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Discussion Paper No. 624

“Welfare Gains and Losses in Sunspot Equilibria”

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October 2006



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KYOTO, JAPAN

Welfare Gains and Losses in Sunspot Equilibria*

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October 23, 2006

Abstract

We study a standard two period exchange economy with one nominal asset. As is well known there is a continuum of sunspot equilibria around each efficient equilibrium. A sunspot equilibrium is inefficient but some household may gain in sunspot equilibria relative to the efficient equilibrium. We show that a household's equilibrium utility level is either locally maximized or locally minimized at the efficient equilibrium, and derive a condition which identifies whether or not a household's utility is locally minimized or maximized.

1 Introduction

Consider a standard two period competitive exchange economy with inside money where households are all risk averse. Using inside money as a medium of exchange, households can borrow or save in the first period. Under standard assumptions, a competitive equilibrium exists and any competitive equilibrium is Pareto efficient. Generically in endowments, there are finitely many such competitive equilibria. However, there may be sunspot equilibria where the second period consumption depends on extrinsic signals.¹

*The financial support by Grant-in-Aid for the 21st Century COE Program and Grant-in-Aid for Scientific Research is gratefully acknowledged. The author thanks Sergio Currarini, Aditya Goenka, Piero Gottardi, Chiaki Hara and for their helpful comments. The auhtor is solely responsible for remaining errors and omissions.

¹For the economic implications of sunspot equilibria, see Cass and Shell (1983).

In fact, it can be shown that a continuum of sunspot equilibria also exists around each of these efficient competitive equilibria.²

A sunspot equilibrium is Pareto inefficient, and therefore in each sunspot equilibrium there must be at least one household who is worse off than in the respective efficient equilibrium. One can consider various scenarios about welfare distributions associated with sunspot equilibria. Of course, the inefficiency does not imply that all the households are worse off, but it certainly seems plausible and intuitive if this is the case. It may even appear that this will be a prevailing case, since a sunspot equilibrium is “contaminated” by extrinsic, welfare irrelevant randomization by construction, and risk averse households do not appreciate such randomization.

In general, it is true that *if* the expected real return from an asset is kept constant, increasing the volatility of its returns is welfare worsening to any household. But notice that there is a general equilibrium effect through changing prices, which is overlooked in the observation above. The expected real returns are determined in equilibrium. In the simple set up we consider where a nominal bond (inside money) is the only asset, if its average real returns in a sunspot equilibria benefits a particular household, and if the benefit is large enough to offset the loss from the increasing volatility, such a household could gain by sunspots.

As far as we know, this important implication of general equilibrium effects on welfare gains and losses in sunspot equilibria is not addressed well in the literature, until Goenka-Préchat (2006): although they considered only a special symmetric model of two households, they derived a simple condition expressed in the derivatives of utility functions, under which the utility level of the borrower (the seller of the asset) is locally *minimized* at the efficient equilibrium, and that of the saver is locally *maximized*.

The Goenka-Préchat condition says that the households are *prudent* enough at the efficient equilibrium, i.e., the third derivative of utility function is positive and large enough relative to its second derivative. The condition is satisfied for the log utility case, as well as for a wide range of popular parametric classes of utility functions. Thus within their setup, the borrower are benefitted from sunspots in a large class of economies.

It is however hard to see the general equilibrium intuition in their condition on the prudence. This is so because of the special symmetric structure of their model. Not only

²See Cass (1992). See also Pietra (1992) and Suda-Tallon-Villanacci (1992).

there are just two households, but also it is assumed that households have *identical preferences* represented by a time separable and time invariant utility function. Moreover, the total endowment is the same in the two periods, and households' endowments are symmetric in the sense that household 1's first period endowment is equal to household 2's second period endowment. Thus although their condition gives an impression that *individual* risk preferences determine the beneficiary from sunspots, the general equilibrium effect is possibly concealed in the special structure. More importantly, it may well be the case that the general class of economies where the borrower is benefitted from sunspots is not as large as it seems.

The purpose of this paper is to obtain a deeper understanding of welfare gains and losses in sunspot equilibria. Given discussions above, it is desirable not to rely on symmetry up front. We therefore study a very general model: except that we keep the assumption of single consumption good in each period, the number of households is arbitrary, and their utility functions and endowments are general. Our analysis takes advantage of symmetry about sunspot states, but nothing else.

In such a general framework with S sunspot states, the equilibrium utility level is expressed by a function of $S - 1$ variables. We find a condition which tells whether or not a household's equilibrium utility level is locally minimized or maximized at the efficient equilibrium (Proposition 6). Our condition says that the net benefit from sunspots is the sum of two terms, where the first negative term corresponds to the risk effect, and the second term represents the general equilibrium effect, so it confirms the intuition we outlined above.

Interestingly enough, even when the equilibrium welfare function as above has more than one variables, it is either locally concave or convex at the efficient equilibrium. Using this condition, we show that either all the savers' or all the borrowers', or all the households' equilibrium utility level is locally maximized at the efficient equilibrium (Corollary 8).

The structure of the paper is as follows. Section 2 sets up the model. The main analysis and the results mentioned above are contained in Section 3. We elaborate the results by relating the model to the standard simple portfolio problem in Section 4. We also discuss in Section 4 why it tends to be a borrower if there is a household who is benefitted from sunspots. The Goenka-Préachac condition is re-examined in the light of

our results in Section 5 to see how the general equilibrium effects and the risk effect are mixed in their simple condition. Section 6 contains a few remarks.

2 The Model

We consider a standard competitive two-period exchange economy. There is one perishable consumption good in each period to be traded. There are $H \geq 2$ households, labelled by $h = 1, 2, \dots, H$. Household h is endowed with e_h^0 units of good in the first period (period 0) and e_h^1 units in the second period (period 1). We write $e_h = (e_h^0, e_h^1)$. In the first period, period 0, a nominal asset which pays off one unit in units of account in the second period is traded. The net supply of the asset is zero, so it is inside money whose real returns are to be determined in the markets.

At the beginning of the second period, a state $s = 1, 2, \dots, S$ occurs which is publicly observed. We assume that these are sunspot states, and they are equally probable.³ Denote by $r^s > 0$ the real return of the asset in units of the first period consumption good when the state is s ; that is, the price of the asset is normalized to be one, and if z units of the asset is held at the end of the first period, $r^s z$ units of consumption good is delivered at the beginning of the second period. Writing z_h for the asset holding of household h , the consumption of household h is therefore $e_h^0 - z_h$ in period 0 and $e_h^1 + z_h r^s$ in state s in period 1.⁴ If $z_h > 0$, then household h is referred to as a *saver*, and if $z_h < 0$, then household h is referred to as a *borrower*.

The preferences of household h are represented by a von Neumann Morgenstern utility function $u_h : \mathbb{R}_{++}^2 \rightarrow \mathbb{R}$; that is, given a vector of returns $r = (r^s)_{s=1}^S \in \mathbb{R}_{++}^S$, if household h chooses z_h such that $e_h^0 - z_h > 0$ and $e_h^1 + z_h r^s > 0$ for every $s = 1, \dots, S$, the level of (normalized) utility is given by

$$\sum_{s=1}^S u_h(e_h^0 - z_h, e_h^1 + z_h r^s). \quad (1)$$

³The assumption of equal probability is not restrictive. See Section 6.

