

Viscosity solutions for the level set formulation of the crystalline mean curvature flow

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Abstract

In this short note we announce the results of [25], coauthored with Y. Giga (University of Tokyo), concerning a new notion of solutions for the level set formulation of the crystalline mean curvature flow of surfaces in three dimensions. We establish a comparison principle, the stability with respect to an approximation by regularized problems, and the unique existence of the level set flow for arbitrary bounded three dimensional crystals. Most of our approach applies to an arbitrary dimension, except for the construction of faceted test functions.

This note is based on a recent joint work [25] with Y. Giga (University of Tokyo). Here we present a simplified exposition of our results concerning the level set formulation of the crystalline mean curvature flow in three dimensions that appeared in [25], but we omit the details and proofs and instead refer the reader to that paper.

1 Introduction

The crystalline mean curvature flow was proposed independently by S. B. Angenent and M. E. Gurtin [3] and by J. Taylor [35] to model the anisotropic motion of antiphase boundaries in material science. It can be regarded as an example of a mean curvature flow under a Finsler metric [12]. Many properties of the crystalline mean curvature were already established by the original authors, and there is a large amount of literature devoted to its study. However, even the local-in-time unique existence of solutions had been a long-standing open problem. For a full bibliography we refer the reader to [6, 22, 25]. It was our aim in [25] to address this issue for surfaces in three dimensions.

Let $n \in \mathbb{N}$ be the dimension. The anisotropic mean curvature $\kappa_\sigma : \Gamma \rightarrow \mathbb{R}$ is defined as the first variation of the surface energy functional

$$\mathcal{F}(\Gamma) := \int_{\Gamma} \sigma(\nu) d\mathcal{H}^{n-1},$$

where $\Gamma \subset \mathbb{R}^n$ is a smooth closed oriented surface with the unit outer normal ν , $\sigma : \mathcal{S}^{n-1} \rightarrow (0, \infty)$ is the surface energy density defined on the unit sphere $\mathcal{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$ and $\mathcal{H}^{n-1}$ is the $(n-1)$ -dimensional Hausdorff measure. We will for convenience suppose that σ is defined on $\mathbb{R}^n$ as the one-homogeneous function

$$\sigma(p) = |p| \sigma\left(\frac{p}{|p|}\right), \quad p \in \mathbb{R}^n \setminus \{0\},$$

and $\sigma(0) = 0$. If σ is C^2 away from the origin such that σ^2 is strictly convex, it can be shown that

$$\kappa_\sigma = -\operatorname{div}_\Gamma(\nabla_p \sigma(\nu)), \quad (1.1)$$

where $\operatorname{div}_\Gamma$ is the surface divergence operator on Γ [21] and $\nabla_p \sigma$ is the gradient of σ , itself a zero-homogeneous function. If $\sigma(p) = |p|$ then κ_σ is nothing but the standard mean curvature.

In this note we are primarily concerned with the *crystalline* case: we assume that σ is a convex, positively one-homogeneous, piece-wise linear function, zero only at the origin, and we refer to such σ as *crystalline*. These functions (not necessarily one-homogeneous) are also known as *polyhedral* functions in convex analysis [32]. We then call κ_σ the *crystalline (mean) curvature* of Γ . Note that (1.1) cannot be used to define κ_σ anymore and κ_σ must be understood in a variational sense [6, 10, 11]. We will give details in Section 2.

Now consider a family of closed surfaces $\{\Gamma_t\}_{t \geq 0}$ evolving under the surface velocity law

$$V = f(\nu, \kappa_\sigma), \quad (1.2)$$

where V , ν , and κ_σ are respectively the normal velocity of Γ_t in the direction of ν , the outer unit normal vector to Γ_t , and the crystalline curvature of Γ_t . We assume that $f : \mathcal{S}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, non-decreasing in the second variable. A typical example is the crystalline curvature flow

$$V = \kappa_\sigma. \quad (1.3)$$

A characteristic feature of the crystalline mean curvature flow (1.2) with crystalline σ is the appearance of flat parts of the surface in the directions in which σ is not differentiable, the so-called *crystal facets*. On these facets, κ_σ is a nonlocal quantity. In general it should be a constant so that facets move with constant velocity and are preserved. This is indeed the case in two dimensions, however, it is known that κ_σ might have jumps and then facet breaking or bending occurs in three dimensions [9]. This makes analysis of the flow rather difficult.

In dimensions $n \geq 3$, even if $\sigma(p) = |p|$ and (1.3) is the standard mean curvature flow, the solution of (1.3) might develop singularities like pinch off in finite time, even when starting with a smooth initial surface. Therefore a suitable notion of a generalized solution is necessary. There are a few standard approaches to the solutions of (1.3) in dimensions three and higher. One of the first ones was a *varifold solution* introduced by K. Brakke [14], further developed by [27] and [34]. However, this approach seems to be still limited to the isotropic mean curvature flow.

Another approach uses the *minimizing movements* scheme developed by Almgren, Taylor and Wang [1] and Luckhaus and Sturzenhecker [30]. This approach provides a global-in-time solution, but uniqueness was shown only in very specific cases like convex initial data [7, 8, 16]. After the work on our paper [25] had been finished, we found out about a result of Chambolle, Morini, Ponsiglione [17] where the authors define a weak solution of

$$V = \sigma(\nu)\kappa_\sigma \quad (1.4)$$

for crystalline σ in terms of the distance function to the evolving surface in the spirit of [33], but in the distributional sense that appeared in [16], and prove a comparison principle in an *arbitrary* dimension. The existence is then shown using the Almgren-Taylor-Wang scheme. However, their approach seems to be currently restricted to the very specific form of the equation (1.4) where the mobility is proportional to σ and does not apply to equations like (1.3) or $V = \kappa_\sigma + 1$.

