

On the Singular Solutions of Nonlinear Singular Partial Differential Equations

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Abstract

Let us consider the following nonlinear singular partial differential equation: $(t\partial_t)^m u = F(t, x, \{(t\partial_t)^j \partial_x^\alpha u\}_{j+|\alpha|\leq m, j<m})$ in the complex domain. Denote by S_+ [resp. S_{log}] the set of all the solutions $u(t, x)$ with asymptotics $u(t, x) = O(|t|^a)$ [resp. $u(t, x) = O(1/|\log t|^a)$] (as $t \rightarrow 0$ uniformly in x) for some $a > 0$. Clearly $S_{log} \supset S_+$. The paper gives a sufficient condition for $S_{log} = S_+$ to be valid.

The paper deals with nonlinear singular partial differential equations of the form

$$(E) \quad (t\partial/\partial t)^m u = F\left(t, x, \{(t\partial/\partial t)^j (\partial/\partial x)^\alpha u\}_{j+|\alpha|\leq m, j<m}\right)$$

in the complex domain. In Gérard-Tahara [1] the author has determined all the singular solutions $u(t, x)$ of (E) under the condition that $u(t, x) = O(|t|^a)$ (as $t \rightarrow 0$ uniformly in x) for some $a > 0$.

The present paper investigates singular solutions $u(t, x)$ of (E) under a weaker condition that $u(t, x) = O(1/|\log t|^a)$ (as $t \rightarrow 0$ uniformly in x) for some $a > 0$.

§1. Equations.

Notations: $t \in \mathbb{C}$, $x = (x_1, \dots, x_n) \in \mathbb{C}^n$, $N = \{0, 1, 2, \dots\}$, and $N^* = \{1, 2, \dots\}$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in N^n$ we write $|\alpha| = \alpha_1 + \dots + \alpha_n$ and

$$\left(\frac{\partial}{\partial x}\right)^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

Let $m \in N^*$, $N = \#\{(j, \alpha) \in N \times N^n; j+|\alpha| \leq m, j < m\}$, and write the variable Z as

$$Z = \{Z_{j,\alpha}\}_{\substack{j+|\alpha|\leq m \\ j < m}} \in \mathbb{C}^N.$$

Let $F(t, x, Z)$ be a function in the variables (t, x, Z) defined in a neighborhood of the origin $(0, 0, 0) \in C_t \times C_x^n \times C_Z^N$, and assume the following:

- (A₁) $F(t, x, Z)$ is holomorphic near $(0, 0, 0)$;
- (A₂) $F(0, x, 0) \equiv 0$ near $x = 0$;
- (A₃) $\frac{\partial F}{\partial Z_{j,\alpha}}(0, x, 0) \equiv 0$ near $x = 0$, if $|\alpha| > 0$.

In this paper we always assume the conditions (A₁), (A₂), (A₃), and we will consider the following nonlinear partial differential equation

$$(E) \quad \left(t \frac{\partial}{\partial t}\right)^m u = F\left(t, x, \left\{\left(t \frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u\right\}_{\substack{j+|\alpha| \leq m \\ j < m}}\right)$$

with $u = u(t, x)$ as an unknown function.

For (E) we set

$$C(\lambda, x) = \lambda^m - \sum_{j < m} \frac{\partial F}{\partial Z_{j,0}}(0, x, 0) \lambda^j$$

and denote by $\lambda_1(x), \dots, \lambda_m(x)$ the roots of the equation $C(\lambda, x) = 0$ in λ . These $\lambda_1(x), \dots, \lambda_m(x)$ are called the *characteristic exponents* of (E).

The following is our basic problem:

Problem. Determine all kinds of local singularities which appear in the solutions of (E).

§2. Gérard-Tahara (1993)

Let us recall the result in Gérard-Tahara [1]. Denote:

- $\mathcal{R}(C \setminus \{0\})$ denotes the universal covering space of $C \setminus \{0\}$;
- $S_\theta = \{t \in \mathcal{R}(C \setminus \{0\}); |\arg t| < \theta\}$;
- $S(\varepsilon(s)) = \{t \in \mathcal{R}(C \setminus \{0\}); 0 < |t| < \varepsilon(\arg t)\}$, where $\varepsilon(s)$ is a positive-valued continuous function on $\mathbf{R}_s$;
- $D_r = \{x \in C^n; |x| \leq r\}$;
- $C\{x\}$ denotes the ring of convergent power series in x , or equivalently the ring of germs of holomorphic functions at the origin of C^n .