⁴This is of course a reduced form and it is equivalent to the standard sequential budget constraints. If we write x^0 and p^0 for the first period consumption and the (nominal) price of good, and x^s and p^s for the consumption in state s and the (nominal) price of good in state s , the sequential budget constraints are: $p^0(x^0 - e^0) + \hat{z} = 0$, and $p^s(x^s - e^1) = \hat{z}$ for $s = 1, \dots, S$. Then setting $r^s = p^s/p^0$ and $z = p^0 \hat{z}$, we get the reduced form in the text.

A household is a price taker by assumption and so household h 's problem is to choose $z_h \in \mathbb{R}$ to maximize the expected utility, which is formally written below:

$$\begin{aligned} \max_{z_h} \sum_{s=1}^S u_h(e_h^0 - z_h, e_h^1 + z_h r^s) \\ \text{subject to } e_h^0 - z_h > 0 \text{ and } e_h^1 + z_h r^s > 0, \quad s = 1, \dots, S. \end{aligned} \quad (2)$$

Note that the problem (2) is nothing but a standard, simple portfolio choice problem between consumption and a risky asset, but with possibly time non-separable utility function.⁵ The inside money constitutes a risky asset since although the nominal return is fixed by assumption, its real return can be random.

It is assumed that u_h is C^3 , differentiable strictly increasing (i.e., for any $x_h \in \mathbb{R}_{++}^2$, the gradient $Du_h(x_h)$ is strictly positive), differentiable strictly concave (i.e., for any $x_h \in \mathbb{R}_{++}^2$, the Hessian $D^2u_h(x_h)$ is negative definite), and the closure in $\mathbb{R}^2$ of each level set is contained in $\mathbb{R}_{++}^2$. The assumption of thrice differentiability is needed since the second derivatives of demand functions are important in our analysis.

Under these assumptions, the objective function in (2) is concave in z_h and the optimal choice is characterized by a solution to the first order condition as follows:

$$-\sum_{s=1}^S \frac{\partial}{\partial x_0} u_h(e_h^0 - z_h, e_h^1 + z_h r^s) + \sum_{s=1}^S \frac{\partial}{\partial x_1} u_h(e_h^0 - z_h, e_h^1 + z_h r^s) r^s = 0, \quad (3)$$

where $\frac{\partial}{\partial x_0} u_h$ and $\frac{\partial}{\partial x_1} u_h$ are derivatives with respect to the first period consumption and the second period consumption, respectively. The solution is unique if it exists by the strict concavity. Since our analysis will be done locally around a competitive equilibrium where the optimal choice is well defined, we will assume that a solution exists in the relevant domain. For a vector of returns $r = (\dots, r^s, \dots) \in \mathbb{R}_{++}^S$, let $Z_h(r)$ be the unique solution to (3); that is, $Z_h(r)$ is the quantity demanded by household h for the asset. Let $Z(r) := \sum_{h=1}^H Z_h(r)$ which is the *market excess demand function* for the asset. It can be shown that each Z_h is a C^2 function, and so is Z . Utilizing the symmetric nature of the model, the following properties can be readily checked: for each

⁵The connection is apparent if we re-write the objective function in (2) equivalently as $E_{\tilde{r}} [u_h(e_h^0 - z_h, e_h^1 + z_h \tilde{r})]$, where E is the expectation operator.

h and any pair of states s and s' , and any positive number ρ ,

$$\frac{\partial}{\partial r^s} Z_h(\rho, \dots, \rho) = \frac{\partial}{\partial r^{s'}} Z_h(\rho, \dots, \rho), \quad (4)$$

$$\frac{\partial}{\partial r^s} Z(\rho, \dots, \rho) = \frac{\partial}{\partial r^{s'}} Z(\rho, \dots, \rho). \quad (5)$$

Definition 1 An equilibrium is a vector of returns $r = (\dots, r^s, \dots) \in \mathbb{R}_{++}^S$ such that $Z(r) = 0$. Equivalently, r is an equilibrium if there exists a vector of asset holdings $z = (z_h)_{h=1}^H \in \mathbb{R}^H$ with $\sum_{h=1}^H z_h = 0$, where each z_h solves the utility maximization problem (2), for $h = 1, \dots, H$.

When $S = 1$, our model is a standard two period model of consumption and saving, and so every equilibrium is efficient. An equilibrium for the case of $S = 1$ is called a *certainty equilibrium*. If $\bar{r} \in \mathbb{R}_+$ is a certainty equilibrium, it can be readily seen that the vector $(\bar{r}, \dots, \bar{r}) \in \mathbb{R}_{++}^S$ is an equilibrium for any $S > 1$: this is an equilibrium where the households think the sunspot states do not affect the real returns, although they know that sunspot states are to be observed. Such an equilibrium is called a *non-sunspot equilibrium* when $S > 1$. By the fundamental theorem of welfare economics and risk aversion, a non-sunspot equilibrium is Pareto efficient. To simplify notation we write $\bar{r}$ instead of $(\bar{r}, \dots, \bar{r})$ whenever it is clear from the context. An equilibrium r is called a *sunspot equilibrium* if $r^s \neq r^{s'}$ for some s and s' .

Example 2 Let $H = 2$, and $u_h(x, y) = v(x) + v(y)$ for both h , where $v' > 0$ and $v'' < 0$. $e_1 = (\alpha, 1 - \alpha)$ and $e_2 = (1 - \alpha, \alpha)$, $\alpha \in (\frac{1}{2}, 1]$. There is a unique certainty equilibrium $\bar{r} = 1$ where both households consume $\frac{1}{2}$ in both periods. Thus $Z_1(\bar{r}) > 0 > Z_2(\bar{r})$. This is the setup Goenka-Préchac (2006) studied.

We are interested in the structure of the set of utility profiles associated with equilibria, especially around a non-sunspot equilibrium. For this purpose it is useful to learn the differential structure of the set. It is known⁶ that for any $S > 1$, *generically in endowments*, there are finitely many non-sunspot equilibria and for any non-sunspot equilibrium $(\bar{r}, \dots, \bar{r})$, $\frac{\partial}{\partial r^s} Z(\bar{r}, \dots, \bar{r}) \neq 0$ for $s = 1, \dots, S$, and moreover $Z_h \neq 0$ for any equilibrium around $\bar{r}$. Thus in particular Z can be solved implicitly around a non-sunspot

⁶This can be established as a simple corollary to the generic regularity result of Cass (1992) on non-sunspot equilibria. See also the leading example of Cass (1989).

equilibrium $(\bar{r}, \dots, \bar{r})$; there is a C^2 function ϕ defined locally around $(\bar{r}, \dots, \bar{r}) \in \mathbb{R}_{++}^{S-1}$ such that $Z(r^1, \dots, r^{S-1}, \phi(r^1, \dots, r^{S-1})) = 0$ for $r^1, \dots, r^{S-1}$ in the domain.