We take the *level set method* approach. It was introduced independently by Chen, Giga and Goto [18] and Evans and Spruck [19] for geometric motions like the anisotropic mean curvature flow (1.2). The idea is to take a function $u : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$ whose zero level set is equal to Γ_t , that is, $\Gamma_t = \{x \in \mathbb{R}^n : u(x, t) = 0\}$. Then u is a solution of a geometric partial differential equation (PDE) of parabolic type. It turns out that one can generalize the theory of viscosity solutions into this context and prove the well-posedness of the initial value problem. The zero level set of any solution of the PDE is then a generalized solution of (1.2). It is unique if the zero level set of u does not *fatten*, that is, if its Lebesgue measure is zero. The main merit of this approach, in contrast to the above variational approaches, is that it allows for very general f in (1.2).

To introduce the *level set formulation* of (1.2), we will use the one-homogeneous extension of f ,

$$F(p, \xi) := -|p|f\left(-\frac{p}{|p|}, \xi\right). \quad (1.5)$$

Suppose that $u : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$ is a function whose every level set moves with the velocity law (1.2), in the sense that $V = u_t/|\nabla u|$ and $\nu = -\nabla u/|\nabla u|$. Here $\nabla u := (\partial u/\partial x_1, \dots, \partial u/\partial x_n)$ is the space gradient and $u_t = \partial u/\partial t$. Then u formally satisfies

$$u_t + F(\nabla u, \operatorname{div} \partial W(\nabla u)) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty), \quad (1.6)$$

where $W(p) := \sigma(-p)$. ∂W denotes the *subdifferential* of W , which in the case of differentiable W is just the gradient of W . We explain how to understand the quantity $\operatorname{div} \partial W(\nabla u)$ in Section 2 below. Since ∂W has jumps as W is piece-wise linear, $\operatorname{div} \partial W(\nabla u)$ has a very strong singularity which makes this operator nonlocal. Therefore (1.6) is not a usual PDE. We say that (1.6) is a level set formulation of the surface evolution problem (1.2).

In [22, 23], we have together with M.-H. Giga and Y. Giga succeeded in extending the theory of viscosity solutions to the equation (1.6) in an arbitrary dimension with W being a positively one-homogeneous function so that the level set $\{W \leq 1\}$ is a smooth strictly convex set containing the origin in its interior. Then (1.6) is an anisotropic total variation flow of non-divergence type. It corresponds to a graph formulation of the crystalline curvature flow with an anisotropy $\sigma(\nu) = |\nu'| + |\nu_n|$, where $|\nu'|$ is the Euclidean norm on $\mathbb{R}^{n-1}$, and $\nu = (\nu', \nu_n)$, $\nu' \in \mathbb{R}^{n-1}$.

The current note summarizes our most recent work [25], coauthored with Y. Giga. For a general crystalline W we introduced a new notion of viscosity solutions for (1.6) and for dimensions $n = 2, 3$ proved the comparison principle Theorem 4.1, stability Theorem 5.2 and well-posedness of the initial value problem. Given an initial Γ_0 that is a boundary of a bounded open set in $\mathbb{R}^n$, $n = 2, 3$, if u_0 is a continuous function, constant outside of a bounded set, with zero level set Γ_0 and $\{\Gamma_t\}_{t \geq 0}$ are the zero level sets of the unique viscosity solution u of (1.6) with initial data u_0 , $\tilde{\Gamma}_t := \{x : u(x, t) = 0\}$, we call $\{\Gamma_t\}_{t \geq 0}$

the level set flow of (1.2). Using the standard arguments, it can be shown that $\{\Gamma_t\}_{t \geq 0}$ is independent of the choice of u_0 . However, it is known that even if Γ_0 is a surface, the level set Γ_t might fatten and thus might not be a surface for some $t > 0$ [18, 21]. We have the following well-posedness theorem (stated for dimension $n = 3$ for simplicity).

1.1 Theorem (Well-posedness). *Let $n = 3$ and let σ be crystalline in $\mathbb{R}^3$. Assume that $f = f(p, \xi)$ is continuous on $S^2 \times \mathbb{R}^3$ and non-decreasing in the second variable. Furthermore, assume that $f(p, \xi)/(|\xi| + 1)$ is bounded in $S^2 \times \mathbb{R}^3$. If Γ_0 is the boundary of a bounded open set in $\mathbb{R}^3$ then there exists a global unique level set flow $\{\Gamma_t\}_{t \geq 0}$ of (1.2).*

Note that the assumption of linear growth of f in its second variable is only for simplicity, and can be removed using the techniques in [21, 26, 28]. We also should point out that it is rather straightforward to allow for f depending on time. On the other hand, the dependence of f on the space variable is much more subtle, and a modification of the definition of κ_σ itself is probably necessary, as in [24].

Now we consider the approximation of (1.2) by problems

$$V = f(\nu, \kappa_{\sigma_\varepsilon}), \quad (1.7)$$

where $\sigma_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}$ are convex positively one-homogeneous surface energy densities that are smooth on $\mathbb{R}^n \setminus \{0\}$. We can prove the following stability result (again stated for dimension $n = 3$).

1.2 Theorem (Stability). *Let $n = 3$. Under the assumption of Theorem 1.1, let u be a viscosity solution of (1.2) with initial data $u_0 \in C(\mathbb{R}^3)$ such that $u_0(x) = -c$ for $|x| \geq R$ with some R and $c > 0$. Assume that σ_ε is smooth in $\mathbb{R}^3 \setminus \{0\}$, convex and one-homogeneous and $\sigma_\varepsilon \rightarrow \sigma$ uniformly on S^2 . Let u^ε be a viscosity solution of (1.2) with $W = W_\varepsilon(p) = \sigma_\varepsilon(-p)$, with initial data u_0^ε such that $u_0^\varepsilon(x) = -c$ for $|x| \geq R$. Assume that $u_0^\varepsilon \rightarrow u_0$ uniformly. Then u^ε converges locally uniformly to u in $\mathbb{R}^3 \times [0, \infty)$.*

Outline

The discussion of our approach follows. Section 2 covers the definition of crystalline curvature on facets. This is then used to introduce the viscosity solutions in Section 3. We then briefly explain the comparison principle and stability in Section 4 and Section 5, respectively. Finally, we present a simple example of test functions in Section 6.

