

Categories of Elements

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1 Elements of a set-valued functor

References: [ML98], [Bo94]

1.1 Elements of a functor

Definition 1.1 An **element** of a set valued functor $F : \mathcal{C} \rightarrow \mathbf{Set}$ is a pair (X, x) of an object $X \in \mathcal{C}$ and $x \in F(X)$. A **morphism** $f : (X, x) \rightarrow (Y, y)$ between elements is a morphism $f : X \rightarrow Y$ in $\mathcal{C}$ such that

$$F(f) : F(X) \rightarrow F(Y); x \mapsto y$$

The elements of F form the **category of elements**, which is denoted by

$$\mathbf{Elts}(F) \text{ or } \mathbf{Elts}(\mathcal{C}, F)$$

with **projection functor**

$$\pi_F : \mathbf{Elts}(F) \rightarrow \mathcal{C}; (X, x) \mapsto X.$$

For a contravariant functor, the category of elements is similarly defined.

See[Yo60], [Bo94, I.p37]. ■

Lemma 1.1 In $\mathbf{Elts}(\mathcal{C}, F)$, the following hold:

- (i) $(X, x) \cong (Y, y)$ if and only if there exists $f : X \cong Y$ in $\mathcal{C}$ such that $y = f(x)$.
- (ii) There is a bijection

$$\mathbf{Obj}(\mathbf{Elts}(\mathcal{C}, F)) / \cong \longleftrightarrow \coprod'_{X \in \mathcal{C}} \mathbf{Aut}(X) \backslash F(X)$$

Here $\coprod'$ is the coproduct over the isomorphisms classes $\mathbf{Obj}(\mathcal{C}) / \cong$

1.2 comma categories and slice categories

Definition 1.2 The **comma category** $(S \downarrow T)$ of a pair of functors $\mathcal{D} \xrightarrow{S} \mathcal{C} \xleftarrow{T} \mathcal{E}$ has as objects all triplets $(X, Y, S(X) \xrightarrow{f} T(Y))$ and as morphisms $(X, Y, S(X) \xrightarrow{f} T(Y)) \rightarrow (X', Y', S(X') \xrightarrow{f'} T(Y'))$ all pairs $(X \xrightarrow{u} X', Y \xrightarrow{v} Y')$ such that

$$\begin{array}{ccc} X & & Y \\ u \downarrow & & v \downarrow \\ X' & & Y' \end{array} \quad \begin{array}{ccc} SX & \xrightarrow{f} & TY \\ Su \downarrow & \circlearrowleft & \downarrow Tv \\ SX' & \xrightarrow{f'} & TY' \end{array}$$

The compositions are given by those of $\mathcal{D}$ and $\mathcal{E}$. [ML98], [Bo94] ■

Definition 1.3 The **slice category** $\mathcal{C}/X$ over an object $X \in \mathcal{C}$ is the category of morphisms into X . A morphism from $(A \xrightarrow{\alpha} X)$ to $(B \xrightarrow{\beta} X)$ is a morphism $f : A \rightarrow B$ in $\mathcal{C}$ such that $\alpha = f\beta$.

Similarly, the **coslice category** $X \backslash \mathcal{C}$ is defined as the category of morphisms from X .

Let $S = \text{Id}_{\mathcal{C}}$ be an identity functor of $\mathcal{C}$, and $T : * := \{*, \text{id}_*\} \rightarrow \mathcal{C}; * \mapsto X$. Then there are equivalences of categories

$$(S \downarrow T) \approx \mathcal{C}/X \text{ and } (T \downarrow S) \approx X \backslash \mathcal{C}$$

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$F : \mathcal{C} \rightarrow \mathbf{Set}$, $S : \{*\} \hookrightarrow \mathbf{Set}$. Then the category of elements of F is presented by a comma category:

$$\mathbf{Els}(\mathcal{C}, F) \cong S \downarrow F$$

■

1.3 Examples

Example 1.1 A monoid M can be identified with a category $\mathbf{M}$ with a single object $*$ and with $\mathbf{Hom}(*, *) = M$. Let X be an M -set with left M -action $M \times X \rightarrow X; (a, x) \mapsto ax$.

Such an M -set X can be viewed as

(i) a functor $X : \mathbf{M} \rightarrow \mathbf{Set}; * \mapsto X$;

and also as

(ii) a category $\mathbf{X}$ with $\mathbf{Obj}(\mathbf{X}) = X$ and with

$$\mathbf{Hom}_{\mathbf{X}}(x, y) = \{a \in M \mid ax = y\}$$

Then the category of elements of the functor X is equivalent to $\mathbf{X}$:

$$\mathbf{Els}(\mathbf{M}, X) \approx \mathbf{X}; (*, x) \longleftrightarrow x$$

■

Example 1.2 Let $X \in \mathcal{C}$. Then

(1) Let $H_X : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}; A \mapsto \mathbf{Hom}(A, X)$ denote the contravariant Hom-functor. Then an element of H_X has the form $(A \xrightarrow{\alpha} X)$, i.e., an object over X , and so the category of elements of H_X is equivalent to the slice category:

$$\mathbf{Els}(\mathcal{C}, H_X) \approx \mathcal{C}/X$$

(2) Similarly, for the covariant Hom-functor $H^X : \mathcal{C} \rightarrow \mathbf{Set}; A \mapsto \mathbf{Hom}(X, A)$, the category of elements is equivalent to the coslice category:

$$\mathbf{Els}(\mathcal{C}, H^X) \approx X \backslash \mathcal{C}$$

■

Example 1.3 Let G be a finite group. Let $\mathbf{set}^G$ denote the category of finite (left) G -sets and G -maps and $\mathbf{trans}^G$ the subcategory of $\mathbf{set}^G$ consisting of transitive G -sets. Then a G -map $f : G/H \rightarrow G/K$ is decided by the image of $H \in G/H$:

$$\mathbf{Map}_G(G/H, G/K) = \{xK \in G/K \mid H \subset {}^xK\}$$

The **subgroup category** $\mathbf{sub}(G)$ has all subgroups of G as objects. A morphism $H \rightarrow K$ is a coset xK such that $H \subset {}^xK := xKx^{-1}$; and the composition is defined by $yL \circ xK = xyL$. Then $\mathbf{sub}(G)$ is equivalent to $\mathbf{trans}^G$ by $H \mapsto G/H$. Two subgroups are isomorphic in $\mathbf{set}(G)$ if and only if they are conjugate, and so $C(G) := \mathbf{sub}(G)/\cong$ is the set of conjugacy classes of subgroups.