Throughout this paper, we assume that the functions are well defined and the asserted properties are true; that is, the endowments are chosen in such a way that a non-sunspot equilibrium exists and these properties hold. As described above, such a choice is generic, which justifies this assumption. From now on, we set $S > 1$ and fix a non-sunspot equilibrium $(\bar{r}, \dots, \bar{r})$. We summarize below the key maintained assumptions throughout this paper:

Regularity Assumption: $\bar{r} > 0$ and $Z(\bar{r}, \dots, \bar{r}) = 0$. $\frac{\partial}{\partial r^s} Z(\bar{r}, \dots, \bar{r}) \neq 0$ for $s = 1, \dots, S$ and so there is a C^2 function ϕ defined a neighborhood $\mathcal{R} \subseteq \mathbb{R}_{++}^{S-1}$ around $(\bar{r}, \dots, \bar{r}) \in \mathbb{R}_{++}^{S-1}$ such that

$$Z(r^1, \dots, r^{S-1}, \phi(r^1, \dots, r^{S-1})) = 0 \quad (6)$$

$$Z_h(r^1, \dots, r^{S-1}, \phi(r^1, \dots, r^{S-1})) \neq 0, \text{ for every } h = 1, \dots, H \quad (7)$$

for any $(r^1, \dots, r^{S-1}) \in \mathcal{R}$

A generic element of $\mathcal{R}$ is denoted by $r_{-S} = (r^1, \dots, r^{S-1})$. By construction, $\phi(\bar{r}_{-S}) = \bar{r}$, and the set of equilibrium asset holdings around a non-sunspot equilibrium $(\dots, \bar{r}, \dots)$ can be found by changing r^s around $\bar{r}$ for $s = 1, \dots, S-1$. For each h , define $\hat{Z}_h$ and $\hat{U}_h$ on $\mathcal{R}$ by the following rule:

$$\begin{aligned} \hat{Z}_h(r_{-S}) &:= Z_h(r_{-S}, \phi(r_{-S})), \\ \hat{U}_h(r_{-S}) &:= \sum_{s=1}^{S-1} u_h(e_h^0 - \hat{Z}_h(r_{-S}), e_h^1 + \hat{Z}_h(r_{-S}) r^s) \\ &\quad + u_h(e_h^0 - \hat{Z}_h(r_{-S}), e_h^1 + \hat{Z}_h(r_{-S}) \phi(r_{-S})). \end{aligned}$$

Namely, $\hat{Z}_h(r_{-S})$ is household h 's asset holding and $\hat{U}_h(r_{-S})$ is the corresponding utility level in equilibrium $(r_{-S}, \phi(r_{-S}))$. Then, the set of profiles of equilibrium asset holdings is $\{(\hat{Z}_h(r_{-S}))_{h=1}^H : r_{-S} \in \mathcal{R}\}$. We shall refer to this set as the *equilibrium manifold* (around the non-sunspot equilibrium), which has dimension $S-1$. The corresponding level of utility is

$$\mathcal{U} := \{(\hat{U}_h(r_{-S}))_{h=1}^H : r_{-S} \in \mathcal{R}\}. \quad (8)$$

Our purpose is to study the local structure of this set around the non-sunspot equilibrium.⁷

3 The Analysis: Characterization in Derivatives

We shall first learn how the equilibrium demand $\hat{Z}_h$ above behaves around $\bar{r}$. It turns out that the first order effect on $\hat{Z}_h$ is null: if returns are changed marginally from the non-sunspot equilibrium, the corresponding consumption remains the same. The following result stating this formally holds from the symmetry.

Lemma 3 *For any state $s = 1, \dots, S - 1$, $\frac{\partial}{\partial r^s} \phi(\bar{r}_{-S}) = -1$. For any household h and state $s = 1, \dots, S - 1$, $\frac{\partial}{\partial r^s} \hat{Z}_h(\bar{r}_{-S}) = 0$ where $\bar{r}_{-S} = (\dots, \bar{r}, \dots) \in \mathbb{R}_{++}^{S-1}$.*

Proof. By the symmetry (5), $\frac{\partial}{\partial r^s} Z(\bar{r}) = \frac{\partial}{\partial r^S} Z(\bar{r})$ holds for $s = 1, \dots, S - 1$. Differentiating (6) with respect to r^s , $s \neq S$, we have, for any $r_{-S} \in \mathcal{R}$,

$$\frac{\partial}{\partial r^s} \hat{Z}(r_{-S}) = \frac{\partial}{\partial r^s} Z(r_{-S}) + \frac{\partial}{\partial r^S} Z(r_{-S}) \frac{\partial}{\partial r^s} \phi(r_{-S}) = 0. \quad (9)$$

Evaluating this at $\bar{r}_{-S} = (\dots, \bar{r}, \dots)$ (thus $\phi(\bar{r}_{-S}) = \bar{r}$), using $\frac{\partial}{\partial r^s} Z(\bar{r}) = \frac{\partial}{\partial r^S} Z(\bar{r})$, we have $\frac{\partial Z(\bar{r})}{\partial r^s} + \frac{\partial Z(\bar{r})}{\partial r^S} \frac{\partial \phi(\bar{r}_{-S})}{\partial r^s} = 0$. So $\frac{\partial}{\partial r^s} \phi(\bar{r}_{-S}) = -1$ must hold for $s = 1, \dots, S - 1$, since $\frac{\partial Z(\bar{r})}{\partial r^s} \neq 0$ by the regularity assumption.

Now similarly to (9), for any h and state $s = 1, \dots, S - 1$, we have $\frac{\partial}{\partial r^s} \hat{Z}_h = \frac{\partial}{\partial r^s} Z_h + \frac{\partial}{\partial r^S} Z_h \frac{\partial}{\partial r^s} \phi$ from (7). Then using the symmetry (4) and $\frac{\partial}{\partial r^s} \phi(\bar{r}_{-S}) = -1$, we have $\frac{\partial}{\partial r^s} \hat{Z}_h(\bar{r}_{-S}) = 0$. ■

To interpret, recall that $\sum_{s=1}^{S-1} r^s + \phi(r^s)$ is (proportional to) the *average* equilibrium returns. So Lemma 3 says that when the return in state s changes, the corresponding equilibrium average returns remain unchanged up to the first order. In fact, the first order effect on the equilibrium utility level is also null. We show this by computing the derivative of the equilibrium utility level $\hat{U}_h(r_{-S})$ at the non-sunspot equilibrium $\bar{r}_{-S} := (\dots, \bar{r}, \dots)$. To simplify notation, we write $\bar{u}_h$ for $\hat{U}_h$ evaluated at $r_{-S} = \bar{r}_{-S}$;

⁷Alternatively, one can directly study the constrained maximization problem of a household's utility given equilibrium system of equations, i.e., the first order conditions and the market clearing condition, analogously to the general method developed in Citanna-Kajii-Villanacci (1998). Indeed, this is the path which Goenka-Préachac (2006) followed. But for the single commodity case, using the excess demand functions appears more tractable, at least for the purpose of this paper.

that is, $\bar{u}_h := u_h \left(e_h^0 - \hat{Z}_h(\bar{r}_{-S}), e_h^1 + \hat{Z}_h(\bar{r}_{-S})\bar{r} \right)$. A similar convention will be used for derivatives of u_h , e.g., $\frac{\partial \bar{u}_h}{\partial x_1}$ and $\frac{\partial^2 \bar{u}_h}{\partial (x_1)^2}$.

