

# Structure of nonnegative solutions for parabolic equations and perturbation theory for elliptic operators

Minoru MURATA

Department of Mathematics  
Tokyo Institute of Technology

This talk is concerned with structure of nonnegative solutions for parabolic equations, perturbation theory for elliptic operators, and their relations.

Let  $M$  be a Riemannian manifold of dimension  $n$ , and  $D$  be a noncompact domain of  $M$ . Let  $L$  be an elliptic operator on  $D$  of the form

$$Lu = -m^{-1} \operatorname{div}(mA \nabla u) + V u.$$

Here  $m$  is a positive measurable function such that  $m, m^{-1} \in L_{\text{loc}}^{\infty}(D)$ ,  $A$  is a symmetric measurable section on  $D$  of

$\operatorname{End}(TM)$  such that  $\forall K \subset\subset D \exists \lambda > 0$

such that

$$\lambda |\xi|^2 \leq \langle A_x \xi, \xi \rangle \leq \lambda^{-1} |\xi|^2, \quad x \in K, \quad (x, \xi) \in TM,$$

and  $V$  is a real-valued function with  $V \in L^p_{loc}(D, m dV)$  for some  $p > \max(\frac{n}{2}, 1)$ , where  $dV$  is the Riemannian measure.

For simplicity of notations, we assume

$$\lambda_0 \equiv \inf \left\{ \int_D (\langle A \nabla u, \nabla u \rangle + V u^2) m dV ; \right.$$

$$\left. u \in C_0^\infty(D), \int_D u^2 m dV = 1 \right\}$$

is positive. (Actually, the condition  $\lambda_0 > -\infty$  suffices.)

We consider nonnegative solutions of the parabolic equation

$$(\#) \quad (\partial_t + L) u = 0 \quad \text{in } Q = D \times (0, T),$$

where  $T > 0$ .

Our problem is the following.

<Problem> Determine all nonnegative solutions of (#).

We put

$$P(Q) = \{ u \geq 0; u \text{ is a solution of } (\#) \}.$$

This problem is closely related to the Cauchy problem

$$(\#^0) \quad \begin{cases} (\partial_t + L) u = 0 & \text{in } Q \\ u(x, 0) = 0 & \text{on } D. \end{cases}$$

We say:

[VP] holds for  $(\#^0)$  when any solution  $u \geq 0$  of  $(\#^0)$  must be identically zero.

[NUP] holds for  $(\#^0)$  when there exists a solution  $u \not\equiv 0$  of  $(\#^0)$ .

< Example > (Widder '44)  $D = M = \mathbb{R}^n$ ,  $L = -\Delta$

$\Rightarrow$  [VP] holds for  $(\#^0)$

When [VP] holds, our problem has an extremely simple answer.

< Fact > (Ancona - Taylor '92)

[UP] holds for  $(\#^0)$

$\Rightarrow \forall u \in \underline{P}(Q) \quad \exists \mu : \text{Borel measure on } D$   
such that

$$u(x, t) = \int_D p(x, y, t) d\mu(y), \quad (x, t) \in Q.$$

Here  $p$  is a minimal fundamental solution for  $\partial_t + L$  with respect to the measure  $m dV$ .

Thus we need to consider what happens in the case [NVP]. For this purpose let us introduce the following condition [SSP] (i.e. the constant function 1 is a semismall perturbation of  $L$ ).

[SSP]  $\forall \varepsilon > 0 \quad \exists K \subset\subset D$  such that

$$\int_{D \setminus K} G(x^0, z) \cdot 1 \cdot G(z, y) m(z) dV(z)$$

$$\leq \varepsilon G(x^0, y), \quad y \in D \setminus K,$$

where  $x^0$  is a fixed reference point in  $D$  and  $G$  is the positive Green function of  $L$  on  $D$ .

Recall that we have assumed  $\lambda_0 > 0$ .  
 Thus the Green function exists. Furthermore we can define the selfadjoint operator  $L_D$  on  $L^2(D; m dV)$  associated with the quadratic form generated by  $L$ .

Let

$\partial_M D$  : Martin boundary of  $D$  for  $L$

$\partial_m D$  : minimal Martin boundary of  $D$  for  $L$

$D_L^* = D \cup \partial_M D$  : Martin compactification of  $D$  for  $L$

$K(x, \xi)$  : Martin kernel with pole  $\xi \in \partial_M D$

Then we see :  $[SSP] \Rightarrow$  The following  
 $1^\circ, 2^\circ, 3^\circ$  hold.

$1^\circ$  The spectrum of  $L_D$  consists of discrete eigenvalues with finite multiplicity.  
 Let  $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$  be the eigenvalues of  $L_D$  repeated according to the multiplicity, and  $\phi_j$  be an eigenfunction for  $\lambda_j$  such that  $\phi_0 > 0$  and  $\{\phi_j\}_{j=0}^\infty$  is a complete orthonormal system of  $L^2(D; m dV)$ .

$2^\circ$  There exists a continuous extension  $[\phi_j / \phi_0]$  of  $\phi_j / \phi_0$  up to  $\partial_M D$ .

3° [NVP] holds for  $(\#^0)$ .