Let $\mathbf{Sub}(G)$ be the **subgroup lattice** of G . Note that any poset can be viewed as a category. Let $\mathbf{hom}(1, -) : H \mapsto G/H$ be the Hom-functor from the trivial subgroup $1 \in \mathbf{set}(G)$. Then

$$\begin{aligned} &\mathbf{Els}(\mathbf{sub}(G), \mathbf{hom}(1, -)), \\ &1 \backslash \mathbf{sub}(G), \\ &\mathbf{Els}(\mathbf{trans}^G, \mathbf{Map}_G(G/1, -)), \\ &(G/1) \backslash \mathbf{trans}^G \end{aligned}$$

are all equivalent to $\mathbf{Sub}(G)$ as categories. In particular, the isomorphism classes of these categories are all bijectively corresponding to the set of subgroups of G .

As a conclusion the subgroup lattice $\mathbf{Sub}(G)$ is categorically viewed as the category of elements of a functor!! ■

Categories of elements are used to prove the following two important theorems. Refer to [Ri14].

Example 1.4 Yoneda's density theorem:

Let $F : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ and let $\widehat{\mathcal{C}} := [\mathcal{C}^{\text{op}}, \mathbf{Set}]$. Then

$$F \cong \varinjlim \left(\mathbf{Els}(F) \xrightarrow{\pi_F} \mathcal{C} \xrightarrow{y} \widehat{\mathcal{C}} \right),$$

where $y : X \mapsto \mathbf{Hom}(-, X)$ denotes the Yoneda embedding.

Example 1.5 Kan extension:

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Then $\widehat{F} : \widehat{\mathcal{D}} \rightarrow \widehat{\mathcal{C}}; Y \mapsto Y \circ F$ has a left adjoint functor and a right adjoint functor:

$$\text{Lan}(F) \dashv \widehat{F} \dashv \text{Ran}(F)$$

The value of $\text{Lan}(F)$ at $X \in \widehat{\mathcal{C}}$ is given by

$$\begin{aligned} \text{Lan}(F)(X) &= \varinjlim \left(F \downarrow J \xrightarrow{\pi} \mathcal{C} \xrightarrow{X} \mathbf{Set} \right) \\ &\cong \varinjlim \left(\mathbf{Els}(H_J \circ F) \xrightarrow{\pi} \mathcal{C} \xrightarrow{X} \mathbf{Set} \right) \end{aligned}$$

Similarly $\text{Ran}(F)(X)$ is obtained by replacing the limit instead of the colimit. [ML98, X.3]

1.4 Operations on set-valued functors

There are some arithmetical operations on categories and functors. We study what categories of the elements of set-valued functors play in such operations. Refer to [Yo01]

Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories, and $F : \mathcal{C} \rightarrow \mathbf{Set}$, $G : \mathcal{D} \rightarrow \mathbf{Set}$, $H : \mathcal{E} \rightarrow \mathbf{Set}$ set-valued functors. Then we define additions and products as follows :

- (i) $\mathcal{C} + \mathcal{D}$: the disjoint union of categories.
- (ii) $\mathcal{C} \times \mathcal{D}$: the Cartesian product of categories.
- (iii) $F + G$: the summation of functors.

$$F + G : \mathcal{C} + \mathcal{D} \rightarrow \mathbf{Set}; Z \mapsto \begin{cases} F(Z) & (Z \in \mathcal{C}) \\ G(Z) & (Z \in \mathcal{D}) \end{cases}$$

- (iv) $F \times G$: the product of functors.

$$F \times G : \mathcal{C} \times \mathcal{D} \rightarrow \mathbf{Set}; (X, Y) \mapsto F(X) \times G(Y)$$

Here $F(X) \times G(Y)$ denotes the disjoint union of sets $F(X)$ and $G(Y)$.

- (v) F^n : the power of a functor.

$$F^n : \mathcal{C}^n \rightarrow \mathbf{Set}; (X_k)_{k=1}^n \mapsto \prod_{k=1}^n F(X_k)$$

Then the 2-category $\mathbf{Cat}$ has a commutative semi-ring structure by $+$ and $\times$. Furthermore,

so is the 2-category $\mathbf{Cat}/\mathbf{Set}$ of set valued functors. For example, the following distributive law holds

$$(F + G) \times H \cong F \times H + G \times H$$

"Zero" and "One" in $\mathbf{Cat}/\mathbf{Set}$ is

$$\begin{aligned} \mathbf{O} : \emptyset &\rightarrow \mathbf{Set}, \\ \mathbf{I} : \mathbf{1} = \{*, \text{id}_*\} &\rightarrow \mathbf{Set}; * \mapsto \{*\} \end{aligned}$$

respectively.

For a functor $F : \mathcal{C} \rightarrow \mathbf{Set}$, define a functor

$$\partial F : \mathbf{Els}(\mathcal{C}, F) \xrightarrow{\pi_F} \mathcal{C} \xrightarrow{F} \mathbf{Set}$$

Then the following hold:

$$\mathbf{Els}(F \times G) \approx \mathbf{Els}(F) \times \mathcal{D} + \mathcal{C} \times \mathbf{Els}(G)$$

$$\partial(F \times G) \cong F \times \partial(G) + \partial(F) \times G$$

$$\mathbf{Els}(F^n) \approx n\mathcal{C}^{n-1} \times \mathbf{Els}(F)$$

$$\partial(F^n) \cong n\mathbb{F}^{n-1} \times \partial(F)$$

These formulas look like Leibniz's product rule for differentiation. This is the reason why we used ∂F for the functor from the category of elements.

Remark. In some literature (e.g., [ML98]), $\mathbf{Els}(\mathcal{C}, F)$ is often denoted by the symbol

$$\int_{\mathcal{C}} F \quad \text{or} \quad \int F.$$

This symbol is not suitable for the category of elements because of Leibniz rule.

2 Generating functions

Reference: [Yo13], [Yo01], [Jo81].

2.1 Universal zeta functions (UZF)

The reason why the category of elements of a functor works like derivation becomes clear by considering generating functions of categories and functors.