Lemma 4 *For any household h and any state $s = 1, \dots, S-1$, $\frac{\partial}{\partial r^s} \hat{U}_h(\bar{r}_{-S}) = 0$.*

Proof. This can be verified by direct computation as follows: notice that the usual envelop argument using (3), which nullifies the effects through $d/d\hat{Z}$, so for any $r_{-S} \in \mathcal{R}$, we have:

$$\begin{aligned} \frac{\partial}{\partial r^s} \hat{U}_h(r_{-S}) &= \frac{\partial}{\partial r^s} \sum_{k=1}^{S-1} u_h \left(e_h^0 - \hat{Z}_h(r_{-S}), e_h^1 + \hat{Z}_h(r_{-S}) r^k \right) \\ &\quad + \frac{\partial}{\partial r^s} u_h \left(e_h^0 - \hat{Z}_h(r_{-S}), e_h^1 + \hat{Z}_h(r_{-S}) \phi(r_{-S}) \right), \\ &= \left(\frac{\partial}{\partial x_1} u_h \left(e_h^0 - \hat{Z}_h(r_{-S}), e_h^1 + \hat{Z}_h(r_{-S}) r^s \right) \right) \hat{Z}_h(r_{-S}) \\ &\quad + \left(\frac{\partial}{\partial x_1} u_h \left(e_h^0 - \hat{Z}_h(r_{-S}), e_h^1 + \hat{Z}_h(r_{-S}) \phi(r_{-S}) \right) \right) \hat{Z}_h(r_{-S}) \frac{\partial}{\partial r^s} \phi(r_{-S}), \end{aligned} \tag{10}$$

where the envelope property is used to derive the second equation. Therefore, from the fact $\frac{\partial}{\partial r^s} \phi(\bar{r}_{-S}) = -1$ shown in Lemma 3, and $\phi(\bar{r}_{-S}) = \bar{r}$, it follows:

$$\begin{aligned} \left. \frac{\partial}{\partial r^s} \hat{U}_h(r_{-S}) \right|_{r_{-S}=\bar{r}_{-S}} &= \left(\frac{\partial}{\partial x_1} \bar{u}_h \right) \hat{Z}_h(\bar{r}_{-S}) + \left(\frac{\partial}{\partial x_1} \bar{u}_h \right) \hat{Z}_h(\bar{r}_{-S}) \frac{\partial}{\partial r^s} \phi(\bar{r}_{-S}) \\ &= 0 \end{aligned}$$

■

So the non-sunspot equilibrium constitutes a local minimum, a local maximum, or a saddle point of $\hat{U}_h$ for all h . To distinguish these cases, we shall check the Hessian matrix of $\hat{U}_h$, denoted by $D^2 U_h(\bar{r}_{-S})$, which will depend on the first and the second order effects though the equilibrium function ϕ . We have already seen the first order effects in Lemma 3. Interestingly enough, the Hessian matrix $D\phi(\bar{r}_{-S})$ at the non-sunspot equilibrium is either negative or positive definite unless it is zero, as is shown in the next result. Let

$$\zeta := \frac{1}{\frac{\partial}{\partial r^1} Z(\bar{r})} \left(\frac{\partial^2}{\partial r^1 \partial r^2} Z(\bar{r}) - \frac{\partial^2}{\partial (r^1)^2} Z(\bar{r}) \right) \tag{11}$$

which is well defined by the regularity assumption.

Lemma 5 *The Hessian matrix $D^2\phi(\bar{r}_{-S})$ is as follows:*

$$D^2\phi(\bar{r}_{-S}) = \zeta \begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 2 \end{bmatrix}. \quad (12)$$

Thus $D^2\phi(\bar{r}_{-S})$ is positive (resp. negative) definite if $\zeta > 0$ (resp. $\zeta < 0$)

Proof. Differentiate the equilibrium identity $\hat{Z}(r_{-S}) = 0$ twice: differentiating the identity (9) and evaluating it at $\bar{z}$, we have, for any $s = 1, \dots, S-1$:

$$\begin{aligned} \frac{\partial^2}{\partial (r^s)^2} \hat{Z}(\bar{r}_{-S}) &= \left\{ \frac{\partial^2 Z(\bar{r})}{\partial (r^s)^2} + \frac{\partial^2 Z(\bar{r})}{\partial r^s \partial r^S} \frac{\partial \phi(\bar{r}_{-S})}{\partial r^s} \right\} + \frac{\partial^2 Z(\bar{r})}{\partial r^s \partial r^S} \frac{\partial \phi(\bar{r}_{-S})}{\partial r^s} \\ &\quad + \frac{\partial^2 Z(\bar{r})}{\partial (r^S)^2} \left(\frac{\partial \phi(\bar{r}_{-S})}{\partial r^s} \right)^2 + \frac{\partial Z(\bar{r})}{\partial r^S} \frac{\partial^2 \phi(\bar{r}_{-S})}{\partial (r^s)^2}, \\ &= 0, \end{aligned} \quad (13)$$

and for $s' = 1, \dots, S-1$ with $s' \neq s$:

$$\begin{aligned} \frac{\partial^2}{\partial r^s \partial r^{s'}} \hat{Z}(\bar{r}_{-S}) &= \left\{ \frac{\partial^2 Z(\bar{r})}{\partial r^s \partial r^{s'}} + \frac{\partial^2 Z(\bar{r})}{\partial r^s \partial r^S} \frac{\partial \phi(\bar{r}_{-S})}{\partial r^{s'}} \right\} + \frac{\partial^2 Z(\bar{r})}{\partial r^{s'} \partial r^S} \frac{\partial \phi(\bar{r}_{-S})}{\partial r^s} \\ &\quad + \frac{\partial^2 Z(\bar{r})}{\partial (r^S)^2} \left(\frac{\partial \phi(\bar{r}_{-S})}{\partial r^s} \frac{\partial \phi(\bar{r}_{-S})}{\partial r^{s'}} \right) + \frac{\partial Z(\bar{r})}{\partial r^S} \frac{\partial^2 \phi(\bar{r}_{-S})}{\partial r^s \partial r^{s'}}, \\ &= 0. \end{aligned} \quad (14)$$

