

Ergodic control in a single product manufacturing system

徳大総科 大橋 守 (Mamoru Ohashi)

Abstract

We study the ergodic control problem related to stochastic production planning in a single product manufacturing system with production constraints. The existence of a solution to the corresponding Bellman equation and the optimal control are shown.

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1 Introduction

This paper deals with the following 1st order differential equation:

$$\lambda = F\left(\frac{\partial v}{\partial x}(x, i)\right) - i \frac{\partial v}{\partial x}(x, i) + Av(x, i) + h(x), \quad x \in R^1, i = 1, 2, \dots, d. \quad (1)$$

Here λ is a constant, $F(x) = kx$ if $x < 0$, $= 0$ if $x \geq 0$ for some positive constant $k > 0$, h is convex function, and A denotes the infinitesimal generator of an irreducible Markov chain $(z(t), P)$ with state space $Z = \{1, 2, \dots, d\}$:

$$Av(x, i) = \sum_{j \neq i} q_{ij}[v(x, j) - v(x, i)], \quad (2)$$

where q_{ij} is the jump rate of $z(t)$ from i to j . The unknown are the pair (v, λ) , where $v(\cdot, i) \in C^1(R^1)$ for every $i \in Z$.

Equation (1) arises in the ergodic control problem of stochastic production planning in a single product manufacturing system and is called the Bellman equation. The inventory level $x(t)$ of stochastic production planning modeled by Sethi and Zhang [11] is governed by the differential equation

$$\frac{dx(t)}{dt} = p(t) - z(t), \quad x(0) = x, \quad z(0) = i, \quad \text{P-a.s.}, \quad (3)$$

for production rate $0 \leq p(t) \leq k$, in which $z(t)$ is interpreted as the demand rate. For ergodic control, the cost $J(p(\cdot) : x, i)$ associated with $p(\cdot)$ is given by

$$J(p(\cdot) : x, i) = \limsup_{T \rightarrow \infty} \frac{1}{T} E \left[\int_0^T h(x(t)) dt \mid x(0) = x, z(0) = i \right], \quad (4)$$

where $h(x)$ represents the convex inventory cost.

The purpose of this paper is to show the existence of a solution of Bellman equation (1) and to present an optimal control minimizing the cost $J(p(\cdot) : x, i)$ subject to (3). In the control problem of manufacturing systems [5], [12] with discounted rate $\alpha > 0$, many authors have investigated the Bellman equation

$$\alpha u_\alpha(x, i) = F\left(\frac{\partial u_\alpha}{\partial x}(x, i)\right) - i \frac{\partial u_\alpha}{\partial x}(x, i) + A u_\alpha(x, i) + h(x). \quad (5)$$

Our method consists in studying the limit of (5) as α tends to 0. This approach develops the technique of Bensoussan-Frehse [2] concerning non-degenerate 2^{nd} order partial differential equations to our degenerate case. We also refer to Ghosh et al. [7], [8] in the case that the Brownian motion is added to (3) as sales returns and a bounded restriction on production rate p is made.

Section 2 is devoted to the existence problem of (1) under the convexity assumption and others on h , and properties of the solution are shown in § 3. In § 4 an optimal control for the ergodic control problem and the value are given. In § 5 we present an example of the solution to (1).

2 Existence

We are concerned with the equation

$$\alpha u_\alpha(x, i) = F\left(\frac{\partial u_\alpha}{\partial x}(x, i)\right) - i \frac{\partial u_\alpha}{\partial x}(x, i) + A u_\alpha(x, i) + h(x) \quad x \in R^1, i \in Z, \quad (6)$$

and make the following assumptions:

$$h(x) \text{ is nonnegative and convex on } R^1, \quad (7)$$

$$\exists C > 0; 0 \leq h(x) \leq C(1 + |x|^\kappa) \text{ for some positive integer } \kappa, \quad (8)$$

$$k - d > 0. \quad (9)$$

Theorem 2.1 *We assume (7), (8) and (9). Then there exists a unique convex solution $u_\alpha(\cdot, i) \in C^1(R^1), i \in Z$ of equation (6) such that*

$$\alpha \|u_\alpha(\cdot, i)\|_{L^\infty(I_r)} \leq K_r, \quad (10)$$

$$\left\| \frac{\partial u_\alpha}{\partial x}(\cdot, i) \right\|_{L^\infty(I_r)} \leq K_r, \quad (11)$$

$$\|A u_\alpha(\cdot, i)\|_{L^\infty(I_r)} \leq K_r, \quad i \in Z, \quad (12)$$

where K_r is a positive constant depending only on r of $I_r = (-r, r)$.