Let $\mathcal{C}$ be a essentially small and locally finite category, and so $\mathcal{C}$ is equivalent to a small category and each hom-set $\text{Hom}(X, Y)$ is a finite set for any $X, Y \in \mathcal{C}$. Then the **universal zeta function** (or **exponential generating function** of $\mathcal{C}$) is defined as a formal series

$$\mathcal{C}(t) := \sum'_{M \in \mathcal{C}} \frac{1}{|\text{Aut}(M)|} t^M$$

where $\sum'$ takes over isomorphism classes of objects of $\mathcal{C}$. The symbols t^M ($M \in \mathcal{C}$) are assumed to satisfy the relations

- (i) $M \cong M' \Rightarrow t^M = t^{M'}$
- (ii) $t^\emptyset = 1$, $t^{M+M'} = t^M \cdot t^{M'}$ if there exist any finite coproducts, where $\emptyset$ is an initial object.

The **universal zeta function** (or **exponential generating function** of a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is

$$F(t) := \sum'_{M \in \mathcal{C}} \frac{1}{|\text{Aut}(M)|} t^{F(M)}$$

Here the summation is well-defined only if the fibers of F are all finite sets, that is, for any $N \in \mathcal{D}$,

$$\#\{M \in \mathcal{C} / \cong \mid F(M) \cong N\} < \infty.$$

Such a functor F is said to have **finite fibers**.

Let **set** be the category of finite sets. We identify the symbol t^N with the monomial polynomial $t^{|N|}$. Thus if $F : \mathcal{C} \rightarrow \text{set}$ is a faithful functor with finite fibers, then the UZF $F(t)$ is the usual formal power series. For example,

$$\text{set}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} = \exp(t) \in \mathbb{Q}[[t]]$$

2.2 $\mathcal{C}$ -structures

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ a faithful functor.

Definition 2.1 An $\mathcal{C}$ -structure on $N (\in \mathcal{D})$ is (X, σ) , where $X \in \mathcal{C}$ and $\sigma : F(X) \xrightarrow{\cong} N$. The isomorphism σ is called a **labeling**. We denote by

$\text{Str}(\mathcal{C}/N) \subset F \downarrow N$ the category of $\mathcal{C}$ -structures on N .

The isomorphism of two $\mathcal{C}$ -structures on N is defined by

$$(X, \sigma) \cong (Y, \tau) \Leftrightarrow \exists f : X \cong Y \text{ s.t. } \tau \circ F(f) = \sigma$$

■

Lemma 2.1 The UZF of F satisfying the following:

$$F(t) = \sum'_{N \in \mathcal{D}} \frac{|\text{Str}(\mathcal{C}/N)/\cong|}{|\text{Aut}(N)|} t^N$$

Furthermore, $|\text{Str}(\mathcal{C}/N)/\cong|$, the number of isomorphism classes of $\mathcal{C}$ -structures on N , is equal to

$$\sum'_{F(X) \cong N} (\text{Aut}(F(X)) : F(\text{Aut}(X))),$$

where the summation is taken over isomorphism classes of $\mathcal{C}$ -structures on N .

2.3 Operations on UZF

The definitions of operations on faithful functors match those on power series, that is, for any faithful functors $F : \mathcal{C} \rightarrow \text{set}$ and $G : \mathcal{D} \rightarrow \text{set}$ into the category of finite sets with finite fibers, we have the equations of formal power series:

$$\begin{aligned} (F + G)(t) &= F(t) + G(t) \\ (FG)(t) &= F(t)G(t) \\ \emptyset(t) &= 0, \quad \mathbf{1}(t) = 1. \end{aligned}$$

As before, let

$$\partial F : \text{Elts}(\mathcal{C}, F) \xrightarrow{\pi_F} \mathcal{C} \xrightarrow{F} \text{set}; (X, x) \mapsto X \mapsto F(X)$$

Then its UZF is

$$(\partial F)(t) = \sum'_{M \in \mathcal{C}} \frac{|F(M)|}{|\text{Aut}(M)|} t^{F(M)} = t \frac{dF(t)}{dt}$$

Remark.

$$F' : \text{Elts}(\mathcal{C}, F) \rightarrow \text{set}; (X, x) \mapsto F(X) - \{x\},$$

gives the usual derivation $F'(t) = dF(t)/dt$. Unfortunately, unless all $F(f)$ are monic, F' is not a functor.

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Let $H^I := \text{Hom}(I, -) : \mathcal{D} \rightarrow \mathbf{set}$ be a Hom-functor associated to $I \in \mathcal{D}$. Then a **partial derivation** of F is defined by

$$\partial_I(F) := \partial(H^I \circ F) : \mathbf{Elts}(H^I \circ F) \xrightarrow{\pi} \mathcal{C} \xrightarrow{H^I} \mathbf{set} \\ ; (X, x) \mapsto \text{Hom}(I, F(X))$$

It is possible to define a so-called plethysm compositions of categories (or functors). Here we only give exponential of categories.

Definition 2.2 For a category $\mathcal{C}$, the **fibred category** $\mathbf{Exp}(\mathcal{C})$ (or often $\mathbf{set}(\mathcal{C})$) is the category with objects all indexed $\mathcal{C}$ -objects $(X_i)_{i \in I}$, where I is a finite set and X_i is an object of $\mathcal{C}$, and with morphisms $(\pi, (f_i)_{i \in I}) : (X_i)_{i \in I} \rightarrow (Y_j)_{j \in J}$, where $\pi : I \rightarrow J$ and $f_i : X_i \rightarrow Y_{\pi(i)}$. The category $\mathbf{Exp}(\mathcal{C})$ has any finite coproducts.

For any functor $F : \mathcal{C} \rightarrow \mathbf{Set}$ can be uniquely extended to

$$\mathbf{Exp}(F) : \mathbf{Exp}(\mathcal{C}) \rightarrow \mathbf{Set}; (X_i)_{i \in I} \mapsto \coprod_{i \in I} F(X_i)$$

which preserves finite coproducts. ■

Let $\mathbf{1}$ be the category with only one object $*$ and only one morphism id_* . Then $\mathbf{Exp}(\mathbf{1}) \approx \mathbf{set}$, the category of finite sets.

Lemma 2.2 (1) $\mathbf{Exp}(\mathcal{C})(t) = \exp(\mathcal{C}(t))$.

