

Exact Multiplicity of Rapidly Decaying Solutions for a Semilinear Elliptic Equation with a Critical Exponent

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1 Introduction

This article is based a joint work with Noriyuki Murai (Tohoku University). We consider radially symmetric solutions of the equation

$$\Delta u + K(|x|)u^p = 0 \quad \text{in } \mathbb{R}^n, \quad (1.1)$$

where $n > 2$, $p > 1$. $K \geq 0$ and $K \in C^1([0, \infty))$, Any radially symmetric solution $u = u(r)$, $r = |x|$, of (1.1) satisfies

$$\begin{cases} u_{rr} + \frac{n-1}{r}u_r + K(r)u^p = 0, & r > 0, \\ u(0) = \alpha > 0, \quad u_r(0) = 0. \end{cases} \quad (1.2)$$

We denote by $u(r; \alpha)$ the unique solution of this initial value problem. According to [7], we can classify the solutions of (1.2) as follows:

- Rapidly decaying solution: $u(r) > 0$ for all $r > 0$ and $r^{n-2}u \rightarrow \beta \in (0, \infty)$ as $r \rightarrow \infty$.
- Slowly decaying solution: $u(r) > 0$ for all $r > 0$ and $r^{n-2}u \rightarrow \infty$ as $r \rightarrow \infty$.
- Crossing solution: $u(z) = 0$ at some $z \in (0, \infty)$.

In what follows, we consider the critical case in the Sobolev sense:

$$p = \frac{n+2}{n-2}.$$

We note that the critical case is related to the Yamabe problem in differential geometry [9]. In the context of the Yamabe problem, any rapidly decaying solution corresponds to the complete metric, while the slowly decaying solution corresponds to the incomplete metric.

Here we collect known facts about the existence of rapidly decaying solutions of (1.2) in the case of $p = \frac{n+2}{n-2}$. First, it is easy to show that if $K \equiv 1$, then

$$u = \varphi(r; \alpha) := \alpha \left\{ 1 + \frac{\alpha^{4/(n-2)}}{n(n-2)} r^2 \right\}^{-(n-2)/2}$$

satisfies (1.2). We note that the solution satisfies

$$r^{n-2} \varphi(r; \alpha) \rightarrow \{n(n-2)\}^{(n-2)/n} \alpha^{-1} \quad (r \rightarrow \infty),$$

so that $u = \varphi(r; \alpha)$ is a rapidly decaying solution for any $\alpha \in (0, \infty)$. Ding-Ni [3] proved that if $K(r)$ is monotone and nonconstant, then there is no rapidly decaying solution. More precisely, they proved the following.

- If $K(r)$ is non-constant and non-increasing in r , then $u(r; \alpha)$ is a slowly decaying solution for all $\alpha > 0$.
- If $K(r)$ is non-constant and non-decreasing in r , then $u(r; \alpha)$ is a crossing solution for all $\alpha > 0$.

When $K(r)$ is not monotone, Bianchi-Egnell [1] showed the existence of a rapidly decaying solution by assuming that K satisfies $K(0) \leq K(\infty)$ and some other asymptotic conditions at $r = 0$ and $r = \infty$. Also, Sasahara-Tanaka [8] studied the case where $K(0) = K(\infty)$ and K has a minimum, and proved that there exists at least one rapidly decaying solution. See also Yanagida-Yotsutani [11] for a sufficient condition on the existence of a rapidly decaying solution.

Concerning the uniqueness, Yanagida-Yotsutani [10] proved that if $K(r)$ is non-constant, non-decreasing in $(0, a)$, non-increasing in $r \in (a, \infty)$, and $K(0) = K(\infty)$, then there exists a unique rapidly decaying solution. In fact, there exists a unique $\alpha^* \in (0, \infty)$ such that

- $u(r; \alpha)$ is a slowly decaying solution for every $\alpha \in (0, \alpha^*)$.
- $u(r; \alpha^*)$ is a rapidly decaying solution.
- $u(r; \alpha)$ is a slowly decaying solution for every $\alpha \in (\alpha^*, \infty)$.

