^2)^{-p} (C_{\gamma}^{\epsilon}(t), 0),$$

we define

$$(2.7) \quad A_{\gamma}^{\epsilon}(t) = \sum_{\alpha=1}^d \left((A, \phi), \tilde{C}_{\gamma}^{\epsilon}(t)^{\alpha} \otimes E_{\alpha} \right)_p E_{\alpha},$$

where $\tilde{C}_{\gamma}^{\epsilon}(t) = \sum \tilde{C}_{\gamma}^{\epsilon}(t)^{\alpha} \otimes E_{\alpha}$, $\{E_{\alpha}\}$ is a basis of $\mathfrak{g}$, and define

$$\bar{A}(t) = \int_{\gamma[0,t]} A.$$

By Chen's iterated integral [4], we define the ϵ -regularization of the holonomy by

$$(2.8) \quad W_{\gamma}^{\epsilon}(A) = I + \sum_{r=1}^{\infty} W_{\gamma}^{\epsilon,r}(A),$$

where

$$W_{\gamma}^{\epsilon,r}(A) = \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{r-1}} d(\bar{A}_0 + A_{\gamma}^{\epsilon})(t_1) d(\bar{A}_0 + A_{\gamma}^{\epsilon})(t_2) \cdots d(\bar{A}_0 + A_{\gamma}^{\epsilon})(t_r),$$

and the ϵ -regularized Wilson line by

$$(2.9) \quad F_{A_0}^\epsilon(A) = \prod_{j=1}^s \text{Tr}_{R_j} W_{\gamma_j}^\epsilon(A),$$

where the trace Tr is taken in the representation R_j of G assigned to each loop γ_j .

Thus adding regularizations of the determinant and holonomy to (2.3), we have

$$(2.10) \quad \frac{1}{Z} \int_{\mathcal{A}} \int_{\Phi} \mathcal{D}(A) \mathcal{D}(\phi) F_{A_0}^\epsilon(A) \\ \times \det_{\text{Reg}}(A) \exp \left[ik((A, \phi), Q_{A_0}(A, \phi))_+ - \frac{ik}{4\pi} \int_M \text{Tr} \frac{2}{3} A \wedge A \wedge A \right],$$

which is a heuristic version of the normalized Chern-Simons integral.

Let us exploit compensations in the numerator and denominator.(see [1]). Then setting

$$\tilde{F}_{A_0}^\epsilon(A) = \det_{\text{Reg}}(A) F_{A_0}^\epsilon(A),$$

we have a heuristic form of (2.10) such that

$$(2.11) \quad \frac{1}{\tilde{Z}} \int_{\mathcal{A}} \int_{\Phi} \mathcal{D}(A) \mathcal{D}(\phi) \tilde{F}_{A_0}^\epsilon\left(\frac{1}{\sqrt[3]{k}}A\right) \\ \times \exp \left[i\sqrt[3]{k}((A, \phi), Q_{A_0}(A, \phi))_+ - \frac{i}{4\pi} \int_M \text{Tr} \frac{2}{3} A \wedge A \wedge A \right],$$

where

$$(2.12) \quad \tilde{Z} = \int_{\mathcal{A}} \int_{\Phi} \mathcal{D}(A) \mathcal{D}(\phi) \exp \left[i\sqrt[3]{k}((A, \phi), Q_{A_0}(A, \phi))_+ \right].$$

Based on the heuristic idea such that the asymptotic expansion up to the order $2N$ of (2.11) may be equal to the asymptotic expansion up to the order $2N$ of

$$(2.13) \quad \frac{1}{\tilde{Z}} \int_{\mathcal{A}} \int_{\Phi} \mathcal{D}(A) \mathcal{D}(\phi) \tilde{F}_{A_0}^\epsilon\left(\frac{1}{\sqrt[3]{k}}A\right) \\ \times \exp \left[i\sqrt[3]{k}((A, \phi), Q_{A_0}(A, \phi))_+ - \frac{i}{4\pi} \int_M \text{Tr} \frac{2}{3} A \wedge A \wedge A \right] \\ \times \exp \left[- \left(\sum_{j=1}^{\infty} |((A, \phi), e_j)_+| \right)^{2N+2} \right],$$

from now on, we discuss the asymptotic expansion of (2.13).

For $x = (A, \phi) \in H_p$,

$$x = \sum_{j=1}^{\infty} (x, h_j)_p h_j,$$

$$(x, Q_{A_0} x)_+ = \sum_{j=1}^{\infty} a_j (x, h_j)_p^2,$$

and

$$\frac{-1}{4\pi} \int_M \text{Tr} \frac{2}{3} A \wedge A \wedge A = \sum_{a,b,c=1}^{\infty} (x, h_a)_p (x, h_b)_p (x, h_c)_p \beta_a \beta_b \beta_c T_{abc},$$

where $T_{abc} = - \int_M \text{Tr} \frac{1}{6\pi} e_a^A \wedge e_b^A \wedge e_c^A$.

By the idea of justifying the Feynman integral due to Itô [7] such that the convergent factors

$$\exp \left[-\frac{(x, x)_m}{2n} \right] \quad \text{with } n > 0$$

implemented into the m -dimensional approximation of (2.13) and using the formula of changing variables, by setting $x = \sqrt{n}y$, we have

$$(2.14) \quad \begin{aligned} & \frac{1}{\hat{Z}_{m,n}} \int_{R^m} \tilde{F}_{A_0}^\epsilon \left(\frac{\sqrt{n}}{\sqrt[3]{k}} y^m \right) \exp \left[in \sqrt[3]{k} (y^m, Q_{A_0} y^m)_+ \right] \\ & \times \exp \left[i \sum_{abc=1}^{\infty} \sqrt{n} y_a \sqrt{n} y_b \sqrt{n} y_c \beta_a \beta_b \beta_c T_{abc} \right] \\ & \exp \left[-\frac{(y, y)_m}{2} - \left(\sum_{j=1}^m \beta_j \sqrt{n} |y_j| \right)^{2N+2} \right] \frac{\nu_m(dy)}{(\sqrt{2\pi})^m}, \end{aligned}$$

where $\nu_m(dy)$ is the m -dimensional Lebesgue measure, $y^m = \sum_{j=1}^m y_j h_j$, and

$$\hat{Z}_{m,n} = \int_{R^m} \exp \left[in \sqrt[3]{k} (y^m, Q_{A_0} y^m)_+ \right] \exp \left[-\frac{(y, y)_m}{2} \right] \frac{\nu_m(dy)}{(\sqrt{2\pi})^m}.$$

Setting

$$\begin{aligned} f_n^L(x_L, x_{L-1}, \dots, x_1) &= \tilde{F}_{A_0}^\epsilon \left(\frac{\sqrt{n}}{\sqrt[3]{k}} x^L \right) \\ & \times \exp \left[i \sum_{abc=1}^L \sqrt{n} x_a \sqrt{n} x_b \sqrt{n} x_c \beta_a \beta_b \beta_c T_{abc} \right] \\ & \exp \left[-\left(\sum_{j=1}^L \beta_j \sqrt{n} |x_j| \right)^{2N+2} \right], \end{aligned}$$

we give the definition of a perturbative Chern-Simons integral (2.13) by setting it as equal to

$$(2.15) \quad \lim_{L \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{\tilde{Z}_{L,n}} \int_{R^L} f_n^L(x_L, x_{L-1}, \dots, x_1) \exp \left[i n \sqrt[3]{k} (x^L, Q_{A_0} x^L)_+ \right] \\ \exp \left[-\frac{(x, x)_L}{2} \right] \frac{\nu_L(dx)}{(\sqrt{2\pi})^L}.$$

Replace the eigenvalues λ_j of Q_{A_0} by λ_j^ζ for large ζ satisfying

Assumption 1. RENORMALIZATION.

$$\sum_{j=1}^{\infty} j^{2(8N^2+12N+4)} \frac{1}{\sqrt{|\lambda_j|}} \leq c < +\infty,$$

which is guaranteed by Lemma 1.6.3 in [6].