2 Crystalline curvature

We take a variational approach to understanding the crystalline curvature κ_σ for a crystalline energy density σ [6, 10, 11]. This is motivated by viewing the mean curvature as the quantity that drives the gradient flow of the surface energy $\mathcal{F}$ with respect to the L^2 -metric on the surface. Consider the classical gradient flow problem for some convex lower-semicontinuous energy $E : H \rightarrow \mathbb{R}$, formulated as the subdifferential inclusion

$$\frac{d}{dt}u(t) \in -\partial E(u(t)), \quad (2.1)$$

where H is a Hilbert space with inner product $(\cdot, \cdot)$ and $\partial E(\psi)$ for $\psi \in H$ denotes the subdifferential of E defined as

$$\partial E(\psi) := \{v \in H : E(\psi + h) - E(\psi) \geq (h, v) \text{ for all } h \in H\}.$$

This set is closed and convex, but possibly nonempty. However, it is known [15, 29] that the right derivative of the solution of (2.1) equals to $-\partial^0 E(u(t))$, where $\partial^0 E(\psi)$ is the *minimal section* (canonical restriction) of the subdifferential,

$$\partial^0 E(\psi) := \arg \min_{v \in \partial E(\psi)} \|v\|_H.$$

That is, it is the unique element of the subdifferential with the smallest norm. In particular, $\partial E(u(t))$ is nonempty for every $t > 0$.

We therefore interpret the quantity $-\operatorname{div} \partial W(\nabla u)$ in (1.6) as the minimal section of the subdifferential of the total variation energy $E : L^2(\Omega) \rightarrow \mathbb{R}$ defined as

$$E(\psi) := \begin{cases} \int_{\Omega} W(D\psi) dx, & \psi \in L^2(\Omega) \cap BV(\Omega), \\ +\infty, & \text{otherwise,} \end{cases} \quad (2.2)$$

for some $\Omega \subset \mathbb{R}^n$, where $BV(\Omega)$ is the space of functions of bounded variation on Ω . The definition of E has to be clarified when $D\psi$ is merely a measure: we therefore understand E as the closure (that is, the lower semi-continuous envelope) of the total variational functional defined for functions on $W^{1,1}(\Omega) \cap L^2(\Omega)$ [13, 20]. In particular, note that the functional defined only for $W^{1,1}$ -functions is not lower semi-continuous and we need to consider BV -functions.

To avoid issues with boundary values, we will always consider only Ω with the flat torus topology. In other words, we only apply E to periodic functions. It turns out that this is sufficient for our purposes since even though the crystalline mean curvature is a nonlocal quantity, its nonlocality is restricted to facets (flat parts) of the surface. Furthermore, we only need to deal with bounded facets.

We shall define the crystalline curvature on *facets* (flat parts) of Lipschitz functions. For that we need a suitable description of facets that is given by the notion of pairs that we have introduced in [22, 23, 25]. A simple closed (compact) set in $\mathbb{R}^n$ is insufficient to specify a facet because we also need the information about the behavior of the function in a neighborhood of the facet to be able to define the crystalline mean curvature.

2.1 Definition. Let $k \in \mathbb{N}$. We say that (A_-, A_+) is a bounded open pair in $\mathbb{R}^k$ if $A_{\pm} \subset \mathbb{R}^k$ are open sets, $A_- \cap A_+ = \emptyset$, and either A_-^c or A_+^c is bounded.

We say that $A_-^c \cap A_+^c$ is the facet of the bounded open pair (A_-, A_+) .

Let (A_-, A_+) and (B_-, B_+) be two bounded open pairs in $\mathbb{R}^k$. We say that

$$(A_-, A_+) \preceq (B_-, B_+)$$

if

$$B_- \subset A_- \quad \text{and} \quad A_+ \subset B_+.$$

The relation $\preceq$ introduces a partial order on the collection of bounded open pairs.

For the purpose of perturbation of facets, we introduce the notion of generalized neighborhood of sets in $\mathbb{R}^k$ and of bounded open pairs.

2.2 Definition (cf. [22, 25]). *For any set $E \subset \mathbb{R}^k$ and $\rho \in \mathbb{R}$ the generalized neighborhood is defined as*

$$\mathcal{U}^\rho(E) := \begin{cases} E + \overline{B}_\rho(0) & \rho > 0, \\ E & \rho = 0, \\ \{x \in E : \overline{B}_{|\rho|}(x) \subset E\} & \rho < 0, \end{cases}$$

where $B_\rho(x) := \{y \in \mathbb{R}^k : |y - x| < \rho\}$ is the open ball and $\overline{B}_\rho(x)$ is its closure.

For a bounded open pair (A_-, A_+) in $\mathbb{R}^k$ we introduce the generalized neighborhood

$$\mathcal{U}^\rho(A_-, A_+) := (\mathcal{U}^{-\rho}(A_-), \mathcal{U}^\rho(A_+)).$$

Finally, to precisely state what we mean by a facet of a function, we introduce the support functions of pairs.

2.3 Definition (cf. [22, 25]). *A Lipschitz function $\psi \in \text{Lip}(\mathbb{R}^k)$ is called a support function of a bounded open pair (A_-, A_+) in $\mathbb{R}^k$ if*

$$\psi(x) \begin{cases} > 0 & x \in A_+, \\ = 0 & x \in A_-^c \cap A_+^c, \\ < 0 & x \in A_-. \end{cases}$$

The crystalline mean curvature is given as the divergence of a certain vector field, the *Cahn-Hoffman vector field*, that lives on the facet, and whose divergence minimizes the L^2 -norm among all possible Cahn-Hoffman vector fields [6, 10, 11].

We make the following definition: given $\hat{p} \in \mathbb{R}^n$, we define V to be the smallest subspace of $\mathbb{R}^n$ such that $V + \xi$ contains $\partial W(\hat{p})$ for some $\xi \in \mathbb{R}^n$. Note that $V + \xi$ is the affine hull of $\partial W(\hat{p})$, denoted as $\text{aff } \partial W(\hat{p})$.

