

GEOMETRIC ASPECTS OF LARGE DEVIATIONS FOR RANDOM WALKS ON A CRYSTAL LATTICE

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The purpose of this talk is to discuss some remarkable relations among convex polyhedra showing up in various circumstances, say Gromov-Hausdorff limits of crystal lattices, homological directions of infinite paths in finite graphs, and the large deviation property (LDP) of random walks on crystal lattices.

Let us start with a simple example. Consider the square lattice $\mathbb{Z}^2$ as a metric space with the graph-distance d . Given a positive constant ϵ , we have the metric space $(\mathbb{Z}^2, \epsilon d)$ homothetic to $(\mathbb{Z}^2, d)$. We then ask what the limit $\lim_{\epsilon \downarrow 0} (\mathbb{Z}^2, \epsilon d)$ is as ϵ tends to zero? The answer is, as we may anticipate, the Euclidean 2-space $\mathbb{R}^2$ with the taxi-cab distance. In this view, it is natural to ask what happens for a more general infinite graph with periodicity. Graphs we would like to consider are *crystal lattices* which are defined to be abelian covering graphs of finite graphs.

Theorem 1. (1) *(a special case of Gromov's result [2]) Let (X, d) be a crystal lattice with the graph-distance. There exists a normed linear space $(L, \|\cdot\|)$ of finite dimension such that*

$$\lim_{\epsilon \downarrow 0} (X, \epsilon d) = (L, d_1),$$

where $d_1(x, y) = \|x - y\|$.

(2) *The unit ball $\overline{D} = \{x \in L \mid \|x\| \leq 1\}$ is a polyhedron.*

Let X_0 be a finite connected graph. We denote the set of all oriented edges by E_0 . Let $c = (e_1, e_2, \dots)$ be an infinite path in X_0 . If the limit

$$\gamma(c) = \lim_{n \rightarrow \infty} \frac{1}{n} (e_1 + \dots + e_n)$$

exists in the 1-chain group $C_1(X_0, \mathbb{R})$, then $\gamma(c)$ is said to be the *homological direction* of c . It is easy to see that $\gamma(c)$ is a 1-cycle so that $\gamma(c) \in H_1(X_0, \mathbb{R})$. To describe the range of homological directions, define the ℓ^1 -norm on $C_1(X_0, \mathbb{R})$ by

$$\left\| \sum_{e \in E_0^+} a_e e \right\|_1 = \sum_{e \in E_0^+} |a_e|,$$

where E_0^+ is an orientation of X_0 .

Theorem 2. *The range of homological directions coincides with*

$$\mathcal{D}_0 = \{\alpha \in H_1(X_0, \mathbb{R}) \mid \|\alpha\|_1 \leq 1\}.$$

Note that $\mathcal{D}_0$ is a convex polyhedron in $H_1(X_0, \mathbb{R})$, symmetric around the origin.

The convex polyhedron $\mathcal{D}_0$ is related to the combinatorics of the finite graph X_0 in the following way.

Theorem 3. 1. $\mathcal{D}_0$ is “rational” in the sense that all extreme points of $\mathcal{D}_0$ are in $H_1(X_0, \mathbb{Q})$.

2. $\alpha \in H_1(X_0, \mathbb{Q})$ is a vertex of $\mathcal{D}_0$ if and only if $\alpha = c/\|c\|_1$ for a circuit (simple closed path) c in X_0 .

We shall go back to crystal lattices. To be exact, a crystal lattice X is a connected infinite graph X on which a free abelian group Γ acts as an automorphism group with a finite quotient $X_0 = \Gamma \backslash X$.