Write $\bar{z}_s := \frac{\partial}{\partial r^s} Z(\bar{r})$ and $\bar{z}_{ss'} := \frac{\partial^2}{\partial r^s \partial r^{s'}} Z(\bar{r})$. Recall that by the symmetry of excess demand functions with respect to sunspot states, we have $\bar{z}_s = \bar{z}_1$ and $\bar{z}_{ss} = \bar{z}_{11}$, and $\bar{z}_{ss'} = \bar{z}_{12}$ for any s, s' with $s \neq s'$. Also $\frac{\partial \phi(\bar{r}_{-S})}{\partial r^s} = -1$ for every s by Lemma 3. Rewrite equations (13) and (14) using these properties, we have

$$\begin{aligned} 2(\bar{z}_{11} - \bar{z}_{12}) &= -\bar{z}_1 \frac{\partial^2 \phi(\bar{r}_{-S})}{\partial (r^s)^2}, \\ (\bar{z}_{11} - \bar{z}_{12}) &= -\bar{z}_1 \frac{\partial^2 \phi(\bar{r}_{-S})}{\partial r^s \partial r^{s'}}, \end{aligned}$$

where $s, s' = 1, \dots, S-1$ and $s \neq s'$. From these equations we have:

$$D^2\phi(\bar{r}_{-S}) = \frac{\bar{z}_{12} - \bar{z}_{11}}{\bar{z}_1} \begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 2 \end{bmatrix},$$

which is (12).

The last part follows since the matrix $\begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 2 \end{bmatrix}$ is positive definite. ■

Recall that the first order effect on the equilibrium average returns is null (Lemma 3). Lemma 5, which identify the sign of the second order effect, then says that the sign of ζ determines the direction of the equilibrium average returns. We shall elaborate more on this in the next section.

Now we are ready to state the main characterization result for the sign of the Hessian matrix $D\hat{U}_h(\bar{r}_{-S})$: for each $h = 1, \dots, H$, let ξ_h be a scalar given by the following formula:

$$\begin{aligned} \xi_h &:= \frac{\partial^2 \bar{u}_h}{\partial (x_1)^2} (Z_h(\bar{r}))^2 + \frac{\partial \bar{u}_h}{\partial x_1} Z_h(\bar{r}) \frac{1}{\frac{\partial}{\partial r^1} Z(\bar{r})} \left(\frac{\partial^2}{\partial r^1 \partial r^2} Z(\bar{r}) - \frac{\partial^2}{\partial (r^1)^2} Z(\bar{r}) \right) \\ &= \frac{\partial^2 \bar{u}_h}{\partial (x_1)^2} (Z_h(\bar{r}))^2 + \frac{\partial \bar{u}_h}{\partial x_1} Z_h(\bar{r}) \zeta, \end{aligned} \quad (15)$$

where ζ is the number defined in (11).

Proposition 6 *For each household h , the Hessian matrix $D^2\hat{U}_h(\bar{r}_{-S})$ is*

$$D^2\hat{U}_h(\bar{r}_{-S}) = \xi_h \times \begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 2 \end{bmatrix}. \quad (16)$$

Thus $D^2\hat{U}_h(\bar{r}_{-S})$ is negative (resp. positive) definite if $\xi_h < 0$ (resp. $\xi_h > 0$). Consequently; (1) the utility level of household h is locally maximized at the non-sunspot equilibrium if $\xi_h < 0$, and it is locally minimized if $\xi_h > 0$. (2) conversely, if the utility level of household h is locally maximized (resp. minimized) at the non-sunspot equilibrium, then $\xi_h \geq 0$ (resp. $\xi_h \leq 0$) must hold.

Proof. We shall show that for each h , the Hessian matrix of $\hat{U}_h$ at $\bar{r}_{-S}$ is as follows:

$$D^2\hat{U}_h(\bar{r}_{-S}) = \kappa_h \begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 2 \end{bmatrix} + \left(\frac{\partial \bar{u}_h}{\partial x_1} \right) Z_h(\bar{r}) D\phi(\bar{r}_{-S}), \quad (17)$$

where $\kappa_h = \left(\frac{\partial^2 \bar{u}_h}{\partial (x_1)^2} \right) \left(\hat{Z}_h(\bar{r}_{-S}) \right)^2$. Once this is shown, substituting $D^2\phi(\bar{r}_{-S})$ in (12) into (17), we find that (16) holds. Then $D^2\hat{U}_h(\bar{r}_{-S})$ is a positive definite matrix multiplied by a scalar ξ_h , so it is negative definite if $\xi_h < 0$, and positive definite if $\xi_h > 0$, as asserted.

Now to see (17) holds, keeping the envelope property (10) in mind, for any s and s' ,

$$\begin{aligned} \frac{\partial^2}{\partial r^s \partial r^{s'}} \hat{U}_h(r_{-S}) &= \left\{ \frac{\partial}{\partial r^{s'}} \left(\frac{\partial}{\partial x_1} u_h \left(e_h^0 - \hat{Z}_h(r_{-S}), e_h^1 + \hat{Z}_h(r_{-S}) r^s \right) \hat{Z}_h(r_{-S}) \right) \right\} \\ &\quad + \left\{ \frac{\partial}{\partial r^{s'}} \left(\frac{\partial}{\partial x_1} u_h \left(e_h^0 - \hat{Z}_h(r_{-S}), e_h^1 + \hat{Z}_h(r_{-S}) \phi(r_{-S}) \right) \hat{Z}_h(r_{-S}) \frac{\partial}{\partial r^s} \phi(r_{-S}) \right) \right\}. \end{aligned} \quad (18)$$

To evaluate (18) at $r_{-S} = \bar{r}_{-S}$, notice that since $\frac{\partial}{\partial r^s} \hat{Z}_h(\bar{r}_{-S}) = 0$ by Lemma 3, the effects through $d/d\hat{Z}$ are all zero; for example, $\left(\frac{\partial}{\partial x_0} \bar{u}_h \right) \left(-\frac{\partial}{\partial r^s} \hat{Z}_h(\bar{r}_{-S}) \right) = 0$ and so all the terms multiplied by $\frac{\partial}{\partial x_0} \bar{u}_h$ vanish. So we have: if $s = s'$,

$$\begin{aligned} \frac{\partial^2}{\partial (r^s)^2} \hat{U}_h(\bar{r}_{-S}) &= \left\{ \left(\frac{\partial^2 \bar{u}_h}{\partial (x_1)^2} \right) \left(\hat{Z}_h(\bar{r}_{-S}) \right)^2 \right\} \\ &\quad + \left(\frac{\partial^2 \bar{u}_h}{\partial (x_1)^2} \right) \left(\hat{Z}_h(\bar{r}_{-S}) \frac{\partial}{\partial r^s} \phi(\bar{r}_{-S}) \right)^2 + \left(\frac{\partial \bar{u}_h}{\partial x_1} \right) \hat{Z}_h(\bar{r}_{-S}) \frac{\partial^2}{\partial (r^s)^2} \phi(\bar{r}_{-S}), \\ &= 2 \left(\frac{\partial^2 \bar{u}_h}{\partial (x_1)^2} \right) \left(\hat{Z}_h(\bar{r}_{-S}) \right)^2 + \left(\frac{\partial \bar{u}_h}{\partial x_1} \right) \hat{Z}_h(\bar{r}_{-S}) \frac{\partial^2}{\partial (r^s)^2} \phi(\bar{r}_{-S}), \end{aligned} \quad (19)$$

where we used the fact $\frac{\partial}{\partial r^s} \phi(\bar{r}_{-S}) = -1$ for the last equation. And similarly, if $s \neq s'$,

$$\frac{\partial^2}{\partial r^s \partial r^{s'}} \hat{U}_h(\bar{r}_{-S}) = \left(\frac{\partial^2 \bar{u}_h}{\partial (x_1)^2} \right) \left(\hat{Z}_h(\bar{r}_{-S}) \right)^2 + \left(\frac{\partial \bar{u}_h}{\partial x_1} \right) \hat{Z}_h(\bar{r}_{-S}) \frac{\partial^2}{\partial r^s \partial r^{s'}} \phi(\bar{r}_{-S}). \quad (20)$$