Then we have

Theorem 1. *Under the renormalization Assumption 1 and the assumption $T_{abc} \leq T < +\infty$, we have (2.15) is equal to*

$$(2.16) \quad \tilde{F}_{A_0}^\epsilon(0) \\ + \lim_{L \rightarrow \infty} \left\{ \sum_{s=1}^N \left(\sum_{r=1}^s \left(\sum_{1 \leq j_1 < j_2 < j_3 \dots < j_r \leq L} \left(\sum_{m_1, m_2, \dots, m_r \geq 1, m_1 + m_2 + \dots + m_r = s} \right. \right. \right. \right. \\ \lim_{n \rightarrow \infty} \left(\left(\prod_{q=1}^r \frac{1}{2^{m_q} m_q! (1 - 2i n \sqrt[3]{k} \beta_{j_q}^2 \lambda_{j_q})^{m_q}} \partial_{x_{j_q}}^{2m_q} \right) \tilde{F}_{A_0}^\epsilon \left(\frac{\sqrt{n}}{\sqrt[3]{k}} x^L \right) \right. \\ \times \exp \left[i \sum_{abc=1}^L \sqrt{n} x_a \sqrt{n} x_b \sqrt{n} x_c \beta_a \beta_b \beta_c T_{abc} \right] \left. \left. \left. \left. \right) \right) \right) \right) \right) \right\} \\ + O \left(\left(\frac{1}{\sqrt[3]{k}} \right)^{N+1} \right),$$

for sufficiently large k , where $\frac{\partial}{\partial x} = \partial_x$.

3 Proof of Theorem

The non-normalized form of (2.15) is equal to

$$(3.1) \quad \lim_{L \rightarrow \infty} \lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{2\pi}} \right)^L \int_{R^L} e^{-\sum_{j=1}^L \frac{1}{2} (1 - 2i \sqrt[3]{k} n a_j) x_j^2} f^L(x_L, x_{L-1}, \dots, x_1) \prod_{j=1}^L dx_j,$$

where $f^L(x_L, x_{L-1}, \dots, x_1)$ is a version of $f_n^L(x_L, x_{L-1}, \dots, x_1)$ such that

$$\exp \left[- \left(\sum_{j=1}^L \sqrt{n} \beta_j |x_j| \right)^{2N+2} \right]$$

in $f_n^L(x)$ is replaced by

$$\begin{cases} \exp \left[-(\sum_{j=1}^L \sqrt{n} \beta_j x_j)^{2N+2} \right], & \text{if } x_j \geq 0, \\ \exp \left[-(-\sum_{j=1}^L \sqrt{n} \beta_j x_j)^{2N+2} \right], & \text{if } x_j \leq 0. \end{cases}$$

When we consider the function $f^L(x_N, x_{L-1}, \dots, x_j, \dots, x_1)$ as a function of j -th coordinate x_j , denote it by $\hat{f}^L(x_j)$.

Since

$$\begin{aligned} \hat{f}^L(x_j) &= \hat{f}^L(0) + x_j \partial_{x_j} \hat{f}^L(0) + \dots + \frac{x_j^{2N}}{(2N)!} \partial_{x_j}^{2N} \hat{f}^L(0) \\ &\quad + x_j^{2N+1} \int_0^1 \frac{(1-\theta_j)^{2N}}{(2N)!} \partial_{x_j}^{2N+1} \hat{f}^L(\theta_j x_j) d\theta_j, \end{aligned}$$

so that set

$$Q_j^0 f^L = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(1-2in\sqrt[3]{ka_j})x_j^2} \left\{ \hat{f}^L(0) + x_j \partial_{x_j} \hat{f}^L(0) + \dots + \frac{x_j^{2N}}{(2N)!} \partial_{x_j}^{2N} \hat{f}^L(0) \right\} dx_j$$

and

$$Q_j^1 f^L = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(1-2in\sqrt[3]{ka_j})x_j^2} x_j^{2N+1} \left(\int_0^1 \frac{(1-\theta_j)^{2N}}{(2N)!} \partial_{x_j}^{2N+1} \hat{f}^L(\theta_j x_j) d\theta_j \right) dx_j.$$

Setting for any well differentiable function $g^L(x_L, x_{L-1}, \dots, x_2, x_1)$,

$$\begin{aligned} D_j^0 g^L &= \frac{1}{\sqrt{1-2in\sqrt[3]{ka_j}}} \hat{g}^L(0), \\ D_j^m g^L &= \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(1-2in\sqrt[3]{ka_j})x_j^2} x_j^m \partial_{x_j}^m \hat{g}^L(0) dx_j, \quad 1 \leq m \leq 2N, \end{aligned}$$

we get

$$\begin{aligned} (3.2) \quad & Q_L^0 \dots Q_2^0 Q_1^0 f^L \\ &= \left(\sum_{m=0}^{2N} D_L^m \right) \left(\sum_{m=0}^{2N} D_{L-1}^m \right) \dots \left(\sum_{m=0}^{2N} D_1^m f^L \right). \end{aligned}$$

Before proceeding to the estimate of the leading terms, we remark

Lemma 1. For any integer $0 \leq m \leq 2N$,

$$\left| D_j^m f^L \right| \leq \frac{1}{|\sqrt{1-2in\sqrt[3]{ka_j}}|} \left| \frac{1}{\sqrt{1-2in\sqrt[3]{ka_j}}} \right|^m \left| \partial_{x_j}^m f^L(x_L, \dots, x_{j+1}, 0, x_{j-1}, \dots, x_1) \right|.$$