Definition 1. We denote by $\tilde{\mathcal{O}}_+$ the set of all $u(t, x)$ satisfying the following conditions i) and ii):

- i) $u(t, x)$ is a holomorphic function on $S(\varepsilon(s)) \times D_r$ for some positive-valued continuous function $\varepsilon(s)$ and some $r > 0$;
- ii) there is an $a > 0$ such that for any $\theta > 0$ we have

$$\max_{|x| \leq r} |u(t, x)| = O(|t|^a) \quad (\text{as } t \rightarrow 0 \text{ in } S_\theta).$$

For the characteristic exponents $\lambda_1(x), \dots, \lambda_m(x)$, we set

$$\mu = \#\{i; \operatorname{Re} \lambda_i(0) > 0\}.$$

When $\mu = 0$, this is equivalent to the fact that $\operatorname{Re} \lambda_i(0) \leq 0$ for all $i = 1, \dots, m$. When $\mu \geq 1$, by a renumeration we may assume

$$(1.1) \quad \begin{cases} \operatorname{Re} \lambda_i(0) > 0 & \text{for } 1 \leq i \leq \mu, \\ \operatorname{Re} \lambda_i(0) \leq 0 & \text{for } \mu + 1 \leq i \leq m. \end{cases}$$

Then we already have:

Theorem 1 (Gérard-Tahara [1]). Denote by $\mathcal{S}_+$ the set of all $\tilde{\mathcal{O}}_+$ -solutions of (E). Then we have:

(I) When $\mu = 0$, we have $\mathcal{S}_+ = \{u_0\}$ where $u_0 = u_0(t, x)$ is the unique holomorphic solution of (E) satisfying $u_0(0, x) \equiv 0$.

(II) When $\mu \geq 1$, under (1.1) and the following additional conditions

- 1) $\lambda_i(0) \neq \lambda_j(0)$ for $1 \leq i \neq j \leq \mu$,
- 2) $C(1, 0) \neq 0$,
- 3) $C(i + j_1 \lambda_1(0) + \dots + j_\mu \lambda_\mu(0), 0) \neq 0$ for any $(i, j) \in \mathbf{N} \times \mathbf{N}^\mu$ satisfying $i + |j| \geq 2$ (where $j = (j_1, \dots, j_\mu)$),

we have

$$\mathcal{S}_+ = \left\{ U(\phi_1, \dots, \phi_\mu); (\phi_1, \dots, \phi_\mu) \in (C\{x\})^\mu \right\},$$

where $U(\phi_1, \dots, \phi_\mu)$ is an $\tilde{\mathcal{O}}_+$ -solution of (E) determined by $(\phi_1, \dots, \phi_\mu) \in (C\{x\})^\mu$ and having the expansion of the following form:

$$\begin{aligned} U(\phi_1, \dots, \phi_\mu) &= \sum_{i \geq 1} u_i(x) t^i \\ &+ \phi_1(x) t^{\lambda_1(x)} + \dots + \phi_\mu(x) t^{\lambda_\mu(x)} \\ &+ \sum_{\substack{i+2m|j| \geq k+2m \\ |j| \geq 1 \\ (i, |j|) \neq (0, 1)}} \varphi_{i, j, k}(x) t^{i+j_1 \lambda_1(x) + \dots + j_\mu \lambda_\mu(x)} (\log t)^k. \end{aligned}$$

§3. Problems.

In Theorem 1 we have restricted ourselves to the study of singular solutions in $\tilde{\mathcal{O}}_+$. But, there seems to be a possibility that (E) has singular solutions which do not belong in the class $\tilde{\mathcal{O}}_+$, as is seen in the following example.