Set $k = \dim V$, $U = V^\perp$ and let $\mathcal{T}_V : \mathbb{R}^k \rightarrow V$, $\mathcal{T}_U : \mathbb{R}^{n-k} \rightarrow U$ be isomorphisms. For $x \in \mathbb{R}^n$ we use x', x'' to denote the unique points $x' \in \mathbb{R}^k$, $x'' \in \mathbb{R}^{n-k}$ such that

$$x = \mathcal{T}_V x' + \mathcal{T}_U x''. \quad (2.3)$$

Note that if $k = 0$ then W is differentiable at $\hat{p}$. However, we are interested in the case $k \geq 1$ when W has a “corner” at $\hat{p}$ and the crystal has a facet in this direction. It turns out that W is locally affine in the directions in U , and has a “corner” in the directions in V . We therefore want to only consider its behavior in V . This is where the sliced energy density comes in. We introduce $W_{\hat{p}}^{\text{sl}} : \mathbb{R}^k \rightarrow \mathbb{R}$ as

$$W_{\hat{p}}^{\text{sl}}(p') := \lim_{\lambda \rightarrow 0} \frac{W(\hat{p} + \lambda \mathcal{T}_V p') - W(\hat{p})}{\lambda} \quad \text{for } p' \in \mathbb{R}^k.$$

It is well known that $W_{\hat{p}}^{\text{sl}}$ is a positively one-homogeneous convex function, see [25] for details.

This energy density then induces the sliced energy functional $E_{\hat{p}}^{\text{sl}} : L^2(\Gamma') \rightarrow \mathbb{R}$ given by

$$E_{\hat{p}}^{\text{sl}}(\psi) := \begin{cases} \int_{\Gamma'} W_{\hat{p}}^{\text{sl}}(D\psi) dx, & \psi \in L^2(\Gamma') \cap BV(\Gamma'), \\ +\infty, & \text{otherwise.} \end{cases}$$

We consider Γ' to be a k -dimensional flat torus, so that ψ is a periodic function on $\mathbb{R}^k$.

With these definitions at hand, we can finally define the crystalline mean curvature on k -dimensional facets with slope $\hat{p}$ as the minimal section of the subdifferential of the sliced energy $E_{\hat{p}}^{\text{sl}}$. The subdifferential can be characterized as divergences of certain vector fields (Cahn-Hoffman vector fields). Let therefore Γ' be a k -dimensional flat torus. We introduce the set of Cahn-Hoffman vector fields for a Lipschitz function $\psi \in \text{Lip}(\Gamma')$ as

$$CH_{\hat{p}}^{\text{sl}}(\psi; \Gamma') := \{z \in X_2(\Gamma') : z(x) \in \partial W_{\hat{p}}^{\text{sl}}(\nabla \psi(x)) \text{ a.e. } x \in \Gamma'\}.$$

Using the notation from [4], the space $X_2(\Gamma')$ is the space of bounded vector fields with L^2 divergence,

$$X_2(\Gamma') := \{z \in L^\infty(\Gamma'; \mathbb{R}^k) : \text{div } z \in L^2(\Gamma')\}.$$

The subdifferential of $E_{\hat{p}}^{\text{sl}}$ then can be characterized for $\psi \in \text{Lip}(\Gamma')$ [2, 31] as

$$\partial E_{\hat{p}}^{\text{sl}}(\psi) = -\text{div } CH_{\hat{p}}^{\text{sl}}(\psi; \Gamma') := \{-\text{div } z : z \in CH_{\hat{p}}^{\text{sl}}(\psi; \Gamma')\}.$$

Suppose that (A_-, A_+) is a bounded open pair in $\mathbb{R}^k$ and $\psi \in \text{Lip}(\mathbb{R}^k)$ is its support function. Since the facet $A_-^c \cap A_+^c$ is bounded, we can find $L > 0$ sufficiently large so that $A_-^c \cap A_+^c \subset (-L/2, L/2)^k$. Then we can find an $2L$ -periodic Lipschitz function $\hat{\psi}$ that $\hat{\psi} = \psi$ on a neighborhood of the facet $A_-^c \cap A_+^c$ and $\hat{\psi} \neq 0$ on $[-L, L]^k \setminus (A_-^c \cap A_+^c)$. We set $\Gamma' = [-L, L]^k$ with the flat torus topology, so that $\hat{\psi} \in \text{Lip}(\Gamma')$.

Now if there exists such a periodic function $\hat{\psi}$ for which $CH_{\hat{p}}^{\text{sl}}(\hat{\psi}; \Gamma')$ is nonempty, we say that ψ is a $\hat{p}$ -admissible support function of pair (A_-, A_+) and we write $\psi \in \mathcal{D}(\Lambda_{\hat{p}})$. We then define the crystalline mean curvature on the facet $A_-^c \cap A_+^c$ for the support function ψ as

$$\Lambda_{\hat{p}}[\psi] := -\partial^0 E_{\hat{p}}^{\text{sl}}(\hat{\psi}; \Gamma') \quad \text{on } A_-^c \cap A_+^c.$$

In other words, the crystalline mean curvature $\Lambda_{\hat{p}}[\psi]$ is given as the divergence of the Cahn-Hoffman vector field for $\hat{\psi}$ that minimizes the L^2 -norm of the divergence.

It takes some work to prove that $\Lambda_{\hat{p}}$ is well-defined, in particular, that it does not depend on the choice of the periodic extension $\hat{\psi}$. We can also show that a Cahn-Hoffman vector field exists for some periodic extension if and only if a Cahn-Hoffman vector field exists for ψ on the neighborhood of the facet $A_-^c \cap A_+^c$. For details we refer the reader to [25].