Now we estimate the leading terms . The absolute value of the above (3.2) is dominated by

$$\begin{aligned}
 (3.3) \quad & \left| \left(\prod_{j=1}^L \frac{1}{\sqrt{1-2in\sqrt[3]{ka_j}}} \right) f^L(0,0,\dots,0) \right| + \sum_{1 \leq j_1 \leq L} \left(\sum_{m_1=1}^{2N} |D_L^0 D_{L-1}^0 \dots D_{j_1}^{m_1} \dots D_2^0 D_1^0 f^L| \right) \\
 & + \sum_{1 \leq j_1 < j_2 \leq L} \left(\sum_{m_1, m_2=1}^{2N} |D_L^0 D_{L-1}^0 \dots D_{j_2}^{m_2} \dots D_{j_1}^{m_1} \dots D_2^0 D_1^0 f^L| \right) \\
 & + \dots + \dots + \dots \\
 & + \sum_{1 \leq j_1 < j_2 < j_3 \dots < j_r \leq L} \left(\sum_{m_1, m_2, \dots, m_r=1}^{2N} |D_L^0 D_{L-1}^0 \dots D_{j_r}^{m_r} \dots D_{j_2}^{m_2} \dots D_{j_1}^{m_1} \dots D_2^0 D_1^0 f^L| \right) \\
 & + \dots + \dots + \dots
 \end{aligned}$$

By Lemma 1,

$$\begin{aligned}
 (3.4) \quad & \left| D_L^0 D_{L-1}^0 \dots D_{j_r}^{m_r} \dots D_{j_2}^{m_2} \dots D_{j_1}^{m_1} \dots D_2^0 D_1^0 f^L \right| \\
 & \leq \frac{1}{|\sqrt{1-2in\sqrt[3]{ka_L}}|} \frac{1}{|\sqrt{1-2in\sqrt[3]{ka_{L-1}}|}} \dots \frac{1}{|\sqrt{1-2in\sqrt[3]{ka_{j_r}}|}} \left| \frac{1}{\sqrt{1-2in\sqrt[3]{ka_{j_r}}}} \right|^{m_r} \\
 & \times \frac{1}{|\sqrt{1-2in\sqrt[3]{ka_{j_{r-1}}}}|} \dots \frac{1}{|\sqrt{1-2in\sqrt[3]{ka_{j_{r-1}}}}|} \left| \frac{1}{\sqrt{1-2in\sqrt[3]{ka_{j_{r-1}}}}} \right|^{m_{r-1}} \\
 & \times \frac{1}{|\sqrt{1-2in\sqrt[3]{ka_{j_{r-1}-1}}|}} \dots \frac{1}{|\sqrt{1-2in\sqrt[3]{ka_{j_1}}|}} \left| \frac{1}{\sqrt{1-2in\sqrt[3]{ka_{j_1}}}} \right|^{m_1} \dots \frac{1}{|\sqrt{1-2in\sqrt[3]{ka_1}}|} \\
 & \times \left| (\partial_{x_{j_r}}^{m_r} \dots \partial_{x_{j_1}}^{m_1} f^L)(0,0,\dots,0,0,\dots,0,0) \right|.
 \end{aligned}$$

Here we state the key lemma will be omitted the proof because of the restriction of the pages.

Lemma 2. For any non-negative integer N and L , there exists some constants $A_{2N} \geq 0$, $B_{2N} \geq 1$ and $\alpha(=2) \geq 1$ such that for any non-negative integers $1 \leq j_1 < j_2 < \dots < j_r \leq L$ and $m_j \leq 2N$,

$$\begin{aligned}
 & \left| \left(\prod_{s=1}^r \left(\frac{1}{\sqrt{1-2in\sqrt[3]{ka_{j_s}}}} \right)^{\alpha_{j_s}} \right) \left(\prod_{j=1, \dots, L, \neq j_1, j_2, \dots, j_r} \left(\frac{1}{\sqrt{1-2in\sqrt[3]{ka_j}}} \right)^{m_j} \right) \right| \\
 & \sup_{2N+1 \leq \alpha_{j_s} \leq 4N+4, x_{j_s}, 1 \leq s \leq r} \left| e^{-\sum_{s=1}^r \frac{1}{2} x_{j_s}^2} \partial_{x_L}^{m_L} \dots \partial_{x_{j_r}}^{\alpha_{j_r}} \right|
 \end{aligned}$$

$$\begin{aligned}
& \left| \dots \partial_{x_{j_2}}^{\alpha_{j_2}} \dots \partial_{x_{j_1}}^{\alpha_{j_1}} \dots \partial_{x_1}^{m_1} f^L(0, \dots, x_{j_r}, 0, \dots, x_{j_1}, 0, \dots, 0) \right| \\
& \leq A_{2N} \left(\{(2N+1) \sum_{j=1, \dots, L, \neq j_1, j_2, \dots, j_r} m_j + (2N+1) \sum_{s=1}^r \alpha_{j_s}\}! \right)^\alpha \\
& \quad \left(\prod_{s=1}^r (B_{2N} \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_s}|}})^{\alpha_{j_s}} \right) \\
& \quad \left(\prod_{j=1, \dots, L, \neq j_1, j_2, \dots, j_r} (B_{2N} \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}})^{m_j} \right).
\end{aligned}$$

Then Lemma 2 implies

$$\begin{aligned}
& \left| \left(\prod_{s=1}^r \left(\frac{1}{\sqrt{1-2in\sqrt[3]{k}a_{j_s}}} \right)^{m_s} \right) (\partial_{x_{j_r}}^{m_r} \dots \partial_{x_{j_1}}^{m_1} f^L)(0, 0, \dots, 0, 0, \dots, 0, 0) \right| \\
& \leq A_{2N} (\{(2N+1) \sum_{s=1}^r m_s\}!)^\alpha \left(\prod_{s=1}^r (B_{2N} \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_s}|}})^{m_s} \right),
\end{aligned}$$

which, together with (3.4), yields

$$\begin{aligned}
& \left| D_L^0 D_{L-1}^0 \dots D_{j_r}^{m_r} \dots D_{j_2}^{m_2} \dots D_{j_1}^{m_1} \dots D_2^0 D_1^0 f^L \right| \\
(3.5) \quad & \leq \left(\prod_{j=1}^L \frac{1}{\sqrt{1-2in\sqrt[3]{k}a_j}} \right) A_{2N} (\{(2N+1) \sum_{s=1}^r m_s\}!)^\alpha \left(\prod_{s=1}^r (B_{2N} \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_s}|}})^{m_s} \right).
\end{aligned}$$

Combining (3.3) and (3.5), we get that the absolute value of (3.2) is dominated by

$$\begin{aligned}
(3.6) \quad & \left| \prod_{j=1}^L \frac{1}{\sqrt{1-2in\sqrt[3]{k}a_j}} \right| \left\{ A_{2N} + A_{2N} \sum_{1 \leq j_1 \leq L} \left(\sum_{m_1=1}^{2N} B_{2N}^{m_1} (\{(2N+1)m_1\}!)^\alpha \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}} \right|^{m_1} \right) \right. \\
& + A_{2N} \sum_{1 \leq j_1 < j_2 \leq L} \left(\sum_{m_1, m_2=1}^{2N} B_{2N}^{m_1+m_2} (\{(2N+1)(m_1+m_2)\}!)^\alpha \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}} \right|^{m_1} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_2}|}} \right|^{m_2} \right) \\
& + \dots + \dots + \\
& + A_{2N} \sum_{1 \leq j_1 < j_2 < j_3 < \dots < j_r \leq L} \left(\sum_{m_1, m_2, \dots, m_r=1}^{2N} B_{2N}^{m_1+m_2+\dots+m_r} \right. \\
& \left. (\{(2N+1)(m_1+m_2+\dots+m_r)\}!)^\alpha \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}} \right|^{m_1} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_2}|}} \right|^{m_2} \dots \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_r}|}} \right|^{m_r} \right) \\
& \left. + \dots + \dots \right\}.
\end{aligned}$$