(2) $\mathbf{Exp}(F)(t) = \exp(F(t))$.

(3) $\mathbf{Exp}(\mathcal{C} + \mathcal{D}) \approx \mathbf{Exp}(\mathcal{C}) \times \mathbf{Exp}(\mathcal{D})$

(4) $\mathbf{Exp}(F + G) \cong \mathbf{Exp}(F) \times \mathbf{Exp}(G)$

(5) $\partial(\mathbf{Exp}(F)) = (\partial F) \cdot \mathbf{Exp}(F)$.

Example 2.1 Tree

2.4 Wohlfahrt formula

Theorem 2.3 Let G be a finitely generated group.. Then the following hold:

(1) $\mathbf{set}^G \approx \mathbf{Exp}(\mathbf{trans}^G)$.

(2) $\mathbf{set}^G(t) = \exp(\mathbf{trans}^G(t))$.

(3) $\mathbf{trans}^G(t) = \sum_{H \leq_f G} \frac{t^{G/H}}{(G:H)},$

where H runs over all subgroups of G of finite index.

(4) Let $F : \mathbf{set}^G \rightarrow \mathbf{set}$ be the forgetful functor.

Then the following identity holds:

$$F(t) = 1 + \sum_{n=1}^{\infty} \frac{|\text{Hom}(G, S_n)|}{n!} t^n \\ = \exp \left(\sum_{H \leq_f G} \frac{t^{(G:H)}}{(G:H)} \right)$$

(1) follows from the unique decomposition of any finite G -set into the disjoint union of its orbits. (3) follows from the fact that a transitive G -set is G -isomorphic to a homogeneous G -set of the form G/H and that (i) $G/H \cong_G G/K$ iff H and K are G -conjugate; (ii) $\text{Aut}(G/H) \cong WH := N_G(H)/H$; (iii) the number of subgroups of G conjugate to H is equal to $(G : N_G(H))$. (4) follows from the existence of a bijection:

$$\mathbf{Str}(\mathbf{set}^G/[n]) / \cong \longleftrightarrow \text{Hom}(G, S_n)$$

remark. The identity in (4) is first published by Wohlfahrt (1977).

Example 2.2 Let $C = \langle \alpha \rangle$ be an infinite cyclic group. For $n \geq 1$, we put $C^n := \langle \alpha^n \rangle \leq C$ and $C(n) := C/C^n$. Then a finite C -set, that is, a finite dynamical system, is uniquely decomposed into a disjoint union of some transitive (connected) C -sets. Thus $\mathbf{set}^C \approx \mathbf{Exp}(\mathbf{trans}^C)$ and so

$$\mathbf{set}^C(t) = \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} t^{C(n)} \right)$$

For any finite C -set X , the substitution $t^N \leftarrow |\mathrm{Hom}_C(N, X)|u^{|N|}$ gives

$$\sum'_{N \in \mathbf{set}^C} \frac{|\mathrm{Hom}(N, X)|}{|\mathrm{Aut}(N)|} u^{|N|} = \exp \left(\sum_{n=1}^{\infty} \frac{|\mathrm{Fix}_X(\alpha^n)|}{n} u^n \right)$$

where the right hand side is the **Artin-Mazur zeta function** of X .

Furthermore, the UZF of the Hom-functor

$$\mathrm{Hom}(C(l), -) : X \mapsto \mathrm{Hom}(C(l), X) \cong \mathrm{Fix}_X(\alpha^l)$$

is the generating function for the numbers of finite C -sets in which α^l fixes exactly l -points.

$$\exp \left(\sum_{n|l} t^n \right)$$

Refer to [DS89]. ■

2.5 Theory of species

There is another categorical theory of generating functions introduced and developed by Joyal([Jo81]).

Definition 2.3 Let $\mathbf{bij}$ be the cat of finite sets and bijections and let S_n be the symmetric group of degree n . Then a (set valued) **species** is a functor $\mathbf{bij} \rightarrow \mathbf{set}$. Thus a species $\mathbf{A}$ is nothing but a series $(\mathbf{A}[n])_{n=0,1,\dots}$ of finite S_n -sets.

The **generating function (series)** of a species $\mathbf{A}$ is

$$\mathbf{A}(t) := \sum_{n=0}^{\infty} |\mathbf{A}[n]| \frac{t^n}{n!}$$

■

Combinatorially, $\mathbf{A}[I]$ means "the set of $\mathbf{A}$ -structures on a finite set I ".

As in the case of Set-valued functors, species also have arithmetical operations, for example, the derivation of $\mathbf{A}$ is defined by

$$\mathbf{A}'[I] := \mathbf{A}[I \cup \{I\}]$$

Then $\mathbf{A}'(t)$ is the derivation of $\mathbf{A}(t)$.

The theory of species is included in those of faithful functors with finite fibers. In fact, given a species $\mathbf{A} : \mathbf{bij} \rightarrow \mathbf{set}$,

$$\mathbf{A} : \mathbf{Elts}(\mathbf{A}) \xrightarrow{\pi} \mathbf{bij} \subset \mathbf{set}; (I, i) \mapsto I$$

is a faithful functor with finite fibers and with the same generating functions $\mathbf{A}(t) = \mathbf{A}'(t)$. Note that $\mathbf{Elts}(\mathbf{A})$ is a groupoid, that is, a category in which all morphisms are isomorphisms. Conversely, given a faithful functor $F : \mathcal{C} \rightarrow \mathbf{set}$ with finite fibers,

$$F : \mathbf{bij} \rightarrow \mathbf{set}; N \mapsto \mathbf{Str}(\mathcal{C}/N)/\cong$$

is a species.

Theorem 2.4 The notion of species is equivalent to those of faithful functors from a groupoid to $\mathbf{set}$ with finite fibers.

Problem. Rewrite the theory of species by using the notion of faithful functors with finite fibers.

