

Primitivity of group rings of groups with non-trivial center

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In [2], we consider the following condition (*): for each subset M of G consisting of a finite number of elements not equal to 1, and for any positive integer m , there exist distinct a, b , and c in G so that if $(x_1^{-1}g_1x_1)\cdots(x_m^{-1}g_mx_m) = 1$, where g_i is in M and x_i is equal to a, b , or c for all i between 1 and m , then $x_i = x_{i+1}$ for some i . This condition is often satisfied by a non-noetherian group which has a non-abelian free subgroup and the trivial center. For a such group G , we have proved that the group ring RG of G over a domain R is primitive if G satisfies (*) and $|R| \leq |G|$. However as long as we deal with groups satisfying (*), since the center of them are always trivial, we can say nothing about primitivity of group rings of groups with nontrivial center. In this note, we consider a more general condition than the above one and give a primitivity result for group rings of groups with non-trivial center.

1 Introduction

Let G be a group and M a subset of G . We denote by $\widetilde{M}$ the symmetric closure of M ; $\widetilde{M} = M \cup \{x^{-1} \mid x \in M\}$. For non-empty subsets $M_1, M_2, \dots, M_n$ of G consisting of elements not equal to 1, we say that $M_1, M_2, \dots, M_n$ are mutually reduced in G if, for each finite number of elements $g_1, g_2, \dots, g_m \in \bigcup_{i=1}^n \widetilde{M}_i$, whenever $g_1g_2\cdots g_m = 1$, there exists $i \in [m]$ and $j \in [n]$ so that $g_i, g_{i-1} \in \widetilde{M}_j$, where $[n] := \{1, 2, \dots, n\}$ for any $n \in \mathbb{N}$. If $M_i = \{x_i\}$ for $i \in [n]$ and $M_1, M_2, \dots, M_n$ are mutually reduced, then we say that $x_1, x_2, \dots, x_n$ are mutually reduced. For a subset M of G and $x \in G$, we denote by M^x the set $\{x^{-1}fx \mid f \in M\}$.

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*Partially supported by Grants-in-Aid for Scientific Research under grant no. 1 2

- (*) For each subset M of G consisting of finite number of elements not equal to 1, there exist distinct $x_1, x_2, x \in G$ such that M^{x_i} ($i \in [3]$) are mutually reduced.

We have showed the following theorem:

Theorem 1.1. ([2, Theorem 1.1]) *Let G be a group which has a non-abelian free subgroup whose cardinality is the same as that of G , and suppose that G satisfies (*). Then, if R is a domain with $|R| \leq |G|$, the group ring RG of G over R is primitive. In particular, the group algebra KG is primitive for any field K .*

By making use of Theorem 1.1, we can get many results for primitivity of group rings (see [1], [2] and [3]). For amalgamated free products, we have showed the following result:

Theorem 1.2. ([2, Corollary 4.5]) *Let R be a domain and suppose that $G = A *_H B$ satisfies $B \neq H$ and there exist elements a and a_* in $A \setminus H$ such that $aa_* \neq 1$ and $a^{-1}Ha \cap H = 1$. If $|R| \leq |G|$, then the group ring RG is primitive. In particular, KG is primitive for any field K .*

For $g \in G$, we denote the centralizer of g in G by $C_G(g)$, and let $\mathcal{C}(G) := \{g \in G \mid [G : C_G(g)] < \infty\}$. Clearly, $\mathcal{C}(G)$ includes the center of G . In Theorem 1.1 (and so in Theorem 1.2), $\mathcal{C}(G)$ is always trivial and so is the center of G , which is needed for KG to be primitive for any field K .

It is easy to see that $\mathcal{C}(G)$ is trivial provided G satisfies (*). In fact, for a nonidentity element g in G , we can see that there exist infinitely many conjugate elements of g . If this is not the case, then the set M of conjugate elements of g in G is a finite set. Since G satisfies (*), for M , there exists $x_1, x_2 \in G$ such that M^{x_1} and M^{x_2} are

mutually reduced. Since g is in M , $(x_1^{-1}gx_1)(x_2^{-1}fx_2)^{-1} \neq 1$ for any $f \in M$, and thus $x_1^{-1}gx_1 \neq x_2^{-1}fx_2$. Hence $(x_1x_2^{-1})^{-1}g(x_1x_2^{-1}) \neq f$ for any $f \in M$, which implies $(x_1x_2^{-1})^{-1}g(x_1x_2^{-1}) \notin M$, a contradiction.