Example 1. The equation

$$t \frac{\partial u}{\partial t} = u \left(\frac{\partial u}{\partial x} \right)^k$$

(where $(t, x) \in \mathbb{C}^2$ and $k \in \mathbb{N}^*$) has a family of singular solutions

$$u(t, x) = \left(\frac{1}{k} \right)^{1/k} \frac{x + \alpha}{(c - \log t)^{1/k}}, \quad \alpha, c \in \mathbb{C},$$

which do not belong in the class $\tilde{\mathcal{O}}_+$.

In order to include this kind of singular solutions in our framework, we introduce the following new class of singular solutions:

Definition 2. We denote by $\tilde{\mathcal{O}}_{log}$ the set of all $u(t, x)$ satisfying the following conditions i) and ii):

- i) $u(t, x)$ is a holomorphic function on $S(\varepsilon(s)) \times D_r$ for some positive-valued continuous function $\varepsilon(s)$ and some $r > 0$;
- ii) there is an $a > 0$ such that for any $\theta > 0$ we have

$$\max_{|x| \leq r} |u(t, x)| = O\left(\frac{1}{|\log t|^a}\right) \quad (\text{as } t \rightarrow 0 \text{ in } S_\theta).$$

Clearly we have $\tilde{\mathcal{O}}_{log} \supset \tilde{\mathcal{O}}_+$. Therefore, if we denote by $\mathcal{S}_{log}$ the set of all $\tilde{\mathcal{O}}_{log}$ -solutions of (E), we have $\mathcal{S}_{log} \supset \mathcal{S}_+$.

We will say that $u(t, x)$ is a solution with temperate singularities if $u(t, x) \in \mathcal{S}_+$, and that $u(t, x)$ is a solution with logarithmic singularities if $u(t, x) \in \mathcal{S}_{log} \setminus \mathcal{S}_+$.

Our next problems can be set up as follows:

Problem 1. When does $\mathcal{S}_{log} = \mathcal{S}_+$ hold ?

Problem 2. When does $\mathcal{S}_{log} \neq \mathcal{S}_+$ hold ?

Note that the problem 1 asserts that new singular solutions do not appear and that the problem 2 asserts that new singular solutions really appear in the solutions of (E).

In this paper we will give a partial answer and a conjecture on the problem 1. The problem 2 will be discussed in the forthcoming paper.

§4. A result and a conjecture.

In this section we will give a result on the problem 1 in a general form.

A function $\mu(t)$ on $(0, T)$ is called a *weight function* if it satisfies the following conditions $\mu_1) \sim \mu_3)$:

$$\begin{aligned} \mu_1) \quad & \mu(t) \in C^0((0, T)), \\ \mu_2) \quad & \mu(t) > 0 \text{ on } (0, T) \text{ and } \mu(t) \text{ is increasing in } t, \\ \mu_3) \quad & \int_0^T \frac{\mu(s)}{s} ds < \infty. \end{aligned}$$

By $\mu_2)$ and $\mu_3)$ the condition $\mu(t) \rightarrow 0$ (as $t \rightarrow +0$) is clear. In this paper we impose the additional condition on $\mu(t)$:

$$(4.1) \quad \mu(t) \in C^1((0, T)) \quad \text{and} \quad \left(t \frac{d\mu}{dt}\right)(t) = o(\mu(t)) \quad (\text{as } t \rightarrow +0).$$

The following functions are typical examples:

$$\mu(t) = \frac{1}{(-\log t)^b}, \quad \frac{1}{(-\log t)(\log(-\log t))^c}$$

with $b > 1$, $c > 1$. Note that the function $\mu(t) = t^d$ with $d > 0$ does not satisfy the condition (4.1).

Definition 3. Let $\mu(t)$ be a weight function.