Proof. According to [11, Theorem 3.1], equation (6) has a viscosity solution [6] given by

$$u_\alpha(x, i) = \inf_{p(\cdot) \in \mathcal{P}(x, i)} \{E[\int_0^\infty e^{-\alpha t} h(x(t)) dt \mid x(0) = x, z(0) = i]\},$$

where $x(t)$ is as in (3), and the infimum is taken over the class $\mathcal{P}(x, i)$ of control processes $p(\cdot)$ such that $0 \leq p(t) \leq k$ and $p(t)$ is adapted to $\mathcal{F}_t = \sigma(z(s), s \leq t)$. Moreover, $u_\alpha(x, i)$ is convex and hence a classical solution of (6) in $C^1(R^1)$. As is well-known [9], for the irreducible Markov chain $(z(t), P)$ there exists a unique equilibrium distribution $\pi = (\pi_1, \pi_2, \dots, \pi_d) > 0$ such that

$$\pi A = 0 \quad \text{and} \quad \sum_{i \in Z} \pi_i = 1. \quad (13)$$

Now, multiplying (6) by π_i and summing up, we have

$$\alpha \sum_i \pi_i u_\alpha(x, i) = \sum_i \pi_i \{F(\frac{\partial u_\alpha}{\partial x}(x, i)) - i \frac{\partial u_\alpha}{\partial x}(x, i)\} + h(x). \quad (14)$$

Since $F(x) - ix \leq 0$ under (9), we have

$$\begin{aligned} \alpha \sum_i \pi_i u_\alpha(x, i) &\leq h(x) \\ &\leq K_r \quad \text{on } I_r. \end{aligned}$$

Thus we can obtain (10) by $u_\alpha(x, i) \geq 0$.

Next, note that

$$F(x) - ix \leq -a|x|, \quad (15)$$

where $a = \min\{k - d, 1\} > 0$. Hence, we have by (14)

$$a \sum_i \pi_i |\frac{\partial u_\alpha}{\partial x}(x, i)| \leq h(x) - \alpha \sum_i \pi_i u_\alpha(x, i).$$

Thus we deduce $|\frac{\partial u_\alpha}{\partial x}(x, i)| \leq K_r$ on I_r by (10) and then (11). Finally, (12) follows from (6), (10) and (11) immediately.

Next we show the behavior of a solution to equation(6) as $\alpha \rightarrow 0$.

Theorem 2.2 *Under the assumptions of Theorem 2.1, there exists a subsequence $\alpha \rightarrow 0$ such that*

$$\begin{aligned} v_\alpha(x, i) &:= u_\alpha(x, i) - u_\alpha(0, i) \quad \rightarrow \quad v_0(x, i) \in C^1(R^1), \\ \mu(\alpha) &:= \alpha u_\alpha(0, i) - A u_\alpha(0, i) \quad \rightarrow \quad \mu_i \in R^1, \end{aligned}$$

uniformly on each $\bar{I}_r$. The limit $(v_0(\cdot, i), \mu_i)$, $i \in Z$, satisfies

$$\mu_i = F\left(\frac{\partial v_0}{\partial x}(x, i)\right) - i \frac{\partial v_0}{\partial x}(x, i) + Av_0(x, i) + h(x), \quad x \in R^1. \quad (16)$$

Proof. Let us note that $(v_\alpha(\cdot, i), \mu(\alpha))$ satisfies

$$\alpha v_\alpha(x, i) + \mu(\alpha) = F\left(\frac{\partial v_\alpha}{\partial x}(x, i)\right) - i \frac{\partial v_\alpha}{\partial x}(x, i) + Av_\alpha(x, i) + h(x). \quad (17)$$

By (11) it is obvious that

$$\|v_\alpha(\cdot, i)\|_{L^\infty(I_r)} + \left\|\frac{\partial v_\alpha}{\partial x}(\cdot, i)\right\|_{L^\infty(I_r)} \leq K_r, \quad i \in Z. \quad (18)$$

Hence $\{v_\alpha(\cdot, i)\}$ is equicontinuous on $\bar{I}_r$.

Let us define

$$B_\alpha(x) = \alpha v_\alpha(x, i) + \mu(\alpha) - Av_\alpha(x, i) - h(x).$$

We recall that, by assumption, $h(x)$ is Lipschitz continuous on $\bar{I}_r$. Then, by (18)

$$|B_\alpha(x) - B_\alpha(y)| \leq C|x - y|, \quad (C > 0 : \text{indep. of } \alpha),$$

From (17) it follows that

$$B_\alpha(x) = F\left(\frac{\partial v_\alpha}{\partial x}(x, i)\right) - i \frac{\partial v_\alpha}{\partial x}(x, i),$$

Then we have

$$\frac{\partial v_\alpha}{\partial x} = \begin{cases} \frac{B_\alpha(x)}{k-i} & \text{if } \frac{\partial v_\alpha}{\partial x} < 0 \\ -\frac{B_\alpha(x)}{i} & \text{if } \frac{\partial v_\alpha}{\partial x} \geq 0, \end{cases}$$

Since $\frac{\partial v_\alpha}{\partial x}$ is nondecreasing, we can see

$$\left|\frac{\partial v_\alpha}{\partial x}(x, i) - \frac{\partial v_\alpha}{\partial x}(y, i)\right| \leq C|x - y|.$$