This means that for group rings of groups with nontrivial center, Theorem 1.1 does not say anything. On the other hand, in her Ph D thesis (1975), C. R. Jordan [4] had given the following result:

Theorem 1.3. ([4, Theorem 2.5.2, 2.5.4]) *Let $G = A *_H B$ be the free product of A and B with H amalgamated.*

(1) *Let R be a ring. Suppose that $R(H)$ is an uncountable domain and that $|G : H|$ is greater than or equal to the cardinality of $R(H)$. Then $R(G)$ is primitive.*

(2) *Let $|A : H| \neq 2$ or $|B : H| \neq 2$. Suppose that $R(H)$ is a domain and has countable cardinality. Then $R(G)$ is primitive.*

Jordan's results hold even if H is the center of G . So we would like to extend our results to one for groups with nontrivial center.

2 A generalization

Let G be a group and N a proper subgroup of G . Let G be a group and M a subset of G . For non-empty subsets $M_1, M_2, \dots, M_n$ of G , consisting of elements of $G - N$, we say that $M_1, M_2, \dots, M_n$ are mutually N -reduced in G , if for each finite number of elements $g_1, g_2, \dots, g_m \in \bigcup_{i=1}^n \widetilde{M_i}$, whenever $g_1g_2 \cdots g_m \in N$, there exists $i \in [m]$ and $j \in [n]$ so that $g_i, g_{i-1} \in \widetilde{M_j}$. We consider the following condition:

- ($* - N$) For each nonempty subset M of G consisting of finite number of elements of $G - N$, there exist distinct $x_1, x_2, x \in G$ such that M^{x_i} ($i \in [3]$) are mutually N -reduced.

If $N = 1$, then ($* - N$) simply means the condition ($*$). We get the following result:

Theorem 2.1. *Let G be a group which has a non-abelian free subgroup whose cardinality is the same as that of G , and suppose that G satisfies ($* - N$) for some proper subgroup of G . Then, if R is a domain with $|R| \leq |G|$, the group ring RG of G over R is primitive. In particular, if $(G) = 1$, then the group algebra KG is primitive for any field K .*

The proof of the above theorem is similar to the one of Theorem 1.1. By the above theorem, even if G has the center $C \neq 1$, if G satisfies ($* - C$) and $|R| \leq |G|$, then the group ring RG is primitivity. In fact, we can easily show the following corollary:

Corollary 2.2. *Let R be a domain and $G = A *_H B$ the free product of A and B with H amalgamated. Suppose that H is a normal subgroup and $|A| \geq |B| > 1$. If $|R| \leq |G|$, $|A : H| \geq \aleph$ and $|A : H| \geq |H|$, then RG is primitive.*

We can easily show that G in the above corollary satisfies the conditions needed in Theorem 2.1. That is, G has a nonabelian free subgroup whose cardinality is the same as that of G and also satisfies ($* - H$).

In fact, since $|A : H| = |G|$, there exist a set I with $|I| = |G|$ and $a_i \in A$ ($i \in I$) such that $\{a_i \mid i \in I\}$ is a complete set of representatives of G/H . We can see then that the subgroup of G generated by $(a_i b)^2$ ($i \in I$) for some $b \in B$ with $b \notin H$ is freely generated by them.

Moreover, for any finite number of elements U_i ($i \in [m]$) in $G - H$, let $U_i = u_{i1} \cdots u_{im_i}$, where either $u_{ij} \in A$ and $u_{i(j-1)} \in B$ or $u_{ij} \in B$ and $u_{i(j-1)} \in A$. We set here that $S_i = \{u_{ij} \mid j \in [m_i], u_{ij} \in A\}$ and $S = \bigcup_{i=1}^m S_i$. Since $|A : H| \geq \aleph$, there are elements a_i in A ($i = 1, 2, 3$) such that $a_i \notin H \cup (\bigcup_{\alpha \in S} \alpha H)$ and $a_i a_j^{-1} \notin H$ if $i \neq j$. For $b \in B$ with $b \notin H$, let $x_i = a_i^{-1} b a_i$ and $M = \{U_i \mid i \in [m]\}$. We have then that $M^{x_1}, M^{x_2}, M^{x_3}$ are mutually H -reduced.

References

- [1] C. R. Abbott and F. Dahmani, *Property P_{naive} for acylindrically hyperbolic groups*, arXiv:1610.04143v2
- [2] J. Alexander and T. Nishinaka, *Non-noetherian groups and primitivity of their group algebras*, J. Algebra **473** (2017), 221-246
- [3] B. B. Solie, *Primitivity of group rings of non-elementary torsion-free hyperbolic groups*, J. Algebra **493** (2018), 438-443.
- [4] C. R. Jordan, *Group Rings of Generalised Free Products*, Ph D thesis (1975), 79-81