3 Abstract Burnside rings (ABR)

References: Yoshida [Yo87], [Yo90]

3.1 Burnside homomorphisms

Let Γ be an essentially finite and locally finite category. $\mathrm{Obj}(\Gamma)/\cong$ or simply $\Gamma/\cong$ denote the finite set of isomorphism classes of objects; $[X]$ or often X denotes the isomorphism class of an object $X \in \Gamma$. Define two abelian groups as follows:

$$\begin{aligned} \Omega(\Gamma) &:= \mathbb{Z}\Gamma := \text{free abelian group on } \Gamma/\cong, \\ \tilde{\Omega}(\Gamma) &:= \mathbb{Z}^{\Gamma} := \mathrm{Map}(\Gamma/\cong, \mathbb{Z}) \cong \prod'_{I \in \Gamma} \mathbb{Z}, \end{aligned}$$

where the product $\prod'$ is taken over isomorphism classes of objects of Γ . The product ring $\mathbb{Z}^{\Gamma}$ (often wrote as $\mathrm{gh}(\Gamma)$) is called the **ghost ring**.

The linear map

$$\varphi = (\varphi_I) : \mathbb{Z}\Gamma \rightarrow \mathbb{Z}^\Gamma; [X] \mapsto (|\text{Hom}(I, X)|)_{I \in \Gamma/\cong}$$

is called the **Burnside homomorphism**, whose representation matrix is the **Hom-set matrix**:

$$H := (|\Gamma(I, J)|)_{I, J \in \Gamma/\cong}.$$

Definition 3.1 $\mathbb{Z}\Gamma$ ($= \Omega(\Gamma)$) is called an **abstract Burnside ring** if $\mathbb{Z}\Gamma$ has a ring structure with 1 and if φ is an injective ring homomorphisms. The abstract Burnside rings with other coefficient rings, for example $\mathbb{Q}, \mathbb{Z}_{(p)}$, etc. can be similarly defined.

Example 3.1 Let $\Gamma := (\text{set}_{\leq n})^{\text{op}}$ be the dual category of the category of finite sets of size at most n . We put $[i] := \{1, 2, \dots, in\}$ and $[0] := \emptyset$.

$$\varphi : \Omega(\Gamma) \rightarrow \tilde{\Omega}(\Gamma); \sum_{i=0}^n a_i [i] \mapsto \left(\sum_{i=0}^n a_i x^i \right)_{0 \leq x \leq n}$$

Thus $\Omega(\Gamma)$ is the module of integral polynomials of degree $\leq n$ and φ is the evaluation map $f(X) \mapsto (f(x))_{0 \leq x \leq n}$

$$\Omega(\Gamma) \cong \mathbb{Z}[X]/(X(X-1) \cdots (X-n)).$$

Example 3.2 Let $\Gamma := \text{set}_{\leq n}^*$ be the category of nonempty sets of size at most n .

$$\varphi : \Omega(\Gamma) \rightarrow \tilde{\Omega}(\Gamma); \sum_{i=1}^n a_i [i] \mapsto \left(\sum_{i=0}^n a_i i^x \right)_{1 \leq x \leq n}$$

Thus $\Omega(\Gamma)$ is the "ring" of finite Dirichlet polynomials of "degree" $\leq n$.

3.2 Möbius rings

Let P be a finite poset, which can be viewed as a finite category such that for any $x, y \in P$, there exists at most one morphism from x to y . Thus the hom-set matrix is a $P \times P$ -matrix $H = (\zeta(x, y))_{x, y \in P}$, where

$$\zeta(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{else} \end{cases}$$

As is well-known, H is invertible, and so

$$\varphi : \mathbb{Z}P \xrightarrow{\cong} \mathbb{Z}^P; [x] \mapsto (\zeta(i, x))_i$$

is isomorphic. Thus $\mathbb{Z}P$ becomes an abstract Burnside ring, which is called a **Möbius ring**.

The inverse matrix of H is presented by the Möbius function:

$$H^{-1} = (\mu(x, y))_{x, y \in P}.$$

Thus we have an inversion formula and an idempotent formula:

$$\varphi^{-1} : \mathbb{Z}^P \rightarrow \mathbb{Z}P; (\chi(i))_i \mapsto \sum_{x, j \in P} \mu(x, j) \chi(j) [x],$$

$$e_t := \sum_{x \in P} \mu(x, t) [x].$$

3.3 Fundamental Theorem for ABR

We assume that two conditions for Γ hold:

- (E) All the morphisms of Γ are epimorphic.
- (C) For any object I and $\sigma \in \text{Aut}(I)$, there exists a coequalizer diagram:

$$I \xrightarrow[\sigma]{1} I \xrightarrow{c_\sigma} I/\sigma$$

Definition 3.2 Define an abelian group and homomorphism

$$\text{Obs}(\Gamma) := \prod'_{I \in \Gamma} (\mathbb{Z}/|\text{Aut}(I)|\mathbb{Z})$$

$$\psi : (\chi : \Gamma \rightarrow \mathbb{Z}) \mapsto \left(\sum_{\sigma \in \text{Aut}(I)} \chi(I/\sigma) \bmod |\text{Aut}(I)| \right)_I$$

$\text{Obs}(\Gamma)$ is called the **group of obstructions** and ψ is called the **Cauchy-Frobenius map**.

Theorem 3.1 The following sequence is exact:

$$0 \rightarrow \mathbb{Z}\Gamma \xrightarrow{\varphi} \mathbb{Z}^\Gamma \xrightarrow{\psi} \text{Obs}(\Gamma) \rightarrow 0.$$

Theorem 3.2 $\mathbb{Z}\Gamma$ is an abstract Burnside ring.

Remark. (1) The condition F can be replaced by (F) the existence of the unique (E, M) -factorization system such that $E \subset \text{Epi}(\Gamma)$. But then $\text{ABR } \mathbb{Z}\Gamma$ is ring isomorphic to another $\text{ABR } \mathbb{Z}\Gamma_e$, where Γ_e is the subcategory of Γ consisting of all epimorphisms of Γ . Thus we may assume that (E) holds at first.

(2) $\mathbb{Q}\Gamma$ is always an ABR isomorphic to $\mathbb{Q}^\Gamma$ via φ under the condition (F) without C .

(3) For a prime p , $\mathbb{Z}_{(p)}\Gamma$ is an ABR under the condition (F) and the following condition

(C_p) For any $I \in \Gamma$ and any p -element $\sigma \in \text{Aut}(I)$, there exists a coequalizer of $1, \sigma$ similarly as (C).

(4) We may assume that Γ is skeletal, i.e., $X \cong Y \Rightarrow X = Y$.