Writing (19) and (20) together in a matrix form, we find that the $S - 1$ dimensional Hessian matrix $D\hat{U}_h(\bar{r}_{-S})$ has the form (17). ■

Remark 7 Notice that given the non-sunspot equilibrium $\bar{r}$, which can be found from the case of $S = 1$, Lemma 6 is a characterization result in the primitives of the model: the derivatives of the excess demand function can be computed, in principle, from the first order condition (3), and so ξ_h can be found without solving the equilibrium system for sunspot equilibria.

Therefore, the key parameter to determine the welfare property around the non-sunspot equilibrium is the sign of ξ_h defined by (15). Recall that $\frac{\partial^2 \bar{u}_h}{\partial (x_1)^2}$ is negative by strict concavity and that $\frac{\partial \bar{u}_h}{\partial x_1}$ is positive by monotonicity. So the first term in (15) is

always negative, and the sign of ξ_h depends on the common parameter ζ and household h 's position $Z_h(\bar{r})$. This observation leads to the following result:

Corollary 8 *If $\zeta \geq 0$, for every household h with $\hat{Z}_h(\bar{r}_{-S}) < 0$, the level of equilibrium utility $\hat{U}_h(r_{-S})$ is locally **maximized** at $r_{-S} = \bar{r}_{-S}$, i.e., at the non-sunspot equilibrium; If $\zeta \leq 0$, for every household h with $\hat{Z}_h(\bar{r}_{-S}) > 0$, the level of equilibrium utility $\hat{U}_h(r_{-S})$ is locally **maximized** at $r_{-S} = \bar{r}_{-S}$. Thus the non-sunspot equilibrium constitutes the (locally) most preferred equilibrium allocation at least for **all** borrowers, or for **all** savers.*

Proof. This follows from Lemmas 4, 6 and 5, since $\zeta \geq 0$ or $\zeta \leq 0$ holds. ■

It is of course not surprising that some household's utility must be maximized at the non-sunspot equilibrium; otherwise we would have a sunspot equilibrium improving upon the efficient allocation. The interesting implication of Corollary 8 is that *all* the households on one side of the market must dislike sunspot equilibria.

As discussed in Introduction, we can give an intuitive reason for this result: since increased volatility is always welfare worsening, thus the welfare improving general equilibrium effect must be through the change in average returns. An increase in average returns will hurt *all* the borrowers and a decrease will hurt *all* the savers. It should be noted here that the sign of changes are determined by the second order terms, as the results in this section have shown. This point is not an obvious point.

Remark 9 *Since our analysis is only local, the assumptions on utility functions need to hold only locally around the consumption vector at the non-sunspot equilibrium. So the results hold for quadratic utility models, for instance.*

There are classes of examples where all the households' level of utility is maximized at the non-sunspot equilibrium. We give one of them as an example below.

Example 10 *Let each u_h be a discounted sum of a quadratic utility function: $u_h(x, y) := (a_h x - x^2) + \delta_h (a_h y - y^2)$ where $a_h > 0$ and $\delta_h > 0$, and assume that a non-sunspot equilibrium exists and the regularity assumptions are satisfied around the non-sunspot equilibrium. By inspection of the first order condition, it can be easily checked that the excess demand function Z_h can be written as $Z_h(r) = \alpha_h + \beta_h / \left(\sum_{s=1}^S r^s \right)$ where α_h and β_h are constants. Then $\frac{\partial^2}{\partial r^1 \partial r^2} Z(\bar{r}) = \frac{\partial^2}{\partial (r^1)^2} Z(\bar{r})$ and so $\zeta = 0$ (and $\xi_h =$*

$\frac{\partial^2 \bar{u}_h}{\partial (x_1)^2} (Z_h(\bar{r}))^2 < 0$). Hence by Corollary 8, for every household, the level of equilibrium utility $\hat{U}_h(r_{-S})$ is locally maximized at $r_{-S} = \bar{r}_{-S}$.

Examples where a household's utility is locally minimized at the non-sunspot equilibrium can be constructed.

Example 11 Let $H = 2$, and set $u_1(x, y) := \log(x) + \log(y)$ and $e_1 = (1, 0)$. Then, by direct calculation, we see that for any $r \gg 0$, $Z_1(r) = \frac{1}{2}$. Thus the derivatives of the market excess demand function coincide with those of Z_2 . Given this freedom, it is then possible to find Z_2 (and underlying utility function u_2) such that $\xi_2 > 0$. For instance, set $u_2(x, y) := \log(x) + \log(y)$ and $e_2 = (0, 1)$, and by direct computations it can be shown that $\xi_2 > 0$: this is the leading example of Goenka-Préachac (2006).

Note that it would be easier to construct an example, if utility functions are not identical to each other in the example above. Since the characterization result (Lemma 6) does not require any symmetry across the households, we contend that the existence of households who are benefitted by sunspots does not depend on the symmetry of households' characteristics.

4 Discussion: Who will be benefitted from sunspots?

Proposition 6 shows that the local property of $\hat{U}_h$ at the non-sunspot equilibrium is captured by a single parameter ξ_h defined in (15). We shall first interpret the parameter ξ_h . Re-writing (15), we have

$$\frac{\xi_h}{\frac{\partial \bar{u}_h}{\partial x_1}} = \frac{\frac{\partial^2 \bar{u}_h}{\partial (x_1)^2}}{\frac{\partial \bar{u}_h}{\partial x_1}} (Z_h(\bar{r}))^2 + Z_h(\bar{r}) \zeta. \quad (21)$$

The first term of the right hand side of (21) is always negative by risk aversion. Notice that this term is determined by household h 's preferences given the (non-random) real return $\bar{r}$ at the non-sunspot equilibrium. So this can be interpreted as the direct negative risk effect of increasing volatility. The second term on the other hand represents the general equilibrium effect through markets, since the parameter ζ defined in (11) is determined by the market excess demand. The parameter ζ is multiplied by individual excess demand, and therefore by Proposition 6, other things being equal, it tends to be those households with large net trade is large who are possibly benefitted from sunspots.