Since

$$\begin{aligned} & (\{(2N+1)(m_1+m_2+\dots+m_r)\})!^\alpha \\ & \leq (\{r(2N+1)2N\})!^\alpha \leq \{((p_N)^{p_N\alpha})^r r!^{p_N\alpha}\}^\alpha \\ & = ((p_N)^{p_N\alpha})^r r!^{p_N\alpha}, \end{aligned}$$

where

$$p_N = (2N+1)2N,$$

we have the above equation is dominated by

$$\begin{aligned} & \left| \prod_{j=1}^L \frac{1}{\sqrt{1-2in\sqrt[3]{k}a_j}} \right| A_{2N} \left\{ 1 + B_{2N}^{2N} \sum_{1 \leq j_1 \leq L} \left(\sum_{m_1=1}^{2N} ((p_N)^{p_N\alpha}) \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}} \right|^{m_1} \right) \right. \\ & + (B_{2N}^{2N})^2 \sum_{1 \leq j_1 < j_2 \leq L} \left(\sum_{m_1, m_2=1}^{2N} ((p_N)^{p_N\alpha})^2 2!^{p_N\alpha} \frac{1}{1^{p_N\alpha} 2^{p_N\alpha}} \right. \\ & \left. j_1^{p_N\alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}} \right|^{m_1} j_2^{p_N\alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_2}|}} \right|^{m_2} \right) \\ & + \dots + \dots \\ & + (B_{2N}^{2N})^r \sum_{1 \leq j_1 < j_2 < j_3 \dots < j_r \leq L} \left(\sum_{m_1, m_2, \dots, m_r=1}^{2N} ((p_N)^{p_N\alpha})^r r!^{p_N\alpha} \frac{1}{1^{p_N\alpha} 2^{p_N\alpha} \dots r^{p_N\alpha}} \right. \\ & \left. j_1^{p_N\alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}} \right|^{m_1} j_2^{p_N\alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_2}|}} \right|^{m_2} \dots j_r^{p_N\alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_r}|}} \right|^{m_r} \right) \\ & \left. + \dots + \dots \right\} \\ & \leq \left| \prod_{j=1}^L \frac{1}{\sqrt{1-2in\sqrt[3]{k}a_j}} \right| A_{2N} \left\{ 1 + B_{2N}^{2N} \sum_{1 \leq j_1 \leq L} \left(\sum_{m_1=1}^{2N} ((p_N)^{p_N\alpha}) \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}} \right|^{m_1} \right) \right. \\ & + (B_{2N}^{2N})^2 \sum_{1 \leq j_1 < j_2 \leq L} \left(\sum_{m_1, m_2=1}^{2N} ((p_N)^{p_N\alpha})^2 \right. \\ & \left. j_1^{p_N\alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}} \right|^{m_1} j_2^{p_N\alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_2}|}} \right|^{m_2} \right) \\ & + \dots + \dots \\ & + (B_{2N}^{2N})^r \sum_{1 \leq j_1 < j_2 < j_3 \dots < j_r \leq L} \left(\sum_{m_1, m_2, \dots, m_r=1}^{2N} ((p_N)^{p_N\alpha})^r j_1^{p_N\alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}} \right|^{m_1} \right. \end{aligned}$$

$$\begin{aligned}
& j_2^{p_N \alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_2}|}} \right|^{m_2} \cdots j_r^{p_N \alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_r}|}} \right|^{m_r} \\
& + \cdots + \cdots \Big\} \\
& \leq \left| \prod_{j=1}^L \frac{1}{\sqrt{1-2in\sqrt[3]{k}a_j}} \right| A_{2N} \left\{ 1 + [B_{2N}^{2N}(p_N)^{p_N \alpha}] \sum_{1 \leq j_1 \leq L} \left(\sum_{m_1=1}^{2N} j_1^{p_N \alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}} \right|^{m_1} \right) \right. \\
& + \frac{1}{2!} [B_{2N}^{2N}(p_N)^{p_N \alpha}]^2 \left(\sum_{j_1=1}^L \left(\sum_{m_1=1}^{2N} j_1^{p_N \alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}} \right|^{m_1} \right) \right)^2 \\
& + \cdots + \cdots \\
& + \frac{1}{r!} [B_{2N}^{2N}(p_N)^{p_N \alpha}]^r \left(\sum_{j_1=1}^L \left(\sum_{m_1=1}^{2N} j_1^{p_N \alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}} \right|^{m_1} \right) \right)^r \\
& \left. + \cdots + \cdots \right\}.
\end{aligned}$$

By the Assumption 1 such that

$$\sum_{j=1}^{\infty} j^{p_N \alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}} \right| \leq c < +\infty,$$

we have

$$\begin{aligned}
& \sum_{j=1}^L \left(\sum_{m=1}^{2N} j^{p_N \alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}} \right|^m \right) \\
& \leq \left(\sum_{j=1}^L j^{p_N \alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}} \right| \right) \left(1 + \sum_{j=1}^L j^{p_N \alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}} \right| \right)^{2N-1} \\
& \leq c(1+c)^{2N-1}.
\end{aligned}$$

Hence we get that the absolute value of (3.2) is dominated by

$$\begin{aligned}
(3.7) \quad & \left| \prod_{j=1}^L \frac{1}{\sqrt{1 - 2in\sqrt[3]{k}a_j}} \right| A_{2N} \left\{ 1 + [B_{2N}^{2N}(p_N)^{p_N\alpha}] (c(1+c)^{2N-1}) \right. \\
& + \frac{1}{2!} [B_{2N}^{2N}(p_N)^{p_N\alpha}]^2 (c(1+c)^{2N-1})^2 + \dots + \dots \\
& \left. + \frac{1}{r!} [B_{2N}^{2N}(p_N)^{p_N\alpha}]^r (c(1+c)^{2N-1})^r + \dots + \dots \right\} \\
& \leq \left| \prod_{j=1}^L \frac{1}{\sqrt{1 - 2in\sqrt[3]{k}a_j}} \right| A_{2N} \exp \left[[B_{2N}^{2N}(p_N)^{p_N\alpha}] c(1+c)^{2N-1} \right]
\end{aligned}$$

and we also have

$$\begin{aligned}
(3.8) \quad & \sum_{r=2N+2}^{\infty} A_{2N} \frac{1}{r!} [B_{2N}^{2N}(p_N)^{p_N\alpha}]^r \left(\sum_{j=1}^L \left(\sum_{m=1}^{2N} j^{p_N\alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}} \right|^m \right) \right)^r \\
& \leq \left(\sum_{j=1}^L j^{p_N\alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}} \right| \right)^{2N+2} A_{2N} (1+c)^{(2N-1)(2N+1)} \exp \left[[B_{2N}^{2N}(p_N)^{p_N\alpha}] (1+c)^{2N} \right].
\end{aligned}$$