Let $H := (|\text{Hom}(I, J)|)_{[I], [J]}$ the Hom-set matrix of Γ . Then the inversion formula and the idempotent formula are given by

$$\begin{aligned} \varphi^{-1} : \mathbb{Q}^\Gamma &\rightarrow \mathbb{Q}\Gamma; \theta \mapsto \sum'_{I \in \Gamma} H_{IK}^{-1} \theta(K) [I] \\ e_K &:= \sum'_{I \in \Gamma} H_{IK}^{-1} [I] \end{aligned}$$

We need to calculate the inverse matrix H^{-1} to obtain an explicit idempotent formula.

Example 3.3 Let G be a finite group. The **Burnside ring** $\Omega(G)$ of G is the Grothendieck ring of set^G . It is canonically isomorphic to the $\text{ABR } \mathbb{Z}\text{trans}^G$. The Burnside homomorphism is defined by

$$\begin{aligned} \varphi : \Omega(G) &\rightarrow \tilde{\Omega}(G) := \prod_{(S) \in C(G)} \mathbb{Z} \\ ; [X] &\mapsto (|X^S|)_{(S)} \end{aligned}$$

Note that there is a bijection

$$X^S := \text{Fix}_S(X) \leftrightarrow \text{Map}_G(G/S, X); x_0 \mapsto (gS \mapsto x_0)$$

The primitive idempotent of $\mathbb{Q}\Omega(G)$ associated to

$H \leq G$ is give by

$$e_H = \frac{1}{|N_G(H)|} \sum_{D \leq H} |D| \mu(D, H) [G/D],$$

where μ is the Möbius function of the subgroup lattice of G .

3.4 Discrete cofibration (DCF)

In order to obtain the inverse matrix H^{-1} of the Hom-set matrix $H = (|\text{Hom}(I, J)|)_{I, J \in \Gamma/\cong}$, we have to construct a poset like the subgroup lattice.

We may assume that all the morphisms of Γ are epimorphic. In this case, H is decomposed as $H = LD$, and so $H^{-1} = D^{-1}L^{-1}$, where

$$\begin{aligned} L &= (|\text{Hom}(I, J)|/|\text{Aut}(J)|)_{I, J \in \Gamma}, \\ D &:= (|\text{Aut}(I)|\delta(I, J)) = \begin{cases} |\text{Aut}(I)| & \text{if } I \cong J \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

$L_{I, J}$ is equal to the number of quotient objects of I isomorphic to J . When Γ is the category of set of size at most n , the number $L(I, J) = S(|I|, |J|)$ is the Stirling number of second kind and $L^{-1}(I, J) = s(|I|, |J|)$ is the Stirling number of first kind.

Now, in the case of trans^G , the subgroup lattice is categorically constructed as follows:

$$\begin{aligned} \text{Sub}(G) &\approx \mathbf{Elts}(\text{trans}^G, \text{Hom}(G/1, -)) \\ &\cong (G/1) \backslash \text{trans}^G. \end{aligned}$$

Thus if the category Γ has a "generator" like $G/1$ using the notion of categories of elements (or coslice categories), we can construct a poset we need.

Definition 3.3 A functor $\pi : \tilde{\Gamma} \rightarrow \Gamma$ is called a **discrete cofibration (DCF)** if

$$\begin{array}{ccc} \text{Mor}(\tilde{\Gamma}) & \xrightarrow{\text{dom}} & \text{Obj}(\tilde{\Gamma}) \\ \pi \downarrow & & \downarrow \pi \\ \text{Mor}(\Gamma) & \xrightarrow{\text{dom}} & \text{Obj}(\Gamma) \end{array}$$

is a fibre product diagram. See [Yo87]. More precisely, this means that for any $\tilde{X} \in \tilde{\Gamma}$, π induces an equivalence between slice categories:

$$\begin{aligned} \tilde{X} \backslash \pi : \tilde{X} \backslash \tilde{\Gamma} &\xrightarrow{\cong} \pi(\tilde{X}) \backslash \Gamma \\ ; (\tilde{X} \rightarrow \tilde{Y}) &\mapsto (\pi(\tilde{X}) \rightarrow \pi(\tilde{Y})) \end{aligned}$$

Note (1) DCF $\pi : \tilde{\Gamma} \rightarrow \Gamma$ is faithful.

(2) Any functor which has a right adjoint is a DCF.

Example 3.4 Let G be a finite group. Let $\mathbf{trans}^G$ be the cat of transitive G -sets and $\mathbf{Sub}(G)$ the subgroup lattice viewed as a category. Then

$$\pi : \mathbf{Sub}(G) \rightarrow \mathbf{trans}^G; I \mapsto G/I$$

is a DCF. The bijection

$$I \backslash \pi : I \backslash \mathbf{Sub}(G) \xrightarrow{\cong} (G/I) \backslash \mathbf{trans}^G$$

is given by

$$\begin{aligned} K(\supset I) &\mapsto (G/I \rightarrow G/K; gI \mapsto gK), \\ (G/I \xrightarrow{\alpha} X) &\mapsto G_{\alpha(I)}(\supset I), \end{aligned}$$

where $G_{\alpha(I)}$ is the stabilizer of $\alpha(I) \in X$.

Let $\mathbf{sub}(G)$ be the subgroup category, which is equivalent to $\mathbf{trans}^G$ by $I \mapsto G/I$. Then $\mathbf{Sub}(G) \rightarrow \mathbf{sub}(G); I \mapsto I$ gives a DCF.

Note that the inverse matrix of the Hom-set matrix $\tilde{H}$ of $\mathbf{Sub}(G)$ is given by the Möbius function:

$$\tilde{H}^{-1} = (\mu(I, J))_{I, J \leq G}$$

3.5 The inverse of the Hom-set matrix

We continue assuming that the morphisms of Γ are all epimorphic. We consider the following conditions for a discrete cofibration $\pi : \tilde{\Gamma} \rightarrow \Gamma$:

- (S) $\pi : \tilde{\Gamma}/\cong \rightarrow \Gamma/\cong$ is surjective on objects.
- (P) $\tilde{\Gamma}/\cong$ is a poset, i.e., $|\mathrm{Hom}(\tilde{X}, \tilde{Y})| \leq 1$ for any $\tilde{X}, \tilde{Y} \in \tilde{\Gamma}$.

For any $G \in \Gamma$, let $G \backslash \Gamma$ be the coslice category, which is equivalent to $\mathbf{Elts}(\mathrm{Hom}_{\Gamma}(G, -))$.