On the other hand, concerning the multiple existence of rapidly decaying solutions, it was shown numerically by Morishita-Yanagida-Yotsutani [6] that for some K , there may exist multiple rapidly decaying solutions. Kabeya [4] obtained a condition on K such that there exist at least two rapidly decaying solutions. Finally, Chen-Lin [2] considered the case $n \geq 7$ and found that for some K , there exists infinitely many rapidly decaying solutions.

The aim of this article is to obtain a condition on K for the exact multiplicity of rapidly decaying solutions of (1.2). The following theorem is our main result.

Theorem 1. *Let $n > 2$ and $p = \frac{n+2}{n-2}$. Assume that*

- (i) $K(r) = 1 + \varepsilon k(r)$, where $k(r) \equiv 0$ for $r \in [0, a]$ and $k(r) \equiv \text{Const.}$ for $r \in [b, \infty]$ with some $0 < a < b < \infty$, and $\varepsilon > 0$ is a parameter, and
- (ii) the function

$$g(\alpha) := \int_0^\infty r^n \varphi(r; \alpha)^{\frac{2n}{n-2}} k_r(r) dr$$

has exactly m simple zeros in $(0, \infty)$.

If $\varepsilon > 0$ is sufficiently small, then the problem (1.2) has exactly m rapidly decaying solutions.

We also have the following result.

Theorem 2. *Let $n > 2$ and $p = \frac{n+2}{n-2}$. Then for any $m \in \mathbf{N}$, there exists $K = K_m(r)$ such that the problem (1.2) has exactly m rapidly decaying solutions.*

2 Preliminaries

In this section we describe some preliminary results about the problem (1.2). Hereafter we always assume that $n > 2$ and $p = \frac{n+2}{n-2}$.

First we introduce the Pohozaev identity, which is obtained by direct computations and (1.2).

Lemma 1. *Define*

$$P[r; u] := \frac{1}{2} r^{n-1} u_r (ru_r + (n-2)u) + \frac{n-2}{2n} r^n K(r) u^{p+1}.$$

Then

$$\frac{d}{dr} P[r; u] \equiv \frac{n-2}{2} r^n K_r(r) u^{p+1}.$$

In particular, this identity implies that if $K(r) \equiv \text{Const.}$ for $r \in [b, \infty)$, then $P[r; u]$ is also constant for $r \in [b, \infty)$.

The following characterization of solutions of (1.2) is proved by Kawano-Yanagida-Yotsutani [5].

Lemma 2. *Suppose that $K(r) \equiv \text{Const.}$ for $r \in [b, \infty)$.*

- (i) *If $P[b; u] > 0$, then $u(r; \alpha)$ is a crossing solution.*
- (ii) *If $P[b; u] = 0$, then $u(r; \alpha)$ is a rapidly decaying solution.*
- (iii) *If $P[b; u] < 0$, then $u(r; \alpha)$ is a slowly decaying solution.*

Using this lemma, we will identify the type of $u(r; \alpha)$ for small $\alpha > 0$, large α and intermediate α as follows.

Lemma 3.

$$\frac{u(r; \alpha)}{\alpha} \rightarrow 1 \quad \text{and} \quad \frac{u_r(r; \alpha)}{\alpha} \rightarrow 0 \quad (\alpha \rightarrow 0)$$

uniformly in $r \in [0, b]$.