Further for any natural number $2 \leq r \leq 2N+1$, we get

$$\begin{aligned}
(3.9) \quad & \sum_{1 \leq j_1 < j_2 < j_3 \dots < j_r \leq L} \left(\sum_{1 \leq m_1, m_2, \dots, m_r \leq 2N+1, \sum_{s=1}^r m_s \geq 2N+2} A_{2N} B_{2N}^{m_1+m_2+\dots+m_r} \right. \\
& \left. ((2N+1)(m_1+m_2+\dots+m_r))! \right)^{\alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}} \right|^{m_1} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_2}|}} \right|^{m_2} \dots \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_r}|}} \right|^{m_r} \\
& \leq \sum_{1 \leq j_1 < j_2 < j_3 \dots < j_r \leq L} \left(\sum_{1 \leq m_1, m_2, \dots, m_r \leq 2N+1, \sum_{s=1}^r m_s \geq 2N+2} A_{2N} [B_{2N}^{2N}(p_N)^{p_N\alpha}]^r j_1^{p_N\alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}} \right|^{m_1} \right. \\
& \quad \left. j_2^{p_N\alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_2}|}} \right|^{m_2} \dots j_r^{p_N\alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_r}|}} \right|^{m_r} \right) \\
& \leq A_{2N} [B_{2N}^{2N}(p_N)^{p_N\alpha}]^{2N+1} \sum_{1 \leq j_1 < j_2 < j_3 \dots < j_r \leq L} \left(\sum_{1 \leq m_1, m_2, \dots, m_r \leq 2N+1, \sum_{s=1}^r m_s \geq 2N+2} \right. \\
& \quad \left. \left(\sum_{j=1}^L j^{p_N\alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}} \right| \right)^{\sum_{s=1}^r m_s} \right) \\
& \leq \left(\sum_{j=1}^{\infty} j^{p_N\alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}} \right| \right)^{2N+2} A_{2N} [B_{2N}^{2N}(p_N)^{p_N\alpha}]^{2N+1} (2N+1)^{2N+1} (1+c)^{(2N+1)^2}.
\end{aligned}$$

Since

$$\partial_{x_{j_r}}^{m_r} \partial_{x_{j_{r-1}}}^{m_{r-1}} \cdots \partial_{x_{j_1}}^{m_1} \exp \left[- \left(\sum_{j=1}^L \sqrt{n} \beta_j |x_j| \right)^{2p+2} \right] (0) = 0$$

if $m_1 + \cdots + m_{r-1} + m_r \leq 2N + 1$, we have, together with (3.6), (3.8) and (3.9),

$$Q_L^0 \cdots Q_2^0 Q_1^0 f^L$$

is rewritten as

$$(3.10) \quad \left(\prod_{j=1}^L \frac{1}{\sqrt{1 - 2in\sqrt[3]{k}a_j}} \right) \left\{ f^L(0, 0, \dots, 0) \right. \\ + \sum_{s=1}^N \left(\sum_{r=1}^s \left(\sum_{1 \leq j_1 < j_2 < j_3 \cdots < j_r \leq L} \left(\sum_{m_1, m_2, \dots, m_r \geq 1, m_1 + m_2 + \dots + m_r = s} \frac{1}{2^{m_1} m_1! (1 - 2in\sqrt[3]{k}a_{j_1})^{m_1}} \partial_{x_{j_1}}^{2m_1} \right. \right. \right. \\ \left. \left. \left. \frac{1}{2^{m_2} m_2! (1 - 2in\sqrt[3]{k}a_{j_2})^{m_2}} \partial_{x_{j_2}}^{2m_2} \cdots \frac{1}{2^{m_r} m_r! (1 - 2in\sqrt[3]{k}a_{j_r})^{m_r}} \partial_{x_{j_r}}^{2m_r} f^L(0, 0, \dots, 0) \right) \right) \right) \\ \left. + O \left(\left(\sum_{j=1}^{\infty} j^{p_N \alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}} \right| \right)^{2N+2} \right) \right\}.$$

Hence letting $L \rightarrow \infty$ in (3.10), we have (2.16).

Now we estimate the remaining terms again by the method similar to that in the leading terms. In this turn, we also begin with the proof of

Lemma 3. *There exists a positive constant C_N such that*

$$(3.11) \quad |Q_j^1 f^L| \leq \frac{1}{|\sqrt{1 - 2in\sqrt[3]{k}a_j}|} C_N \left(\frac{1}{|\sqrt{1 - 2in\sqrt[3]{k}a_j}|} \right)^{\alpha_j} \\ \sup_{2N+1 \leq |\alpha_j| \leq 4N+4, x_j} e^{-\frac{x_j^2}{2}} \left| \partial_{x_j}^{\alpha_j} \hat{f}^L(x_j) \right|.$$

Proof. Apply the Fubini theorem and use the integration by parts formula, we have

$$Q_j^1 f^L = \int_0^1 d\theta_j \frac{(1 - \theta_j)^{2N}}{(2N)!} \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(1 - 2in\sqrt[3]{k}a_j)x_j^2} x_j^{2N+1} \\ \partial_{x_j}^{2N+1} \hat{f}^L(\theta_j x_j) dx_j \\ = \int_0^1 d\theta_j \frac{(1 - \theta_j)^{2N}}{(2N)!} \frac{1}{\sqrt{2\pi}} \int \frac{-\partial_{x_j}}{1 - 2in\sqrt[3]{k}a_j} (e^{-\frac{1}{2}(1 - 2in\sqrt[3]{k}a_j)x_j^2}) x_j^{2N}$$

$$\begin{aligned}
& \partial_{x_j}^{2N+1} \hat{f}^L(\theta_j x_j) dx_j \\
&= \int_0^1 d\theta_j \frac{(1-\theta_j)^{2N}}{(2N)!} \frac{1}{(1-2in\sqrt[3]{ka_j})\sqrt{2\pi}} \int e^{-\frac{1}{2}(1-2in\sqrt[3]{ka_j})x_j^2} 2N x_j^{2N-1} \\
& \partial_{x_j}^{2N+1} \hat{f}^L(\theta_j x_j) dx_j \\
&+ \int_0^1 d\theta_j \frac{(1-\theta_j)^{2N}}{(2N)!} \frac{1}{(1-2in\sqrt[3]{ka_j})\sqrt{2\pi}} \int e^{-\frac{1}{2}(1-2in\sqrt[3]{ka_j})x_j^2} x_j^{2N} \\
& \theta_j \partial_{x_j}^{2N+2} \hat{f}^L(\theta_j x_j) dx_j.
\end{aligned}$$