It is easy to see that any vector field $z_{\min} \in CH_{\hat{p}}^{\text{sl}}(\hat{\psi}; \Gamma')$ that minimizes the L^2 -norm of $\text{div } z$ should satisfy the Euler-Lagrange equation $\nabla(\text{div } z_{\min}) = 0$ on the facet $A_-^c \cap A_+^c$. If such a vector field exists, then $\Lambda_{\hat{p}}[\psi] = \text{div } z_{\min}$ is a constant on the facet. However, it is known that there might not be any such vector field, in which case $\Lambda_{\hat{p}}[\psi]$ is not a constant on the facet, and in fact it might have jumps. This leads to facet breaking or

bending. In general, $\Lambda_{\hat{p}}[\psi]$ is only a BV-function [10, 11]. This is not a problem for our viscosity solutions, however, we have to be careful when trying to use the value of $\Lambda_{\hat{p}}[\psi]$ at a point. This is one of the reasons for giving the faceted test in (3.2) in terms of the essential infimum or supremum of $\Lambda_{\hat{p}}[\psi]$ over a small ball.

A pair that has at least one $\hat{p}$ -admissible support function is called $\hat{p}$ -admissible.

2.4 Definition. Let $\hat{p} \in \mathbb{R}^n$ and $k = \dim \text{aff } \partial W(\hat{p})$. A bounded open pair (A_-, A_+) in $\mathbb{R}^k$ is called $\hat{p}$ -admissible if there exists a support function ψ of this pair such that $\psi \in \mathcal{D}(\Lambda_{\hat{p}})$.

It turns out that the crystalline mean curvature on a facet of an admissible pair does not depend on the choice of the admissible support function. This follows from the comparison principle for the crystalline curvature $\Lambda_{\hat{p}}$.

2.5 Proposition (Comparison principle for $\Lambda_{\hat{p}}$). Let $\hat{p} \in \mathbb{R}^n$ and $k = \dim \text{aff } \partial W(\hat{p})$. Suppose that $(A_{1,-}, A_{1,+})$ and $(A_{2,-}, A_{2,+})$ are two $\hat{p}$ -admissible pairs in $\mathbb{R}^k$. If the pairs are ordered in the sense of

$$(A_{1,-}, A_{1,+}) \preceq (A_{2,-}, A_{2,+}),$$

then for any two $\hat{p}$ -admissible support functions ψ_1 and ψ_2 of the respective pairs we have

$$\Lambda_{\hat{p}}[\psi_1](x) \leq \Lambda_{\hat{p}}[\psi_2](x) \quad \text{a.e. } x \in A_{1,-}^c \cap A_{1,+}^c \cap A_{2,-}^c \cap A_{2,+}^c. \quad (2.4)$$

This result follows from the comparison principle for the resolvent problems for the corresponding energy $E_{\hat{p}}^{\text{sl}}$, and the resolvent approximation of the minimal section of the subdifferential. For details again see [25].

3 Viscosity solutions

Viscosity solutions are defined as the functions that satisfy a comparison principle with a certain class of sufficiently “regular” test functions for which the differential equation can be understood in the classical sense. The particular choice of test functions is crucial for the well-posedness of the Cauchy problem for (1.6). Note that smooth functions are not sufficient for this purpose since they might not be admissible in the sense of Section 2 and therefore we cannot define the crystalline curvature for them. For example, consider a function in one dimension that has a strict local minimum or maximum. Therefore the test functions from the following definition play a crucial role. Recall the notation x', x'' from (2.3).

3.1 Definition. Let $(\hat{x}, \hat{t}) \in \mathbb{R}^n \times \mathbb{R}$ and $\hat{p} \in \mathbb{R}^n$, $V \subset \mathbb{R}^n$ be the subspace parallel to $\text{aff } \partial W(\hat{p})$, $U = V^\perp$, $k = \dim V$. We say that a function $\varphi(x, t)$ is a stratified faceted test function at $(\hat{x}, \hat{t})$ with gradient $\hat{p}$ if

$$\varphi(x, t) = \bar{\psi}(x' - \hat{x}') + f(x'' - \hat{x}'') + \hat{p} \cdot x + g(t),$$

where

- $\bar{\psi} : \mathbb{R}^k \rightarrow \mathbb{R}$ is a support function of a $\hat{p}$ -admissible pair (A_-, A_+) in $\mathbb{R}^k$ with $0 \in \text{int}(A_-^c \cap A_+^c)$ and $\bar{\psi} \in \mathcal{D}(\Lambda_{\hat{p}})$,

- $f \in C^2(\mathbb{R}^{n-k})$, $f(0) = 0$ and $\nabla f(0) = 0$,
- $g \in C^1(\mathbb{R})$.

With this notion of test functions, we define viscosity solutions on $Q := \mathbb{R}^n \times (0, \infty)$ in a straightforward way, while carefully extending the notion of “local minima/maxima” of $u - \varphi$ to allow for the proof of both the comparison principle and the stability.

3.2 Definition. An upper semi-continuous function $u : \overline{Q} \rightarrow \mathbb{R}$ is a viscosity subsolution of (1.6) if the following hold:

- (i) (faceted test) Let φ be a stratified faceted test function at $(\hat{x}, \hat{t}) \in Q$ with gradient $\hat{p} \in \mathbb{R}^n \setminus \{0\}$ and pair (A_-, A_+) . Then if there is $\rho > 0$ such that

$$u(x + w, t) - \varphi(x, t) \leq u(\hat{x}, \hat{t}) - \varphi(\hat{x}, \hat{t}) \quad (3.1)$$

for all

$$|w'| \leq \rho, \quad w'' = 0, \quad \text{and } x' - \hat{x}' \in \mathcal{U}^\rho(A_-^c \cap A_+^c), \quad |x'' - \hat{x}''| \leq \rho, \quad |t - \hat{t}| \leq \rho,$$

then there exists $\delta > 0$ such that $B_\delta(\hat{x}') \subset \text{int}(A_-^c \cap A_+^c)$ and

$$\varphi_t(\hat{x}, \hat{t}) + F(\hat{p}, \text{ess inf}_{B_\delta(0)} \Lambda_{\hat{p}}[\bar{\psi}]) \leq 0. \quad (3.2)$$

- (i-cf) (curvature-free test) Let $g \in C^1(\mathbb{R})$, $\varphi(x, t) := g(t)$ and suppose that $u - \varphi$ has a local maximum at $(\hat{x}, \hat{t})$. Then

$$g'(\hat{t}) + F(0, 0) = g'(\hat{t}) \leq 0.$$

- (ii) (off-facet test) Let $\varphi \in C^1(\mathcal{U})$ where $\mathcal{U}$ is a neighborhood of some point $(\hat{x}, \hat{t}) \in Q$ and suppose that $\dim \partial W(\nabla \varphi(\hat{x}, \hat{t})) = 0$. If $u - \varphi$ has a local maximum at $(\hat{x}, \hat{t})$ then

$$\varphi_t(\hat{x}, \hat{t}) + F(\nabla \varphi(\hat{x}, \hat{t}), 0) \leq 0.$$

Viscosity supersolutions are defined analogously by reversing the appropriate inequalities.