Proof. Setting $v(r) = \alpha^{-1}u(r; \alpha)$, we have

$$\begin{cases} v_{rr} + \frac{n-1}{r}v_r + \alpha^{p-1}K(r)v^p = 0, & r > 0, \\ v(0) = 1 > 0, & v_r(0) = 0 \end{cases}$$

Hence $v = \alpha^{-1}u \rightarrow 1$ and $v_r = \alpha^{-1}u_r \rightarrow 0$ as $\alpha \rightarrow 0$ uniformly in $r \in [0, b]$. ■

Lemma 4.

$$\frac{P[b; u]}{\alpha^{p+1}} \rightarrow \frac{n-2}{2} \int_0^b r^n K_r(r) dr \quad (\alpha \rightarrow 0).$$

Proof. By the Pohozev identity and Lemma 3, we have

$$\begin{aligned} \frac{P[b; u]}{\alpha^{p+1}} &= \frac{n-2}{2} \int_0^b r^n K_r(r) \left\{ \frac{u(r; \alpha)}{\alpha} \right\}^{p+1} dr \\ &\rightarrow \frac{n-2}{2} \int_0^b r^n K_r(r) dr \quad (\alpha \rightarrow 0) \end{aligned}$$

Thus for small α , $P[b; u]$ has the same sign as $\int_0^b r^n K_r(r) dr$. ■

Lemma 5. Suppose that $K(r) \equiv 1$ for $r \in [0, a]$. Then

$$\alpha u(r; \alpha) \rightarrow C_0(n)r^{2-n} \quad \text{and} \quad \alpha u_r(r; \alpha) \rightarrow -(n-2)C_0(n)r^{2-n}$$

as $\alpha \rightarrow \infty$ uniformly in $[a, b]$, where $C_0(n) := \{n(n-2)\}^{(n-2)/2} > 0$.

Proof. Setting $w(r) = \alpha u(r; \alpha)$, we have for $r \in [0, a]$

$$\begin{aligned} w(r) &= \alpha \varphi(r; \alpha) = \alpha^2 \left\{ 1 + \frac{\alpha^{4/(n-2)}}{n(n-2)} r^2 \right\}^{-(n-2)/2} \\ &= \left\{ \alpha^{-4/(n-2)} + \frac{1}{n(n-2)} r^2 \right\}^{-(n-2)/2} \\ &\rightarrow C_0(n)r^{2-n} \quad (\alpha \rightarrow \infty) \end{aligned}$$

uniformly $r \in [\delta, a]$. Similarly

$$\begin{aligned} w_r(r) &= \alpha \varphi_r(r; \alpha) = -\frac{1}{n} \alpha^{2n/(n-2)} r \left\{ 1 + \frac{\alpha^{4/(n-2)}}{n(n-2)} r^2 \right\}^{-n/2} \\ &\rightarrow -(n-2)C_0(n)r^{1-n} \quad (\alpha \rightarrow \infty) \end{aligned}$$

uniformly in $r \in [\delta, a]$. On the other hand, from (1.2) we see that w satisfies

$$\begin{cases} w_{rr} + \frac{n-1}{r} w_r + \alpha^{-4/(n-2)} K(r) w^p = 0, & r > 0, \\ w(a) \rightarrow C_0(n), \quad w_r(a) \rightarrow C_0(n)a^{2-n} & (\alpha \rightarrow \infty). \end{cases}$$

This implies that $w \rightarrow C_0(n)r^{2-n}$ uniformly in $r \in [a, b]$ as $\alpha \rightarrow \infty$. ■

As a consequence of this lemma, we have

Lemma 6.

$$\alpha^{p+1} P[b; u] \rightarrow \frac{n-2}{2} C_0(n)^{p+1} \int_0^a r^{-n} K_r(r) dr \quad \text{as } \alpha \rightarrow \infty.$$

Proof.

$$\begin{aligned} \alpha^{p+1} P[b; u] &= \frac{n-2}{2} \int_0^b r^n K_r(r) \{\alpha u(r; \alpha)\}^{p+1} dr \\ &\rightarrow \frac{n-2}{2} \int_a^b r^n K_r(r) \{C_0(n)r^{2-n}\}^{p+1} dr \quad (\alpha \rightarrow \infty) \\ &= \frac{n-2}{2} C_0(n)^{p+1} \int_0^b r^{-n} K_r(r) dr. \end{aligned}$$

Thus for large α , $P[b; u]$ has the same sign as $\int_0^b r^{-n} K_r(r) dr$. ■

We write K as $K(r) = 1 + \varepsilon k(r)$.