Repeating this process until x_j vanishes, we have

$$\begin{aligned}
(3.12) \quad Q_j^1 f^L &= \int_0^1 d\theta_j \frac{(1-\theta_j)^{2N}}{(2N)!} \frac{1}{(1-2in\sqrt[3]{ka_j})^{N+1}\sqrt{2\pi}} 2N(2N-2)(2N-4)\cdots 2 \\
&\left\{ -\partial_{x_j}^{2N+1} \hat{f}^L(0) + \int e^{-\frac{1}{2}(1-2in\sqrt[3]{ka_j})x_j^2} \right. \\
&\times \theta_j \partial_{x_j}^{2N+2} \hat{f}^L(\theta_j x_j) dx_j \Big\} \\
&+ \cdots + \cdots \\
&+ \int_0^1 d\theta_j \frac{(1-\theta_j)^{2N}}{(2N)!} \frac{1}{(1-2in\sqrt[3]{ka_j})^{2N+1}\sqrt{2\pi}} \int e^{-\frac{1}{2}(1-2in\sqrt[3]{ka_j})x_j^2} \\
&\theta_j^{2N+1} \partial_{x_j}^{4N+2} \hat{f}^L(\theta_j x_j) dx_j.
\end{aligned}$$

Setting

$$M_j = \frac{1 - x_j \partial_{x_j}}{1 + (1 - 2in\sqrt[3]{ka_j})x_j^2},$$

we have

$$M_j e^{-\frac{1}{2}(1-2in\sqrt[3]{ka_j})x_j^2} = e^{-\frac{1}{2}(1-2in\sqrt[3]{ka_j})x_j^2}.$$

Noticing M_j^* denotes the adjoint operator of M_j in $L^2(dx_j)$, we get for $G(x)$ of being well differentiable function and of satisfying

$$\lim_{|x_j| \rightarrow \infty} e^{-\frac{1}{2}x_j^2} |G(x)| = 0,$$

$$M_j^* G(x) = \frac{2G(x)}{(1 + (1 - 2in\sqrt[3]{ka_j})x_j^2)^2} + \frac{x_j \partial_{x_j} G(x)}{1 + (1 - 2in\sqrt[3]{ka_j})x_j^2},$$

so that we have

(3.13)

$$\begin{aligned}
|(M_j^*)^2 G(x)| \leq & \left\{ \left| \frac{4G(x)}{1 + (1 - 2in\sqrt[3]{ka_j})x_j^2} \right| + 8 \left| G(x) \right| \left| \frac{(1 - 2in\sqrt[3]{ka_j})x_j^2}{(1 + (1 - 2in\sqrt[3]{ka_j})x_j^2)^4} \right| \right. \\
& + 4 \left| \frac{x_j}{1 + (1 - 2in\sqrt[3]{ka_j})x_j^2} \right| \left| \frac{\partial_{x_j} G(x)}{(1 + (1 - 2in\sqrt[3]{ka_j})x_j^2)^2} \right| + \left| \frac{x_j}{1 + (1 - 2in\sqrt[3]{ka_j})x_j^2} \right| \left| \frac{\partial_{x_j} G(x)}{1 + (1 - 2in\sqrt[3]{ka_j})x_j^2} \right| \\
& + 2 \left| \partial_{x_j} G(x) \right| \left| \frac{x_j}{1 + (1 - 2in\sqrt[3]{ka_j})x_j^2} \right| \left| \frac{(1 - 2in\sqrt[3]{ka_j})x_j^2}{(1 + (1 - 2in\sqrt[3]{ka_j})x_j^2)^2} \right| \\
& \left. + \left| \frac{x_j^2}{1 + (1 - 2in\sqrt[3]{ka_j})x_j^2} \right| \left| \frac{\partial_{x_j}^2 G(x)}{1 + (1 - 2in\sqrt[3]{ka_j})x_j^2} \right| \right\}.
\end{aligned}$$

Therefore for any positive integer q , we have

$$\begin{aligned}
& |e^{-\frac{1}{2}(1-2in\sqrt[3]{ka_j})x_j^2} (M_j^*)^2 (\partial_{x_j}^q \hat{f}^L)(\theta_j x_j)| \\
& \leq 20 \left(\sup_{|\alpha_j| \leq 2, x_j, |\theta_j| \leq 1} \left(\frac{1}{|\sqrt{1 - 2in\sqrt[3]{ka_j}}|} \right)^{\alpha_j} \left| e^{-\frac{1}{2}x_j^2} \partial_{x_j}^{\alpha_j+q} \hat{f}^L(\theta_j x_j) \right| \right) \times \left| \frac{1}{1 + (1 - 2in\sqrt[3]{ka_j})x_j^2} \right|.
\end{aligned}$$

Notcing that

$$\int \left| \frac{1}{1 + (1 - 2in\sqrt[3]{ka_j})x_j^2} \right| dx_j \leq \frac{1}{|\sqrt{1 - 2in\sqrt[3]{ka_j}}|} \int \frac{1}{\sqrt{1 + y^4}} dy$$

and taking

$$\begin{aligned}
& \sup_{2N+2 \leq |\alpha_j| \leq 4N+4, x_j, |\theta_j| \leq 1} \left(\frac{1}{|\sqrt{1 - 2in\sqrt[3]{ka_j}}|} \right)^{\alpha_j} \left| e^{-\frac{1}{2}x_j^2} \partial_{x_j}^{\alpha_j} \hat{f}^L(\theta_j x_j) \right| \\
& \leq \sup_{2N+2 \leq |\alpha_j| \leq 4N+4, x_j, |\theta_j| \leq 1} \left(\frac{1}{|\sqrt{1 - 2in\sqrt[3]{ka_j}}|} \right)^{\alpha_j} \left| e^{-\frac{1}{2}\theta_j^2 x_j^2} \partial_{x_j}^{\alpha_j} \hat{f}^L(\theta_j x_j) \right| \\
& \leq \sup_{2N+2 \leq |\alpha_j| \leq 4N+4, x_j} \left(\frac{1}{|\sqrt{1 - 2in\sqrt[3]{ka_j}}|} \right)^{\alpha_j} \left| e^{-\frac{1}{2}x_j^2} \partial_{x_j}^{\alpha_j} \hat{f}^L(x_j) \right|
\end{aligned}$$

and for example,

$$\begin{aligned}
& \left| \frac{1}{(1 - 2in\sqrt[3]{ka_j})^{N+1} \sqrt{2\pi}} \left\{ -\partial_{x_j}^{2N+1} \hat{f}^L(0) \right\} \right| \\
& \leq \left| \frac{1}{\sqrt{1 - 2in\sqrt[3]{ka_j}}} \right| \left| \frac{1}{\sqrt{1 - 2in\sqrt[3]{ka_j}}} \right|^{2N+1} \sup_{x_j} \left| e^{-\frac{1}{2}x_j^2} \partial_{x_j}^{2N+1} \hat{f}^L(x_j) \right|
\end{aligned}$$

into account, we have the desired inequality from (3.12). $\square$

Now we return to the estimate the remaining terms such that

$$(3.14) \quad \left| \sum_{k_L, \dots, k_2, k_1=0,1, (k_L, \dots, k_2, k_1) \neq (0,0, \dots, 0)} Q_L^{k_L} \dots Q_2^{k_2} Q_1^{k_1} f^L \right| \\ \leq \sum_{r=1}^L \sum_{1 \leq j_1 < j_2 < \dots < j_r \leq L} \left| Q_L^0 \dots Q_{j_r}^1 \dots Q_{j_2}^1 \dots Q_{j_1}^1 \dots Q_1^0 f^L \right|.$$