A continuous function that is both a subsolution and a supersolution is called a viscosity solution.

4 Comparison principle

We formulate the comparison principle theorem for solutions on the whole $\mathbb{R}^n$ that are constant outside of a compact set. This allows us to prove the theorem without having to deal with unbounded facets. Since the crystalline curvature quantity is nonlocal, this can lead to potential technical difficulties that we are trying to avoid. On the other hand, it does not significantly diminish the applicability of our result since the statement still covers the evolution of arbitrary bounded crystals.

4.1 Theorem (Comparison principle). *Let $W : \mathbb{R}^n \rightarrow \mathbb{R}$ be a positively one-homogeneous convex polyhedral function such that the conclusion of Proposition 4.2 holds for $1 \leq k \leq n - 1$, and let F be as defined in (1.5). Suppose that u and v are a subsolution and a supersolution of (1.6) on $\mathbb{R}^n \times [0, T]$ for some $T > 0$, respectively. Moreover, suppose that there exist a compact set $K \subset \mathbb{R}^n$ and constants $c_u \leq c_v$ such that $u \equiv c_u$, $v \equiv c_v$ on $(\mathbb{R}^n \setminus K) \times [0, T]$. Then $u(\cdot, 0) \leq v(\cdot, 0)$ on $\mathbb{R}^n$ implies $u \leq v$ on $\mathbb{R}^n \times [0, T]$.*

The requirement that both u and v are equal to constants outside of a compact set guarantees that they might cross only inside this compact set and also that we will need to only construct stratified faceted test functions with bounded facets described by bounded open pairs.

To prove the comparison principle, we need to be able to construct stratified faceted test functions at a contact point of the subsolution and the supersolution. This unfortunately appears to be a nontrivial problem. Currently we only know how to construct such test functions in one and two dimensions, which still allows us to prove the comparison principle for the level set flow of a three dimensional crystal. The existence of such test functions is asserted by the following proposition. It is a density result, stating that any bounded open pair can be approximated arbitrarily well in the Hausdorff distance by an admissible pair. We expect this to hold in any dimension, but we have been unable to prove it in dimensions $k \geq 3$.

4.2 Proposition. *Let $W : \mathbb{R}^n \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, be a polyhedral convex function finite everywhere. Suppose that $\hat{p} \in \mathbb{R}^n$ such that $\dim \partial W(\hat{p}) = k$ for $k = 1$ or 2 . Then for any bounded open pair (A_-, A_+) in $\mathbb{R}^k$ and any $0 \leq \rho_1 < \rho_2$ there exists a $\hat{p}$ -admissible pair (G_-, G_+) satisfying*

$$\mathcal{U}^{\rho_1}(A_-, A_+) \preceq (G_-, G_+) \preceq \mathcal{U}^{\rho_2}(A_-, A_+), \quad (4.1)$$

To prove the comparison principle, Theorem 4.1, we follow the standard doubling of variables argument with an extra parameter. That is, we study the maxima of the function

$$\Phi_{\zeta, \varepsilon}(x, t, y, s) := u(x, t) - v(y, s) - \frac{|x - y - \zeta|^2}{2\varepsilon} - S_\varepsilon(t, s),$$

over $Q \times Q := \mathbb{R}^n \times [0, T] \times \mathbb{R}^n \times [0, T]$ for arbitrary $\zeta \in \mathbb{R}^n$ and sufficiently small $\varepsilon > 0$ where

$$S_\varepsilon(t, s) := \frac{|t - s|^2}{2\varepsilon} + \frac{\varepsilon}{T - t} + \frac{\varepsilon}{T - s}. \quad (4.2)$$

If the comparison theorem fails, there exists a point where $u > v$ and this will imply that $\Phi_{\zeta, \varepsilon}$ also has a positive maximum for small $\varepsilon > 0$ for all sufficiently small ζ . We fix one such small $\varepsilon > 0$ and define the maximum of $\Phi_{\zeta, \varepsilon}$,

$$\ell(\zeta) := \max_{Q \times Q} \Phi_{\zeta, \varepsilon}.$$

We observe that since W is a polyhedral convex function, the space $\mathbb{R}^n$ can be decomposed into a finite union of relatively open convex sets on which the subdifferential of W is constant [25, 32]. The Baire category theorem then allows us to choose one of

these sets Ξ and a small open ball $B_{2\lambda}(\zeta_0) \subset \mathbb{R}^n$ such that for any $\zeta \in B_{2\lambda}(\zeta_0)$ there always exists a point of maximum $(\hat{x}, \hat{t}, \hat{y}, \hat{s})$ of $\Phi_{\zeta, \varepsilon}$ with the property that $\hat{x} - \hat{y} - \zeta \in \Xi$. This has consequences for the behavior of $\ell(\zeta)$, the maximum of $\Phi_{\zeta, \varepsilon}$, when changing ζ in the direction orthogonal to $\text{aff } \Xi$. We denote this subspace orthogonal to $\text{aff } \Xi$ by V , and it happens that V is parallel to $\text{aff } \partial W(p)$ for $p \in \Xi$. Specifically, the function $\ell(\zeta)$ is affine in $(\zeta_0 + V) \cap B_{2\lambda}(\zeta_0)$ with the slope given by the projection of some element of Ξ into V . Since W is positively one-homogeneous, $0 \in \text{aff } \Xi$. Therefore ℓ is constant on $(\zeta_0 + V) \cap B_{2\lambda}(\zeta_0)$.