Lemma 7. *Let $0 < \alpha_1 < \alpha_2 < \infty$ be fixed. Then*

$$u(r; \alpha) \rightarrow \varphi(r; \alpha) \quad \text{and} \quad u_r(r; \alpha) \rightarrow \varphi_r(r; \alpha)$$

as $\varepsilon \rightarrow 0$ uniformly in $(r, \alpha) \in [0, b] \times [\alpha_1, \alpha_2]$.

Proof. Since

$$\begin{cases} u_{rr} + \frac{n-1}{r}u_r + \{1 + \varepsilon k(r)\}u^p = 0, & r > 0, \\ u(0) = \alpha > 0, & u'(0) = 0, \end{cases}$$

the proof is clear. $\blacksquare$

As a consequence of this lemma, we have

Lemma 8.

$$\frac{P[b : u]}{\varepsilon} \rightarrow \frac{n-2}{2} \int_0^b r^n k_r(r) \varphi(r; \alpha)^{p+1} dr.$$

as $\varepsilon \rightarrow 0$.

Proof. By the Pohozaev identity, we have

$$P[b : u] = \frac{n-2}{2} \int_0^b r^n \varepsilon k_r(r) u(r; \alpha)^{p+1} dr.$$

Since $u(r; \alpha) \rightarrow \varphi(r; \alpha)$ as $\varepsilon \rightarrow 0$, we obtain the conclusion. Thus for intermediate α and small $\varepsilon > 0$, $P[b : u]$ has the same sign as $\int_0^b r^n k_r(r) \varphi(r; \alpha)^{p+1} dr$. $\blacksquare$

3 Outline of proofs

Proof of Theorem 1.

Step 1: For small $\alpha \in (0, \alpha_1)$, we can identify the type of $u(r; \alpha)$ by examining the sign of

$$\int_0^b r^n k_r(r) dr \quad (\text{the same sign as } g(0)).$$

Step 2: For large $\alpha \in (\alpha_2, \infty)$, we can identify the type of $u(r; \alpha)$ by examining the sign of

$$\int_0^a r^{-n} k_r(r) dr \quad (\text{the same sign as } g(\infty)).$$

Step 3: Fix $0 < \alpha_1 < \alpha_2 < \infty$ as above. If we take $\varepsilon > 0$ sufficiently small, then we can identify the type of solutions for $\alpha \in (\alpha_1, \alpha_2)$ by examining the sign of

$$g(\alpha) := \int_0^\infty r^n \varphi(r; \alpha)^{p+1} k_r(r) dr$$

From these considerations and the simplicity of zeros of $g(\alpha)$, we can count the exact number of rapidly decaying solutions by counting the number of zeros of $g(\alpha)$.

In fact, if $\varepsilon > 0$ is sufficiently small, then the number of rapidly decaying solutions is the same as the number of zeros of $g(\alpha)$. ■

Proof of Theorem 2.

If we rewrite $g(\alpha)$ as

$$g(\alpha) := - \int_0^\infty \{r^n \varphi(r; \alpha)^{p+1}\}_r k(r) dr,$$

we can handle the case where $K(r) = 1 + \varepsilon k(r)$ is piece-wise constant (or $K_r = \varepsilon k_r(r)$ is a superposition of the delta functions). Then for every $m \in \mathbf{N}$, we may control the locations of discontinuous points and gaps to find a piece-wise constant $K(r) = 1 + \varepsilon k(r)$ such that $g(\alpha)$ has exactly m simple zeros.

In the last step, we approximate the piece-wise function by a smooth function. Then the number of zeros does not change because the zeros are simple. ■

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