A piecewise linear map Φ of X into $\Gamma \otimes \mathbb{R} \cong \mathbb{R}^k$ ($k = \text{rank } \Gamma$) is said to be a *periodic realization* if it satisfies $\Phi(\sigma x) = \Phi(x) + \sigma$. We consider a random walk on X given by a Γ -invariant transition probability p . Given a periodic realization Φ , we put $\xi_n(c) = \Phi(x_n(c))$ for an infinite path c . We thus obtain a $\Gamma \otimes \mathbb{R}$ -valued process $\{\xi_n\}_{n=0}^\infty$.

Now comes a discussion about large deviations principle for the process $\{\xi_n\}$.

Theorem 4. *A large deviation property holds for $\{\xi_n\}$. Namely, there exists $I : \Gamma \otimes \mathbb{R} \rightarrow [0, \infty]$, (which is called entropy function) and satisfies, for $A \subset \Gamma \otimes \mathbb{R}$,*

$$\begin{aligned} -I(\text{int} A) &\leq \liminf_{n \rightarrow \infty} \frac{1}{n} \log P_x\left(\frac{1}{n} \xi_n \in \text{int} A\right) \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log P_x\left(\frac{1}{n} \xi_n \in \overline{A}\right) \leq -I(\overline{A}), \end{aligned}$$

where $I(K) = \inf\{I(z) \mid z \in K\}$ for $K \subset \Gamma \otimes \mathbb{R}$.

To give more details, we let

$$\langle \cdot, \cdot \rangle : (\Gamma \otimes \mathbb{R}) \times \text{Hom}(\Gamma, \mathbb{R}) \rightarrow \mathbb{R}$$

be the pairing map between $\Gamma \otimes \mathbb{R}$ and its dual $(\Gamma \otimes \mathbb{R})^* = \text{Hom}(\Gamma, \mathbb{R})$, and let $\rho : H_1(X_0, \mathbb{Z}) \rightarrow \Gamma$ be the surjective homomorphism coming from the covering map $X \rightarrow X_0$.

Lemma 5. *Let $\chi \in \text{Hom}(\Gamma, \mathbb{R})$.*

1. The limit $\lim_{n \rightarrow \infty} \frac{1}{n} \log E(e^{\langle \xi_n, \chi \rangle}) = c(\chi)$ exists. Here $e^{c(\chi)}$ is the maximal positive eigenvalue of the “twisted” transition operator associated with χ .
2. The function c is real analytic, and the hessian of c is strictly positive definite everywhere. Thus the correspondence $\chi \mapsto (\nabla c)(\chi)$ is a diffeomorphism of $\text{Hom}(\Gamma, \mathbb{R})$ onto an open subset U in $\Gamma \otimes \mathbb{R}$.

By using a general recipe in the theory of large deviation (see [1]), with the entropy function $I : \Gamma \otimes \mathbb{R} \rightarrow [0, \infty]$ defined by

$$I(\mathbf{z}) = \sup_{\chi} (\langle \mathbf{z}, \chi \rangle - c(\chi)),$$

we have the LDP for our R.W. It should be noted that the function I assumes finite values on U . We also see

Proposition 6. $\bar{U} = \rho_{\mathbb{R}}(\mathcal{D}_0)$, and hence is independent of p . Moreover

$$\bar{U} = \{\mathbf{x} \in \Gamma \otimes \mathbb{R} \mid \|\mathbf{x}\|_1 \leq 1\},$$

where

$$\|\mathbf{x}\|_1 = \inf \{\|\alpha\|_1 \mid \alpha \in H_1(X_0, \mathbb{R}), \rho_{\mathbb{R}}(\alpha) = \mathbf{x}\}.$$

Therefore $\bar{U}$ is a convex polyhedron, symmetric around the origin, and rational in the sense that the vertices of $\bar{U}$ are in $\Gamma \otimes \mathbb{Q}$.

Finally, we come back to the theorem mentioned in the begining. As an application of the LDP, we have

Theorem 7.

$$\lim_{\epsilon \downarrow 0} (X, \epsilon d) = (\Gamma \otimes \mathbb{R}, d_1),$$

where $d_1(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_1$.

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