This fact has a very important implication, namely, there exists $\hat{p} \in \Xi$ such that the functions u and v have certain flatness property in the directions in V in a neighborhood of the maximum of $\Phi_{\zeta, \varepsilon}$ and gradient $\hat{p}$. Now we proceed depending on the dimension of V . In particular, if $0 < \dim V < n$, using Proposition 4.2 we can find two ordered $\hat{p}$ -admissible pairs in $\mathbb{R}^k$, $k = \dim V$, with $\hat{p}$ -admissible support functions that belong to $\mathcal{D}(\Lambda_{\hat{p}})$. Moreover, we can build stratified faceted test functions in the sense of Definition 3.1 that are ordered with respect to u and v as in (3.1). Since they then have ordered values of $\Lambda_{\hat{p}}$ due to the comparison principle for $\Lambda_{\hat{p}}$, Proposition 2.5, we obtain a contradiction with the faceted test (3.2) in the definition of viscosity solutions, Definition 3.2. The cases $\dim V = 0$ and $\dim V = n$ yield contradiction with the other conditions in Definition 3.2. We conclude that $u \leq v$.

5 Stability

There are two natural ways how to regularize the problem (1.6), each with different merits. On the one hand, we can take the PDE point of view and approximate W by smooth energy densities with quadratic growth, yielding a uniformly elliptic operator $\psi \mapsto \text{div } \partial W(\nabla \psi)$ and turning the approximating PDE into a degenerate parabolic equation that falls into the scope of the classical viscosity solution theory. On the other hand, since (1.6) is the level set formulation of a surface evolution equation (1.2), we want to show that the evolution can be approximated by a evolution with smooth anisotropic mean curvature, that is, we approximate W by a sequence of positively one-homogeneous surface energy densities with smooth uniformly convex level sets. We obtain stability results in both cases.

Let us first recall the definition of *half-relaxed limits* (semi-continuous limits) of a sequence of functions $\{u_m\}_{m \in \mathbb{N}}$:

$$\begin{aligned} \star\text{-}\limsup_{m \rightarrow \infty} u_m(x, t) &:= \lim_{k \rightarrow \infty} \sup_{m > k} \sup_{|y-x| < \frac{1}{k}} \sup_{|t-s| < \frac{1}{k}} u_m(y, s), \\ \star\text{-}\liminf_{m \rightarrow \infty} u_m(x, t) &:= - \star\text{-}\limsup_{m \rightarrow \infty} (-u_m(x, t)). \end{aligned}$$

Recall that both limits are equal if and only if the convergence is uniform.

We have the following two theorems on stability in either mode of approximation.

5.1 Theorem. *Suppose that $\{W_m\}_{m \in \mathbb{N}} \subset C^2(\mathbb{R}^n)$ is a decreasing sequence of convex functions such that $W_m \searrow W$ as $m \rightarrow \infty$ locally uniformly. Moreover, suppose that there exist positive numbers a_m such that $a_m^{-1}I \leq \nabla_p^2 W_m(p) \leq a_m I$ for all $p \in \mathbb{R}^n$, $m \in \mathbb{N}$, where I is the $n \times n$ identity matrix. Let $\{u_m\}_{m \in \mathbb{N}}$ be a locally bounded sequence of viscosity*

solutions of

$$u_t + F(\nabla u, \text{tr}[(\nabla_p^2 W_m)(\nabla u) \nabla^2 u]) = 0. \quad (5.1)$$

Then $\star\text{-}\limsup_{m \rightarrow \infty} u_m$ is a viscosity subsolution of (1.6) and $\star\text{-}\liminf_{m \rightarrow \infty} u_m$ is a viscosity supersolution of (1.6).

5.2 Theorem. Suppose that F is of the form given in (1.5) and that $\{W_m\}_{m \in \mathbb{N}} \subset C(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0\})$ are positively one-homogeneous functions with bounded, strictly convex sub-level sets $\{W_m \leq 1\}$ such that $W_m \rightrightarrows W$ uniformly on $\overline{B}_1(0)$. Let u_m be for each $m \in \mathbb{N}$ the unique viscosity solution of

$$\begin{cases} u_t + F(\nabla u, \text{div} \nabla_p W_m(\nabla u)) = 0, & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(\cdot, 0) = u_{0,m}, & \text{in } \mathbb{R}^n, \end{cases}$$

where $u_{0,m} \in C(\mathbb{R}^n)$ are uniformly bounded. Then $\star\text{-}\limsup_{m \rightarrow \infty} u_m$ is a viscosity subsolution of (1.6) and $\star\text{-}\liminf_{m \rightarrow \infty} u_m$ is a viscosity supersolution of (1.6).

The classical viscosity solutions are stable with respect to the half-relaxed limits, and the above stability theorems are natural extensions of this stability. However, there is a significant difference between the stratified faceted test functions for (1.6) and the smooth test functions for the regularized problems. Another major difficulty in proving the stability is the nature of the differential operators: we are approximating a nonlocal operator by a sequence of local operators. Therefore there needs to be a way how to recover the nonlocal information about the shape of the facet that determines the value of the crystalline curvature Λ_p from the fundamentally local regularized problems.