Example 3.5 (1) For any $G \in \Gamma$,

$$\pi_G : G \backslash \Gamma \rightarrow \Gamma; (G \xrightarrow{\alpha} X) \mapsto X$$

is a DCF satisfying (P). It satisfies (S) if and only if any $X \in \Gamma$ has a morphism from G . Such a G exists uniquely up to isomorphism if it exists.

(2) Let $\mathbf{G}$ be a set of objects of Γ such that any $X \in \Gamma$ has a morphisms from some $G \in \mathbf{G}$. Then

$$\pi_{\mathbf{G}} := \coprod_{G \in \mathbf{G}} \pi_G : \mathbf{G} \backslash \Gamma := \coprod_{G \in \mathbf{G}} G \backslash \Gamma \rightarrow \Gamma$$

is a DCF satisfying (S) and (P).

(3) For finite group G , $\pi : \mathbf{Sub}(G) \rightarrow \mathbf{trans}^G; I \mapsto G/I$ and $\pi \mathbf{Sub}(G) \rightarrow \mathbf{sub}(G); I \mapsto I$ are both DCF satisfying (S) and (P).

Let $\pi : \tilde{\Gamma} \rightarrow \Gamma$ be a DCF satisfying (S) and (P). Let μ be the Möbius function of the poset $\tilde{\Gamma}/\cong$, which value at the isomorphism classes $[\tilde{I}], [\tilde{J}]$ is simply wrote as $\mu(\tilde{I}, \tilde{J})$. For any $I \in \Gamma$, we define

$$\begin{aligned} N_I &:= \#\{[\tilde{I}] \in \tilde{\Gamma}/\cong \mid \pi(\tilde{I}) \cong I\}, \\ \mathrm{ind}(I) &:= N_I |\mathrm{Aut}(I)| \end{aligned}$$

Example 3.6 When

$$\pi : \tilde{\Gamma} = \mathbf{Sub}(G) \rightarrow \Gamma = \mathbf{sub}(G); I \mapsto I,$$

we have

$$\begin{aligned} N_I &= \#\{\tilde{I} \leq G \mid \tilde{I} \sim_G I\} = (G : N_G(I)), \\ \mathrm{Aut}(I) &\cong N_G(I)/I, \end{aligned}$$

and so $\mathrm{ind}(I) = (G : I)$. ■

Theorem 3.3 The inverse of the Hom-set matrix $\tilde{H} := (|\mathrm{Hom}(I, J)|)_{I, J \in \Gamma/\cong}$ of Γ is given by

$$H_{IJ}^{-1} = \frac{1}{\mathrm{ind}(I)} \sum'_{\pi(\tilde{I}) \cong I} \sum'_{\pi(\tilde{J}) \cong J} \mu(\tilde{I}, \tilde{J})$$

Theorem 3.4 The primitive idempotent associated to $J \in \Gamma$ is given by

$$e_J = \sum'_{\tilde{I}} \frac{1}{\mathrm{ind}(\pi(\tilde{I}))} \sum'_{\pi(\tilde{J}) \cong J} \mu(\tilde{I}, \tilde{J}) [\pi(\tilde{I})]$$

Theorem 3.5 Let $\theta \in \mathbb{Q}^\Gamma$. Then

$$\varphi^{-1}(\theta) = \sum_{\tilde{I}, \tilde{J}}' \frac{\mu(\tilde{I}, \tilde{J}) \theta(\pi(\tilde{J}))}{\text{ind}(\pi(\tilde{I}))} [\pi(\tilde{I})] \in \mathbb{Q}\Gamma$$

4 Abstract monomial Burnside rings

Refer to [Dr71], [Sn88], [Sn94], [Ta10].

4.1 Definition of AMBR

Definition 4.1 As before, let Γ denote an essentially finite and locally finite category. Let

$$\wedge : \Gamma^{\text{op}} \rightarrow \mathbf{mon}; I \mapsto \hat{I}$$

be a functor to the category of finite monoids. Thus an $f : I \rightarrow J$ induces a monoid homomorphism $\hat{f} : \hat{J} \rightarrow \hat{I}$, which we often extend to a ring homomorphism $\hat{f} : \mathbb{Z}[\hat{J}] \rightarrow \mathbb{Z}[\hat{I}]$ between monoid rings. In particular, $\hat{I}$ is a right $\text{Aut}(I)$ -set, and so $\text{Aut}(I)$ acts the monoid algebra $\mathbb{Z}[\hat{I}]$. We can consider the centralizer algebra $\mathbb{Z}[\hat{I}]^{\text{Aut}(I)}$ under this action. Then the **monomial ghost ring** is defined as the product algebra

$$\tilde{\Omega}(\Gamma, \wedge) := \prod_{I \in \Gamma/\cong} \mathbb{Z}[\hat{I}]^{\text{Aut}(I)}$$

Let $\Omega(\Gamma, \wedge) := \mathbb{Z}[\text{Elts}(\Gamma, \wedge)]$ be the free abelian group generated by $\text{Elts}(\Gamma, \wedge)/\cong$. ■

Definition 4.2 The **monomial Burnside homomorphism** is the linear map defined by

$$\varphi : \Omega(\Gamma, \wedge) \rightarrow \tilde{\Omega}(\Gamma, \wedge); [X, x] \mapsto \left(\sum_{f: I \rightarrow X} \hat{f}(x) \right)_I$$

$\Omega(\Gamma, \wedge)$ is called an **abstract monomial Burnside ring** (AMBR) if

- (a) $\Omega(\Gamma, \wedge)$ has a ring structure, and
- (b) φ is an injective ring homomorphism. ■

Example 4.1 (1) Let G be a finite group. and $\Gamma = \mathbf{sub}(G)$, the subgroup category, and $\wedge : H \mapsto \hat{H} := \text{Hom}(H, \mathbb{C}^*)$. the linear character functor.