Dividing both sides of (21) by $(Z_h(\bar{r}))^2$, and applying Proposition 6, we know that the beneficiaries are *exactly* those households for which the condition

$$-\frac{\frac{\partial^2 \bar{u}_h}{\partial (x_1)^2}}{\frac{\partial \bar{u}_h}{\partial x_1}} < \frac{1}{Z_h(\bar{r})} \zeta \quad (22)$$

holds. Note that $-\left(\frac{\partial^2 \bar{u}_h}{\partial (x_1)^2} / \frac{\partial \bar{u}_h}{\partial x_1}\right) Z_h(\bar{r})$ is the coefficient of relative risk aversion, relative to no trade. So for instance when $\zeta < 0$ and $Z_h(\bar{r}) < 0$, it is the households with high enough coefficient of risk aversion who is benefitted from sunspots, other things being equal.

As is shown in Corollary 8, all the households on one side of the market dislike sunspots, and the households on the other side may or may not be benefitted from sunspots. Indeed, in the Goenka-Préachac model of symmetric two households, the borrower, i.e., the household with $Z_h(\bar{r}) < 0$, is benefitted under some assumption.

Is this general? That is, does it tend to be borrowers who are benefitted from sunspots? We shall study this question in the rest of this section.

The question is whether $\zeta \leq 0$ holds under some reasonable conditions, in view of (22). Recall that by Lemma 5, we know $\zeta \leq 0$ holds if and only if $D^2\phi(\bar{r}_{-S})$ is negative semi-definite, which should be equivalent to the average equilibrium returns falling at the margin by Lemma 3. We shall formally state this point below.

Lemma 12 *$D^2\phi(\bar{r})$ is negative semi-definite if $\phi(r_{-S}) \leq S\bar{r} - \sum_{s=1}^{S-1} r^s$ holds for any small enough r_{-S} in $\mathcal{R}$. $D^2\phi(\bar{r})$ is positive semi-definite if $\phi(r_{-S}) \geq S\bar{r} - \sum_{s=1}^{S-1} r^s$ holds for any small enough r_{-S} in $\mathcal{R}$.*

Proof. If for any small enough r_{-S} in $\mathcal{R}$, $\phi(r_{-S}) \leq S\bar{r} - \sum_{s=1}^{S-1} r^s$ holds, $r_{-S} = \bar{r}_{-S}$ is a maximizer of the function $\phi(r_{-S}) + \sum_{s=1}^{S-1} r^s$, since $\phi(\bar{r}_{-S}) = \bar{r}$. Thus the Hessian matrix of this function, which is just $D^2\phi(\bar{r})$, must be negative semi-definite. The other statement can be shown analogously. ■

This result suggests the following comparative statics question. Starting from a fixed, sure return $\bar{r}$, suppose that the returns get slightly risky in the sense that the average returns is unchanged. Notice that any of such a small risk around $\bar{r}$ can be written as $(r_{-S}, S\bar{r} - \sum_{s=1}^{S-1} r^s)$. We shall discuss two conditions to determine the sign of ζ .

The first condition is about $\frac{\partial}{\partial r^1} Z$. First recall that that $t \mapsto Z(t, t, \dots, t)$ corresponds the standard market excess demand function in an exchange economy with two goods.

The standard law of demand corresponds to Z is increasing in r 's: increasing r means that the relative price of period 0 consumption increases, and thus household saves more. So let us say that the excess demand *respects the law of demand* at the non-sunspot equilibrium if $\frac{\partial}{\partial r^1} Z(\bar{r}) > 0$. Note that by continuity, this means that Z is increasing function of r around $\bar{r}$. Obviously the law of demand is respected for all household, i.e., $\frac{\partial}{\partial r^1} Z_h(\bar{r}) > 0$ for all h , then clearly $\frac{\partial}{\partial r^1} Z(\bar{r}) > 0$ holds.

The law of demand is not a general property in our general equilibrium model. But if there is a *unique* certainty equilibrium, i.e., $Z(t, t, \dots, t) = 0$ if and only if $t = \bar{r}$, then the graph of this function cut zero from below once, and so $\frac{d}{dt} Z(\bar{r}) = \sum_{s=1}^S \frac{\partial Z(\bar{r})}{\partial r^s} > 0$ holds. In general, there exists at least one certainty equilibrium where the law of demand is respected.

The second condition states the behavior of the excess demand against small risks. Let us say that the excess demand *exhibits risk-sensitivity* if for any r in a neighborhood of $\bar{r}$, $Z\left(r_{-S}, S\bar{r} - \sum_{s=1}^{S-1} r^s\right) \geq Z(\bar{r})$ holds.

This condition does not hold in general, and we believe it tends to be more stringent than the law of demand. However, a simple foundation can be given from the view point of an individual portfolio choice problem. To elaborate on this, let us study the simple portfolio problem (2) when u is a discounted sum of a concave utility function v_h ; that is $u_h(x, y) = v_h(x) + \delta_h v_h(y)$.⁸ Then the first order condition (3) simplifies and it can be written as follows:

$$-v'_h(e_h^0 - z_h) + \sum_{s=1}^S \frac{1}{S} v'_h(e_h^1 + z_h r^s) r^s = 0. \quad (23)$$

The second term in the left hand side of (23) is the expected value of the function $x \mapsto v'_h(e_h^1 + z_h x) x$ with respect to a random variable r . Now suppose the expected value of r is $\bar{r}$, i.e., $\sum_{s=1}^S r^s = S\bar{r}$. By the usual argument of risk aversion, if the function $\eta_h(x; z_h) := v'_h(e_h^1 + z_h x) x$ is *convex* in x , $\sum_{s=1}^S \frac{1}{S} v'_h(e_h^1 + z_h r^s) r^s \geq \sum_{s=1}^S \frac{1}{S} v'_h(e_h^1 + z_h \bar{r}) \bar{r}$. On the other hand, given vector $r \gg 0$, the left hand side of (23) is decreasing in z_h . So we conclude that $Z_h(r) > \bar{z}_h := Z_h(\bar{r})$ if r is close enough to $\bar{r}$, if $\eta_h(x; z_h)$ is convex in a neighborhood of $\bar{z}_h$.

By differentiating twice, we see that a sufficient condition for the convexity of $\eta_h(x; z_h)$

⁸The following comparative statics analysis is very standard. See for instance Chapter II of Gollier (2001).

in x around $\bar{r}$ is $v_h''' (e_h^1 + \bar{z}_h \bar{r}) (\bar{z}_h)^2 \bar{r} + 2v_h'' (e_h^1 + \bar{z}_h \bar{r}) \bar{z}_h > 0$. When household h is *prudent*, that is, $v_h''' > 0$, then the inequality is automatically satisfied if $\bar{z}_h < 0$, i.e., h is a borrower. When $\bar{z}_h > 0$, then the convexity depends on the size of relative prudence: dividing the inequality by z_h and collecting terms, we have

$$-\frac{v_h''' (e_h^1 + \bar{z}_h \bar{r})}{v_h'' (e_h^1 + \bar{z}_h \bar{r})} \bar{z}_h \bar{r} > 2. \quad (24)$$

That is, the coefficient of relative prudence (relative to net trade) at the equilibrium consumption is more than 2.