We overcome this difficulty by adapting the standard perturbed test function method due to Evans. Suppose that we have a stratified faceted test function ψ for the faceted test in Definition 3.2. Since this function is in general only Lipschitz, it cannot be used as a test function for the regularized problem (5.1). And even if this function were C^2 , application of the elliptic operator $\text{tr}[(\nabla_p^2 W_m)(\nabla \psi) \nabla^2 \psi]$ would yield zero on the facet (flat part) of ψ for any m . We therefore perturb this test function by solving the resolvent problem for the energy $E_m(\psi) := \int W_m(\nabla \psi) dx$ and the total variational energy (2.2) in the class of periodic functions,

$$\begin{aligned} \psi_{a,m} + a \partial E_m(\psi_{a,m}) &\ni \psi, \\ \psi_a + a \partial E(\psi_a) &\ni \psi, \end{aligned}$$

for $a > 0$. This is indeed a uniform perturbation in the sense that for subsequences $m \rightarrow \infty$ and then $a \rightarrow 0$ the solutions converge uniformly $\psi_{a,m} \rightarrow \psi_a$ and $\psi_a \rightarrow \psi$ due to uniform Lipschitz estimates for the resolvent problems. By elliptic regularity $\psi_{a,m} \in C^{2,\alpha}$ and therefore it is a test function for the regularized problem. Moreover, by the standard result [5], we have $(\psi_a - \psi)/a \rightarrow -\partial^0 E(\psi)$ in L^2 . This provides the missing nonlocal information and allows us to pass in the limit $m \rightarrow \infty$ and prove that the uniform limit of solutions of the regularized problems (5.1) is a solutions of the crystalline curvature flow (1.6). Heuristically, the resolvent problem is the one step of the implicit Euler discretization for the gradient flow of the energy E and will therefore carry the information about the crystalline curvature Λ_p which is given as the subdifferential of E .

To prove Theorem 5.2, the stability with respect to one-homogeneous W_m , we use the above idea with an extra regularization step by uniformly elliptic operators.

The stability results together with the comparison principle, using barriers at $t = 0$, yield the well-posedness of (1.6) for arbitrary continuous initial data, constant outside of some compact set, in three dimensions.

6 Whole space test function

In this section we mention an important example of “test functions” that are useful to prove some properties of solutions.

Let $W : \mathbb{R}^n \rightarrow \mathbb{R}$ be a positively one-homogeneous convex function, zero only at the origin. We define the *polar* of W as

$$W^\circ(x) = \sup \{x \cdot p : W(p) \leq 1\}.$$

It is easy to see that W° is again a positively one-homogeneous convex function, zero only at the origin. Moreover,

$$W^{\circ\circ} = W.$$

The Wulff shape associated with W is defined as

$$\mathcal{W} := \{x \in \mathbb{R}^n : W^\circ(x) \leq 1\}.$$

$\mathcal{W}$ is a compact subset of $\mathbb{R}^n$ containing the origin in its interior.

For given $c > 0$, define

$$\psi(x) := \max \{W^\circ(x) - c, 0\}.$$

ψ is clearly a Lipschitz continuous function on $\mathbb{R}^n$. It is in fact a support function of the pair $(\emptyset, c\mathcal{W}^c)$.

We shall show that ψ is an admissible support function and compute the crystalline curvature Λ_0 on its facet $c\mathcal{W}$.

Define the vector field

$$z(x) := \frac{x}{\max(c, W^\circ(x))} = \begin{cases} \frac{x}{c}, & x \in c\mathcal{W}, \\ \frac{x}{W^\circ(x)}, & x \in c\mathcal{W}^c. \end{cases}$$

Note that z is Lipschitz continuous. By Lemma 6.1 and Lemma 6.2 below we see that $z(x) \in \partial W(\nabla \psi(x))$ for almost every $x \in \mathbb{R}^n$ (that is, wherever ψ is differentiable). On the other hand, $\operatorname{div} z \in L^2_{loc}(\mathbb{R}^n)$ and

$$\operatorname{div} z \equiv \frac{n}{c}, \quad \text{on } c\mathcal{W}.$$

Since $\operatorname{div} z$ is constant on the facet, we conclude the crystalline mean curvature is the constant

$$\Lambda_0(\psi) \equiv \frac{n}{c}, \quad \text{on } c\mathcal{W}.$$

Note that as the size of the facet increases, the curvature decreases as $\frac{\mathcal{H}^{n-1}(c\partial\mathcal{W})}{|c\mathcal{W}|} \sim \frac{1}{c}$, the ratio of the surface area of the facet boundary to the facet's volume.

We now state the two technical lemmas that were used above.

6.1 Lemma. *If $p \neq 0$ and $x \in \partial W(p)$ then $W^\circ(x) = 1$ and $x \cdot p = W(p)$. Similarly, if $x \neq 0$ and $p \in \partial W^\circ(x)$ then $W(p) = 1$ and $x \cdot p = W^\circ(x)$. Suppose now that $x \neq 0$ and $p \neq 0$. Then*

$$\frac{x}{W^\circ(x)} \in \partial W(p) \quad \Leftrightarrow \quad \frac{p}{W(p)} \in \partial W^\circ(x).$$

Proof. Write $\hat{x} = \frac{x}{W^\circ(x)}$ and $\hat{p} = \frac{p}{W(p)}$. We have $W^\circ(\hat{x}) = 1$, $W(\hat{p}) = 1$ and therefore $W^*(\hat{x}) = 0$, $W^{\circ*}(\hat{p}) = 0$. Since W and W° are one-homogeneous, we have $\partial W(\hat{p}) = \partial W(p)$ and $\partial W(\hat{x}) = \partial W(x)$. Suppose that $\hat{x} \in \partial W(p) = \partial W(\hat{p})$. Then by the above and the characterization of the subdifferential, for instance [32, Theorem 23.5], we have

$$1 = W(\hat{p}) + W^*(\hat{x}) = \hat{x} \cdot \hat{p}.$$

Therefore

$$W^\circ(\hat{x}) + W^{\circ*}(\hat{p}) = 1 \leq \hat{x} \cdot \hat{p},$$

which by the characterization of the subdifferential yields $\hat{p} \in \partial W^\circ(\hat{x}) = \partial W^\circ(x)$.

The other direction can be proved by reversing the steps. □

6.2 Lemma. *Suppose that W is positively one-homogeneous convex function on $\mathbb{R}^d$. Then $\partial W(p) \subset \partial W(0)$ for any $p \in \mathbb{R}^d$. We also have $(x - y) \perp p$ for any $x, y \in \partial W(p)$ and any $p \in \mathbb{R}^d$.*

Proof. The proof is straightforward, see [32] for details. □

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