(1) For $a > 0$ we denote by $\tilde{\mathcal{O}}_a(\mu(t))$ the set of all $u(t, x)$ satisfying the following conditions i) and ii):

- i) $u(t, x)$ is a holomorphic function on $S(\varepsilon(s)) \times D_r$ for some positive-valued continuous function $\varepsilon(s)$ and some $r > 0$;
- ii) for any $\theta > 0$ we have

$$\max_{|x| \leq r} |u(t, x)| = O(\mu(|t|)^a) \quad (\text{as } t \rightarrow 0 \text{ in } S_\theta).$$

(2) We define $\tilde{\mathcal{O}}_+(\mu(t))$ by

$$\tilde{\mathcal{O}}_+(\mu(t)) = \bigcup_{a>0} \tilde{\mathcal{O}}_a(\mu(t)).$$

Lemma 1. (1) $\tilde{\mathcal{O}}_{\log} = \tilde{\mathcal{O}}_+(\mu(t))$ if $\mu(t) = 1/(-\log t)^b$ with $b > 1$.

(2) If $\mu(t)$ satisfies (4.1) we have $\tilde{\mathcal{O}}_+ \subset \tilde{\mathcal{O}}_1(\mu(t)) (\subset \tilde{\mathcal{O}}_+(\mu(t)))$.

Proof. (1) is clear. (2) is verified as follows. By (4.1), for any $\varepsilon > 0$ there is a $\delta > 0$ such that $t\mu'_t(t) \leq \varepsilon\mu(t)$ holds on $(0, \delta]$ and therefore we have

$$\frac{d}{dt}(t^{-\varepsilon}\mu(t)) \leq 0 \quad \text{for } 0 < t \leq \delta.$$

Integrating this from t to δ we have

$$\delta^{-\varepsilon} \mu(\delta) \leq t^{-\varepsilon} \mu(t) \quad \text{for } 0 < t \leq \delta$$

and so

$$(4.2) \quad \left(\frac{\mu(\delta)}{\delta^\varepsilon} \right) t^\varepsilon \leq \mu(t) \quad \text{for } 0 < t \leq \delta.$$

Since $\varepsilon > 0$ is arbitrary, (4.2) leads us to the conclusion of (2). $\square$

Denote by $\mathcal{S}_+(\mu(t))$ (resp. $\mathcal{S}_a(\mu(t))$) the set of all $\tilde{\mathcal{O}}_+(\mu(t))$ -solutions of (E) (resp. $\tilde{\mathcal{O}}_a(\mu(t))$ -solutions of (E)). By (2) of Lemma 1 we have

$$\mathcal{S}_+ \subset \mathcal{S}_1(\mu(t)) \subset \mathcal{S}_+(\mu(t)).$$

The following theorem gives a sufficient condition for $\mathcal{S}_+(\mu(t)) = \mathcal{S}_+$ to be valid.

Theorem 2. *Let $\mu(t)$ be a weight function satisfying (4.1). Then, $\mathcal{S}_+(\mu(t)) = \mathcal{S}_+$ is valid if*

$$(4.3) \quad \operatorname{Re} \lambda_i(0) < 0 \quad \text{for all } i = 1, \dots, m$$

or if

$$(4.4) \quad \operatorname{Re} \lambda_i(0) > 0 \quad \text{for all } i = 1, \dots, m.$$

In the case (4.3), by Theorem 1 we have $\mathcal{S}_+ = \{u_0\}$ and therefore the condition $\mathcal{S}_+(\mu(t)) = \mathcal{S}_+$ is equivalent to the fact that the local uniqueness of the solution is valid in $\mathcal{S}_+(\mu(t))$ which is already proved in Tahara [4],[5].

In the case (4.4) the proof of Theorem 2 consists of the following two parts:

C₁) if $u \in \mathcal{S}_+(\mu(t))$ we have $u \in \mathcal{S}_m(\mu(t))$;

C₂) if $u \in \mathcal{S}_m(\mu(t))$ we have $u \in \mathcal{S}_+$.

The proofs of thses C₁) and C₂) will be published in Tahara [6].

Corollary. *If (4.3) or (4.4) holds, we have $\mathcal{S}_{log} = \mathcal{S}_+$.*

Remark. The author believes that the following conjecture is true, though at present he has no idea to prove this conjecture:

Conjecture. $\mathcal{S}_{log} = \mathcal{S}_+$ is valid if

$$(4.5) \quad \operatorname{Re} \lambda_i(0) \neq 0 \quad \text{for all } i = 1, \dots, m.$$

References

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