First we discuss the case of $r \geq 2$.

$$(3.15) \quad \left| Q_L^0 \dots Q_{j_r}^1 \dots Q_{j_2}^1 \dots Q_{j_1}^1 \dots Q_1^0 f^L \right| \\ = \left| \left(\sum_{m=0}^{2N} D_L^m \right) \dots Q_{j_r}^1 \dots Q_{j_2}^1 \dots Q_{j_1}^1 \dots \left(\sum_{m=0}^{2N} D_1^m \right) f^L \right| \\ \leq \left| D_L^0 \dots Q_{j_r}^1 \dots Q_{j_2}^1 \dots Q_{j_1}^1 \dots D_1^0 f^L \right| \\ + \dots + \dots + \dots \\ + \left(\sum_{1 \leq l_1 < l_2 < \dots < l_r \leq L, l_1, l_2, \dots, l_r \neq j_1, j_2, \dots, j_r} \right) \\ \left(\sum_{m_{l_1}, m_{l_2}, \dots, m_{l_r}=1, m_i=0, i \neq \{l_1, l_2, \dots, l_r\}}^{2N} \right) \\ \left| D_L^{m_L} \dots Q_{j_r}^1 \dots Q_{j_2}^1 \dots Q_{j_1}^1 \dots D_1^{m_1} f^L \right| + \dots + \dots.$$

Next we estimate

$$(3.16) \quad \left| D_L^{m_L} \dots D_{j_r+1}^{m_{j_r+1}} Q_{j_r}^1 D_{j_r-1}^{m_{j_r-1}} \dots Q_{j_2}^1 \dots Q_{j_1}^1 \dots D_1^{m_1} f^L \right|.$$

By the manner similar to that in (3.4), we arrive at (3.16) is dominated by

$$(3.17) \quad \left(\prod_{j=1}^L \frac{1}{|\sqrt{1-2i\sqrt[3]{k}na_j}|} \right) \left(\prod_{j \neq j_1, j_2, \dots, j_r} \frac{1}{|\sqrt{1-2i\sqrt[3]{k}na_j}|}^{m_j} \right) \\ \times C_N^r \sup_{2N+1 \leq |\alpha_{j_s}| \leq 4N+4, x_{j_s}, 1 \leq s \leq r} \left(\prod_{s=1}^r \left(\frac{1}{|\sqrt{1-2i\sqrt[3]{k}na_j}|} \right)^{\alpha_{j_s}} \right) \\ \times \left| e^{-\sum_{s=1}^r \frac{1}{2} x_{j_s}^2} \left(\partial_{x_1}^{m_1} \dots \partial_{x_{j_1}}^{\alpha_{j_1}} \dots \partial_{x_{j_2}}^{\alpha_{j_2}} \dots \partial_{x_{j_r-1}}^{m_{j_r-1}} \partial_{x_{j_r}}^{\alpha_{j_r}} \partial_{x_{j_r+1}}^{m_{j_r+1}} \dots \partial_{x_L}^{m_L} \right. \right. \\ \left. \left. f^L((0, \dots, 0, x_{j_r}, \dots, x_{j_2}, \dots, x_{j_1}, \dots, 0)) \right|.$$

Suppose that

$$1 \leq l_1 < l_2 < \cdots < l_t \leq L, l_1, l_2, \dots, l_t \neq j_1, j_2, \dots, j_r$$

and

$$m_{l_j} \geq 1, j = 1, 2, \dots, s, m_i = 0, i \neq \{l_1, l_2, \dots, l_t\}.$$

Setting

$$\hat{p}_N = (2N + 1)(4N + 4),$$

we have , by Lemma 2,

$$\begin{aligned}
 & \left(\prod_{j \neq j_1, j_2, \dots, j_r} \frac{1}{|\sqrt{1 - 2in\sqrt[3]{k}a_j}|} \right)^{m_j} \\
 & \sup_{2N+1 \leq |\alpha_{j_s}| \leq 4N+4, 1 \leq s \leq r} \left(\prod_{s=1}^r \left(\frac{1}{|\sqrt{1 - 2i\sqrt[3]{k}na_j}|} \right)^{\alpha_{j_s}} \right) \\
 & \times \left| e^{-\sum_{s=1}^r \frac{1}{2} x_{j_s}^2} (\partial_{x_L}^{m_L} \dots \partial_{x_{j_r}}^{\alpha_{j_r}} \dots \partial_{x_{j_2}}^{\alpha_{j_2}} \dots \partial_{x_{j_1}}^{\alpha_{j_1}} \dots \partial_{x_1}^{m_1} f^L((0, \dots, 0, x_{j_r}, \dots, x_{j_2}, \dots, x_{j_1}, \dots, 0)) \right| \\
 & \leq A_{2N} \sup_{2N+1 \leq |\alpha_{j_s}| \leq 4N+4} (\{(2N + 1) \sum_{s=1}^r \alpha_{j_s} + (2N + 1) \sum_{j=1}^t m_{l_j}\})^\alpha \\
 (3.18) \quad & \left(\prod_{s=1}^r (B_{2N} \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_s}|}})^{\alpha_{j_s}} \right) \left(\prod_{j=1}^t (B_{2N} \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}})^{m_{l_j}} \right) \\
 & \leq A_{2N} \hat{p}_N^{\hat{p}_N \alpha(t+r)} \{(t+r)!\}^{\hat{p}_N \alpha} \\
 & \frac{1}{1^{\hat{p}_N \alpha} 2^{\hat{p}_N \alpha} \dots (t+r)^{\hat{p}_N \alpha}} \\
 & \times \left(\prod_{s=1}^r j_s^{\hat{p}_N \alpha} (B_{2N} \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}})^{\alpha_{j_s}} \right) \\
 & \left(\prod_{j=1}^t l_j^{\hat{p}_N \alpha} (B_{2N} \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}})^{m_{l_j}} \right).
 \end{aligned}$$

Summing up, (3.16), (3.17) and (3.18), and setting $c_N = \max\{1, C_N \hat{p}_N^{\hat{p}_N \alpha} B_{2N}^{4N+4} (1+c)^{2N+3}\}$, we have for sufficiently large k ,

$$\begin{aligned}
& \left| D_L^{m_L} D_{L-1}^{m_{L-1}} \cdots D_{j_r+1}^{m_{j_r+1}} Q_{j_r}^1 D_{j_r-1}^{m_{j_r-1}} \cdots Q_{j_2}^1 \cdots Q_{j_1}^1 \cdots D_1^{m_1} f^L \right| \\
& \leq \left(\prod_{j=1}^L \frac{1}{\sqrt{1-2i\sqrt[3]{k}na_j}} \right) A_{2N} \hat{p}_N^{\hat{p}_N \alpha t} \left(\prod_{j=1}^t l_j^{\hat{p}_N \alpha} \left| B_{2N} \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}} \right|^{m_{l_j}} \right) \\
& \times C_N^r \hat{p}_N^{\hat{p}_N \alpha r} \left(\prod_{s=1}^r j_s^{\hat{p}_N \alpha} B_{2N}^{4N+4} (1+c)^{2N+3} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_s}|}} \right|^{2N+1} \right) \\
(3.19) \quad & \leq \left(\prod_{j=1}^L \frac{1}{\sqrt{1-2i\sqrt[3]{k}na_j}} \right) A_{2N} \hat{p}_N^{\hat{p}_N \alpha t} \left(\prod_{j=1}^t l_j^{\hat{p}_N \alpha} \left| c_N \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}} \right|^{m_{l_j}} \right) \\
& \times c_N^r \left(\prod_{s=1}^r j_s^{\hat{p}_N \alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_s}|}} \right|^{2N+1} \right).
\end{aligned}$$