To sum up, if all the house holds are prudent, and the condition (24) holds for every h , then every household's excess demand exhibits risk-sensitivity, and so does the market excess demand.

The next result gives a condition where *if* there is a household who is benefitted from sunspots, it must be a borrower.

Proposition 13 *Suppose at the non-sunspot equilibrium $\bar{r}$, the excess demand function respects the law of demand, and exhibits risk-sensitivity. Then $D^2\phi(\bar{r})$ is negative semi-definite, i.e., $\zeta \leq 0$.*

Proof. Since every household is a risk-sensitive investor, $Z\left(r_{-S}, S\bar{r} - \sum_{s=1}^{S-1} r^s\right) \geq Z(\bar{r}) = 0$, that is, there is an excess demand for the asset for small risk $\left(r_{-S}, S\bar{r} - \sum_{s=1}^{S-1} r^s\right)$. Since around $\bar{r}$ the law of demand holds and Z is locally an increasing function, the return in state S must become less attractive: i.e., $\phi(r_{-S}) \leq S\bar{r} - \sum_{s=1}^{S-1} r^s$. Thus the conclusion follows from Lemma 12. ■

5 Relation to The Goenka-Préachac Condition.

With our results in hand, we shall now examine the Goenka-Préachac model of symmetric two households (see examples 2) more closely. Goenka and Préachac (2006) have shown that household 2's equilibrium utility is locally minimized at the non-sunspot equilibrium, hence household 2 is benefitted from sunspots, if

$$v''\left(\frac{1}{2}\right) + \left(\alpha - \frac{1}{2}\right) v''' \left(\frac{1}{2}\right) > 0, \quad (25)$$

holds.

By symmetry, there is a unique equilibrium in this model, where both households consume $\frac{1}{2}$ in both periods. Write $\varepsilon := (\alpha - \frac{1}{2}) > 0$ which is nothing but the equilibrium excess demand of household 1, the saver. Recall that $v'' < 0$ and $\alpha > \frac{1}{2}$, dividing both sides of (25) by $-v''(\frac{1}{2}) > 0$, we get

$$-\frac{v'''(\frac{1}{2})}{v''(\frac{1}{2})}\varepsilon > 1. \quad (26)$$

That is, one way to read the Goenka-Préchat condition is that the coefficient of *relative prudence* (in terms of net trade) of the *saver* is higher than 1. Goenka and Préchat interpret (26) as *the borrower's* coefficient of *absolute prudence* is greater than $1/\varepsilon$, but then the threshold depends on the size of trade. These observations are very curious since the relevant parameter should be the *borrower's* degree of *risk aversion* relative to the general equilibrium effect by Proposition 6.

We shall demonstrate below that the condition (25) is in fact the sum of the risk aversion term and the general equilibrium effect term, and therefore the Goenka-Préchat condition does correspond to our characterization. We therefore conclude that the two observations given above are misleading.

In their model the first order condition (3) is reduced to the following:

$$-Sv'(e_h^0 - z) + \sum_{s=1}^S v'(e_h^1 + zr^s) = 0. \quad (27)$$

Since the equilibrium consumption is $\frac{1}{2}$ for both households and both periods, all the derivatives of v are to be evaluated at $\frac{1}{2}$, so from now on we omit the reference to $\frac{1}{2}$ to simplify the notation. By differentiating (27) with respect to r^1 , evaluating at the unique equilibrium $\bar{r} = 1$, we find the first derivative of the excess demand function of household h is $-(v' + v''Z_h(\bar{r}))/2Sv''$. Adding these up we have

$$\frac{\partial Z}{\partial r^1}(\bar{r}) = \frac{-v'}{Sv''}, \quad (28)$$

which is always positive. This is of course not surprising since there is a unique non-sunspot equilibrium in their model and so the law of demand must be satisfied at the equilibrium.

Differentiating (27) again in r^s , $s = 1, 2$, and evaluating them at $\bar{r}$, we find the second derivatives of households' excess demand functions. Adding them up, omitting tedious

calculations, we have

$$\frac{\partial^2 Z}{\partial (r^1)^2}(\bar{r}) = \frac{2v' - v''' \varepsilon^2 (S - 1)}{S^2 v''}, \quad (29)$$

$$\frac{\partial^2 Z}{\partial r^1 \partial r^2}(\bar{r}) = \frac{2v' - v''' \varepsilon^2}{S^2 v''}. \quad (30)$$

Using (28), (29) and (30), the general equilibrium effect ζ in (11) is given as follows:

$$\zeta = -\frac{v'''}{v'} \varepsilon^2. \quad (31)$$

Notice that $\zeta < 0$ given prudence $v''' > 0$, so it must be the borrower (household 2) if some household is ever benefitted from sunspots under prudence. The condition (24) does not matter here because of the symmetry of the model; when $v''' > 0$, household 2 may *decrease* the demand against risks. But it turns out that household 1's demand *increases* more because they have the same utility function v , and that the aggregate demand exhibits risk-sensitivity.

Now from (31), we can find ξ_2 in (15), which is shown below:

$$\begin{aligned} \xi_2 &= v'' (-\varepsilon)^2 + v' (-\varepsilon) \zeta \\ &= \varepsilon^2 (v'' + \varepsilon v'''). \end{aligned}$$

By Proposition 6, household 2 is locally benefitted from sunspots if (and almost only if) $\xi_2 > 0$, but this is exactly the condition (31) says. This is what we wanted to demonstrate.

6 Concluding Remarks

To conclude, let us provide a few remarks concerning the restriction of our analysis.

First of all, the assumption of equally probable sunspots is not restrictive. Notice that our analysis does not exclude sunspot equilibria where the return r^s is constant on a subset of states. Since the method of our proofs does not depend on the number of states directly, our results can be readily translated for the case where probabilities are rational numbers. We believe that applying continuity, the case of irrational numbers can be treated as well.

If there are multiple consumption goods, the set of sunspot equilibria is still parametrized by $S - 1$ variables. A complication arises due to changes in equilibrium relative prices of

goods within each spot markets. This will make the analysis potentially involved, but it appears to us that the nature of the analysis will not change. Thus we conjecture that similar results obtain if the assumption of a single consumption good is relaxed.

In the case of multiple goods, however, there is an extension which is not covered in our analysis, which is the case of a real asset. In our case the sunspot equilibria are parametrized by real returns, but this will not happen if the real return is fixed independent of sunspots. Indeed, when there is one consumption good, if the real asset is fixed, there is no sunspot equilibrium as shown in Mas-Colell (1992).⁹ But when there are multiple goods, Gottardi and Kajii (1999) established an existence result of a sunspot equilibrium: a sunspot equilibrium exists because relative prices of goods may depend on sunspots, which in effect makes the real return of the single real asset dependent on sunspots. It is not clear at this point whether or not the technique developed in this paper can be applied in this case.

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⁹See also the discussion in Goenka-Préchac (2006).

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