Furthermore, we have the following

< Theorem 1 > [SSP] holds

$$\Rightarrow \forall \xi \in 2_M D$$

$$\exists \lim_{D \ni y \rightarrow \xi} \frac{P(x, y, t)}{\phi_0(y)} \equiv q(x, \xi, t), \quad x \in D, t \in \mathbb{R}$$

$q$  is a continuous function on  $D \times 2_M D \times \mathbb{R}$  such that

$$q > 0 \quad \text{on } D \times 2_M D \times (0, \infty)$$

$$q = 0 \quad \text{on } D \times 2_M D \times (-\infty, 0]$$

$$(\partial_t + L) q(\cdot, \xi, \cdot) = 0 \quad \text{in } D \times \mathbb{R}$$

We are now ready to give an answer to our problem.

< Theorem 2 > [SSP] holds

$$\Rightarrow \forall u \in \mathcal{P}(\mathbb{R}) \quad \exists_1 (\mu, \lambda) \text{ such that } \mu \text{ is a Borel measure on } D$$

$\lambda$  is a Borel measure on  $\partial_M D \times [0, T)$   
which is supported by  $\partial_M D \times [0, T)$

$$u(x, t) = \int_D p(x, y, t) d\mu(y) \\ + \int_{\partial_M D \times [0, T)} g(x, \xi, t-s) d\lambda(\xi, s), \\ (x, t) \in Q$$

I should mention that Theorems 1 and 2  
are results of [Mendez - Murata '18, preprint].

The proof of Theorem 2 is based upon  
the abstract parabolic Martin representation  
theorem and Choquet's theorem. Its  
key step is to identify the parabolic  
Martin boundary.

Here let's see simple examples.

< Example 1 >  $D \subset \mathbb{R}^2$  with  $|D| < \infty$   
 $L = -\Delta$

$\Rightarrow$  [SSP] holds and so Theorem 2 holds.

For the higher dimensional case, we need some regularity of a boundary.

<Example 2>  $D \subset \mathbb{R}^n$  : bounded John domain  
 $L = -\Delta$

$\Rightarrow$  [SSP] holds and so Theorem 2 holds.

<Example 3>  $L = -\Delta + 1$  on  $\mathbb{R}^n$ ,  $\beta \in \mathbb{R}$ ,  
 $D = \{ (x_1, x') \in \mathbb{R}^n ; x_1 > 1, |x'| < x_1^\beta \}$

Then

[SSP] holds  $\Leftrightarrow \beta < -1$

Now let us introduce a condition which is weaker than [SSP].

Put  $E_L(D) = \{ h > 0 ; Lh = 0 \text{ on } D \}$ .

Fix  $h \in E_L(D)$ .

[N $h$ B] (i.e. 1 is non- $h$ -big)

$\exists v \in E_{L+1}(D)$  s.t.  $0 < v \leq h$  on  $D$

By virtue of a nice characterization of [N $h$ B] by Grigor'yan - Hansen '98, we can show

[SSP]  $\Rightarrow$  [N $h$ B]  $\forall h \in E_L(D)$

Furthermore, we have

<Theorem> Fix  $h \in E_L(D)$ . Then

$$[NhB] \Leftrightarrow \exists u : \text{solution of } (\#^0) \text{ such that} \\ 0 < u(x, t) < h(x) \text{ on } D \times (0, \infty)$$

Obviously, this implies  $[NVP]$ .

We say  $[hB]$  (i.e. 1 is  $h$ -big) holds when

$$(L+1)v = 0, \quad 0 \leq v \leq h \text{ on } D \Rightarrow v \equiv 0$$

Then a direct consequence of the above theorem is the following

<Corollary>  $[VP] \Rightarrow [hB] \quad \forall h \in E_L(D)$

This observation is useful in showing  $[hB]$  since we have a general and sharp sufficient condition for  $[VP]$  by Ishige-Murata '01.

As for  $[SSP]$ , we have powerful theorems by Ancona '97.

Summing up, we have :

$$[SSP] \Rightarrow [N\&B] \Rightarrow [NUP]$$

$$[VP] \Rightarrow [hB]$$

$[NUP] \Leftrightarrow 1$  is a "small" perturbation

$[VP] \Leftrightarrow 1$  is a "big" perturbation

Now let us see some more examples.

< Example 4 > (generalized Poincaré disc)

$$D = \{x \in \mathbb{R}^2 : |x| < 1\}, \quad \gamma \in \mathbb{R}$$

$$L = -(1-|x|^2)^2 \left( \log \frac{2}{1-|x|^2} \right)^\gamma \Delta_{\mathbb{R}^2}$$

Then

$$(i) [SSP] \Leftrightarrow \gamma > 1$$

$$(ii) [VP] \Leftrightarrow \gamma \leq 1$$

$$< \text{Example 5} > \quad L = - \sum_{i,j=1}^n \partial_i (a_{ij}(x) \partial_j) + V(x)$$

: uniformly elliptic operator on  $D = \mathbb{R}^n$

$0 < c < 1$ ,  $\exists \rho$  : positive continuous increasing function on  $[0, \infty)$

such that

$$c [\rho(|x|)]^2 \leq V(x) \leq [\rho(|x|)]^2, \quad x \in \mathbb{R}^n$$

$$c \rho\left(r + \frac{c}{\rho(r)}\right) \leq \rho(r), \quad r \geq 0$$

Then

$$(i) \quad [SSP] \Leftrightarrow \int_1^{\infty} \frac{dr}{p(r)} < \infty$$

$$(ii) \quad [UP] \Leftrightarrow \int_1^{\infty} \frac{dr}{p(r)} = \infty$$