Therefore by (3.19) and (3.15), we have

$$\begin{aligned}
& \left| Q_L^0 \cdots Q_{j_r}^1 \cdots Q_{j_2}^1 \cdots Q_{j_1}^1 \cdots Q_1^0 f^L \right| \\
& \leq \left(\prod_{j=1}^L \frac{1}{\sqrt{1-2i\sqrt[3]{k}na_j}} \right) c_N^r \left(\prod_{s=1}^r j_s^{\hat{p}_N \alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_s}|}} \right|^{2N+1} \right) A_{2N} \left\{ 1 + \right. \\
(3.20) \quad & + \hat{p}_N^{\hat{p}_N \alpha} \left(\sum_{1 \leq l_1 \leq L, l_1 \neq j_1, j_2, \dots, j_r} \right) \left(\sum_{m_{l_1}=1, m_i=0, i \neq l_1}^{2N} \right) l_1^{\hat{p}_N \alpha} \left| c_N \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{l_1}|}} \right|^{m_{l_1}} \\
& + \cdots + \cdots \\
& + \hat{p}_N^{\hat{p}_N \alpha t} \left(\sum_{1 \leq l_1 < l_2 < \cdots < l_t \leq L, l_i \neq j_1, j_2, \dots, j_r} \right) \left(\sum_{m_{l_1}, m_{l_2}, \dots, m_{l_t}=1, m_i=0, i \neq \{l_1, l_2, \dots, l_t\}}^{2N} \right) \\
& \left. \left(\prod_{j=1}^t l_j^{\hat{p}_N \alpha} \left| c_N \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{l_j}|}} \right|^{m_{l_j}} \right) + \cdots + \cdots \right\}.
\end{aligned}$$

Since $m_{l_i} \leq 2N$, setting

$$M = A_{2N} \sum_{t=0}^{\infty} \frac{1}{t!} \left(c_N^{2N} \hat{p}_N^{\hat{p}_N \alpha} \right)^t \left(\sum_{j=1}^{\infty} \left(\sum_{m=1}^{2N} j^{\hat{p}_N \alpha} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}} \right|^m \right) \right)^t,$$

by the manner similar to that in estimating the leading terms, (3.20) is dominated by

$$(3.21) \quad \left(\prod_{j=1}^L \frac{1}{|\sqrt{1-2i\sqrt[3]{k}na_j}|} \right) M c_N^r \left(\prod_{s=1}^r j_s^{\hat{p}_{N\alpha}} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_s}|}} \right|^{2N+1} \right).$$

Therefore by (3.14) and (3.21), we arrive at for sufficiently large k ,

$$\begin{aligned} & \left| \sum_{k_L, \dots, k_2, k_1=0,1, (k_L, \dots, k_2, k_1) \neq (0,0, \dots, 0), \sum_{j=1}^L k_j \geq 2} Q_L^{k_L} \dots Q_2^{k_2} Q_1^{k_1} f^L \right| \\ & \leq \left(\prod_{j=1}^L \frac{1}{|\sqrt{1-2i\sqrt[3]{k}na_j}|} \right) \left\{ \sum_{r=2}^L \frac{1}{r!} M c_N^r \left(\sum_{j=1}^L j^{\hat{p}_{N\alpha}} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}} \right| \right)^{(2N+1)r} \right\} \\ & \leq \left(\prod_{j=1}^L \frac{1}{|\sqrt{1-2i\sqrt[3]{k}na_j}|} \right) M \left(\sum_{j=1}^L j^{\hat{p}_{N\alpha}} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_s}|}} \right| \right)^{(2N+2)} c_N^2 c^{2N} \exp[c_N c^{2N+1}] \\ & \leq \left(\prod_{j=1}^L \frac{1}{|\sqrt{1-2i\sqrt[3]{k}na_j}|} \right) \left\{ O \left(\left(\frac{1}{\sqrt[3]{k}} \right)^{N+1} \right) \right\}. \end{aligned}$$

Next we discuss the case of $r = 1$. Since

$$\partial_{x_j}^{2N+1} f^L(0) = 0,$$

so that the discussion after (3.12) implies there exists a constant C such that for sufficiently large k ,

$$\begin{aligned} & |D_L^0 \dots Q_{j_1}^1 \dots D_1^0 f^L| \\ & \leq C \left(\prod_{j=1}^L \frac{1}{|\sqrt{1-2i\sqrt[3]{k}na_j}|} \right) \left(j_1^{\hat{p}_{N\alpha}} \left| \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}} \right| \right)^{(2N+2)}. \end{aligned}$$

Further, since

$$\begin{aligned} & \left| Q_L^0 \dots Q_{j_1}^1 \dots Q_1^0 f^N - D_L^0 \dots Q_{j_1}^1 \dots D_1^0 f^L \right| \\ & \leq \left(\prod_{j=1}^L \frac{1}{|\sqrt{1-2i\sqrt[3]{k}na_j}|} \right) \left[A_{2N} \sum_{r=1}^{\infty} \frac{1}{r!} \left(c_N^{2N} \hat{p}_N^{\hat{p}_{N\alpha}} \right)^r \left(\sum_{j=1}^{\infty} \left(\sum_{m=1}^{2N} j^{\hat{p}_{N\alpha}} \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}} \right)^m \right)^r \right] \\ & \quad \times c_N j_1^{\hat{p}_{N\alpha}} \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_{j_1}|}}^{2N+1}, \end{aligned}$$

and

$$\left(\sum_{j=1}^{\infty} \left(\sum_{m=1}^{2N} j^{\hat{p}_{N\alpha}} \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}} \right)^m \right)^r$$

$$\leq \left(\left(\sum_{j=1}^{\infty} j^{\hat{p}_{N\alpha}} \frac{1}{\sqrt{2\sqrt[3]{k}|\lambda_j|}} \right) (1+c)^{2N-1} \right)^r,$$

we have for sufficiently large k ,

$$\sum_{j_1=1}^L |Q_L^0 \cdots Q_{j_1}^1 \cdots Q_1^0 f^L| \leq \left(\prod_{j=1}^L \frac{1}{\sqrt{1-2i\sqrt[3]{k}na_j}} \right) \left\{ O\left(\left(\frac{1}{\sqrt[3]{k}} \right)^{N+1} \right) \right\},$$

which completes the proof.

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Itaru Mitoma
Department of Mathematics
Saga University
Saga 840-8502
Japan
E-mail address: mitoma@ms.saga-u.ac.jp