Then as the AMBR, we have a classical monomial Burnside ring $\Omega(G, \wedge)$ which is an abelian group generated by the symbols $[H, \lambda]$, where $H \leq G$ and $\lambda \in \hat{H}$, a linear character, and with relation $[H^g, \lambda^g] = [H, \lambda]$. The multiplication is defined by

$$[H, \lambda] \cdot [K, \mu] = \sum_{HgK} [H^g \cap K, \lambda^g \mu_{H^g \cap K}]$$

There is a ring homomorphism into the character ring:

$$\Omega(G, \wedge) \rightarrow R(G); [H, \lambda] \mapsto \text{ind}^G(\lambda)$$

(2) Let G be a finite group and S a monoid with right G -action. Take the centralizer functor $C_S : \mathbf{sub}(G) \rightarrow \mathbf{mon}; H \mapsto C_S(H)$. Then the AMBR $\Omega(\mathbf{sub}(G), C_S)$ is the crossed Burnside ring $\Omega(G, S)$. In general, this ring is not commutative, but when $S = G^c$, the group G with G -action by G -conjugation, $\Omega(G, G^c)$ is commutative.

(3) Let A be a finite abelian group with G -action. Then $\Omega(\mathbf{sub}(G), H^1(-, A)) = \Omega(G, A)$ is the Dress monomial BR.

4.2 The fundamental theorems for AMBR

As before, we assume that Γ satisfies the following two conditions:

- (E) All the morphisms of Γ are epimorphic.
- (C) For any object I and $\sigma \in \text{Aut}(I)$, there exists a coequalizer diagram:

$$I \xrightarrow[\sigma]{1} I \xrightarrow{c_\sigma} I/\sigma$$

By (C), we have a monoid homomorphism

$$\hat{c}_\sigma : \widehat{I/\sigma} \rightarrow \hat{I}^{(\sigma)} := \{i \in \hat{I} \mid \hat{\sigma}(i) = i\}$$

Furthermore, if $f : I \rightarrow X$ satisfies $f \circ \sigma = f$, then there exists a unique $g : I/\sigma \rightarrow X$ such that $g \circ c_\sigma = f$, and so $\hat{c}_\sigma \circ \hat{g} = \hat{f}$.

By (C), the coequalizer $c_\sigma : I \rightarrow I/\sigma$ of $1, \sigma \in \text{Aut}(I)$ induces a monoid homomorphism $\widehat{c}_\sigma : \widehat{I/\sigma} \rightarrow \widehat{I^{(\sigma)}} \hookrightarrow \widehat{I}$, which furthermore induces

$$\widehat{c}_\sigma : \mathbb{Z}[\widehat{I/\sigma}] \rightarrow \mathbb{Z}[\widehat{I^{(\sigma)}}] \rightarrow \mathbb{Z}[\widehat{I}]$$

Define the **group of obstructions** by

$$\text{Obs}(\Gamma, \wedge) := \prod_{I \in \Gamma/\cong} ((\mathbb{Z}/|\text{Aut}(I)|)[\widehat{I}])^{\text{Aut}(I)}$$

and define a module endmorphism $\tilde{\psi} = (\tilde{\psi}_I)$ of $\tilde{\Omega}(\Gamma, \wedge)$ by

$$\tilde{\psi}_I(\theta) := \sum_{\sigma \in \text{Aut}(I)} \widehat{c}_\sigma \theta(I/\sigma).$$

Finally define the **Cauchy-Frobenius map** by

$$\psi : \tilde{\Omega}(\Gamma, \wedge) \xrightarrow{\tilde{\psi}} \tilde{\Omega}(\Gamma, \wedge) \xrightarrow{\text{pr}} \text{Obs}(\Gamma, \wedge).$$

Theorem 4.1 The following is an exact sequence of modules:

$$0 \rightarrow \Omega(\Gamma, \wedge) \xrightarrow{\varphi} \tilde{\Omega}(\Gamma, \wedge) \xrightarrow{\psi} \text{Obs}(\Gamma, \wedge) \rightarrow 0$$

Theorem 4.2 $\Omega(\Gamma, \wedge)$ is an AMBR.

4.3 Monomial G -sets

It is often more convenient to use the notion of monomial G -set than of $\mathbf{sub}(G)$. The category of monomial G -sets is equivalent to $\mathbf{Exp}(\mathbf{Elts}(\mathbf{sub}(G)))$.

Then any functor $\wedge : \mathbf{sub}(G)^{\text{op}} \rightarrow \mathbf{mon}$ can be extend to $\mathbf{set}^G$. In fact, the monoid $\widehat{X}$ for any G -set X is defined by the set of X -indexed family $(\lambda_x)_{x \in X}$ such that $\lambda_x \in \widehat{G_x}$ and $\lambda_{gx} \cdot^g \lambda_x$ for any $x \in X$ and $g \in G$.

Then the AMBR $\Omega(\mathbf{sub}(G), \wedge)$ is isomorphic to the Grothendieck ring of monomial G -sets with respect to disjoint union and multiplication defined by

$$(X, (\lambda_x)) \otimes (Y, (\mu_y)) = (X \times Y, (\lambda_x \downarrow_{G_{xy}} \cdot \mu_y \downarrow_{G_{xy}})_{(x,y)})$$

In this notation, the monomial Burnside homomorphism $\varphi = (\varphi_I)$ ($I \leq G$) is given by

$$\varphi_I : [X, (\lambda_x)] \mapsto \sum_{x \in X^I} \lambda_x|_I \in (\mathbb{Z}[\widehat{I}])^{N_G(I)}$$

4.4 Idempotent formula

Theorem 4.3 (Takegahara) The primitive idempotent of the complex coefficient MBR $\mathbb{C}\Omega(G, \wedge)$ associated to (H, t) is given by

$$\begin{aligned} e_{H,t} &= \frac{1}{|N_G(H)| \cdot |H|} \sum_{D \leq H} \sum_{\lambda \in \widehat{H}} |D| \mu(D, H) \overline{\lambda(t)} [D, \lambda|_D] \\ &= e_t \otimes e_H, \end{aligned}$$

where

$$e_t := \frac{1}{|H|} \sum_{\lambda \in \widehat{H}} \overline{\lambda(t)} \lambda$$

is the primitive idempotent of the complex coefficient character ring $\mathbb{C}R(H)$ associated to $t \in H$, and

$$e_H := \frac{1}{|N_G(H)|} \sum_{D \leq H} |D| \mu(D, H) [D],$$

is the primitive idempotent of the Burnside ring $\mathbb{C} \otimes \Omega(G)$. Furthermore, we used the notation $\lambda \otimes [D] := [D, \lambda|_D]$.

Corollary 4.4 (Snaith, Boltje) Explicit Brauer induction theorem!

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