Thus $\{\frac{\partial v_\alpha}{\partial x}(\cdot, i)\}$ is also equicontinuous on $\bar{I}_r$. By the Ascoli-Arzelà theorem, there exists a subsequence $\alpha \rightarrow 0$ such that

$$v_\alpha(x, i) \rightarrow v_0(x, i), \quad (19)$$

$$\frac{\partial v_\alpha}{\partial x}(x, i) \rightarrow \frac{\partial v_0}{\partial x}(x, i), \quad \text{uniformly on } \bar{I}_r. \quad (20)$$

By a standard argument, we can choose a subsequence $\alpha \rightarrow 0$, independent of r , such that (19) and (20) are fulfilled on every $\bar{I}_r$. Further, by (10) and (12)

$$\mu(\alpha) \rightarrow \mu_i.$$

Letting $\alpha \rightarrow 0$ in (17), we deduce (16). The proof is complete.

Now let us show the existence of a solution to equation (1).

Theorem 2.3 *We assume (7), (8) and (9). Then there exists a solution (v, λ) of equation (1) such that $v(x, i)$ is convex on R^1 and $v(\cdot, i) \in C^1(R^1)$.*

Proof. Let us define

$$\begin{aligned} v(x, i) &= v_0(x, i) + f(i), \\ \lambda &= \sum_i \pi_i \mu_i, \end{aligned}$$

where $(v_0(\cdot, i), \mu_i)$ is as in (16) and $f(i)$ is a solution of

$$Af(i) = -\mu_i + \lambda, \quad i \in Z. \quad (21)$$

Then it is easily seen that (v, λ) satisfies (1). The convexity of $v(x, i)$ and $v(\cdot, i) \in C^1(R^1)$ are immediate from Theorems 2.1 and 2.2.

To complete the proof, it is sufficient to check the existence of $f(i)$. By the irreducible Markov chain $(z(t), P)$ it follows that for any $g \in R^d$

$$E[g(z(s/\alpha))] \rightarrow \sum_i \pi_i g(i) \quad \text{as } \alpha \rightarrow 0.$$

Hence

$$\begin{aligned} \alpha G_\alpha g(i) &= \alpha E\left[\int_0^\infty e^{-\alpha t} g(z(t)) dt\right] \\ &= \int_0^\infty e^{-s} E[g(z(s/\alpha))] ds \\ &\rightarrow \sum_i \pi_i g(i) = \pi g, \end{aligned}$$

where G_α denotes the resolvent operator of the Markov chain $(z(t), P)$. According to [4, Lemma 7.3(c,d), p.39], we can obtain the relation:

$$\{g \in R^d \mid \pi g = 0\} = \{Ag \in R^d \mid g \in R^d\}.$$

We notice by (13) that

$$\pi(-\mu + \lambda) = 0.$$

Therefore we conclude that equation (21) admits a solution $f(i)$.

3 Properties

We investigate properties of a solution to the Bellman equation (1). Now we make the assumption:

$$h(x)/|x| \rightarrow \infty \quad \text{as } |x| \rightarrow \infty. \quad (22)$$

Lemma 3.1 Under (22), the convex solution $v(\cdot, i) \in C^1(R^1)$ of equation (1) satisfies

$$\left| \frac{\partial v}{\partial x}(x, i) \right| \rightarrow \infty \text{ as } |x| \rightarrow \infty. \quad (23)$$

Proof. It is sufficient to show (23) in the case $x \rightarrow -\infty$. By the convexity of $v(x, i)$, we can define M_i by

$$M_i = - \lim_{x \rightarrow -\infty} \frac{\partial v}{\partial x}(x, i).$$

For any sequence $x_n \rightarrow -\infty$, we can easily see

$$\frac{v(x_n, i)}{|x_n|} \rightarrow M_i.$$

Suppose that $M_i < \infty$ for some $i \in Z$. Then, dividing (1) by $|x_n|$ and passing to the limit, we have by (22)

$$\begin{aligned} \lambda/|x_n| = [F(\frac{\partial v}{\partial x}(x_n, i)) - i \frac{\partial v}{\partial x}(x_n, i) + \sum_{j \neq i} q_{ij} v(x_n, j) \\ - \sum_{j \neq i} q_{ij} v(x_n, i) + h(x_n)]/|x_n| \rightarrow \infty, \end{aligned}$$

since $v(x, j) \geq ax + b$ for some constants a and b . This is a contradiction. Hence $M_i = \infty$ for all $i \in Z$, and thus the assertion follows.

Lemma 3.2 For the convex solution $v(\cdot, i) \in C^1(R^1)$ of equation (1), there is a constant $C > 0$ such that

$$|v(x, i)| \leq C(1 + |x|^{\kappa+1}). \quad (24)$$

Proof. From (1) and (15) it follows that

$$\lambda \leq -a \left| \frac{\partial v}{\partial x}(x, i) \right| + Av(x, i) + h(x).$$

If $\frac{\partial v}{\partial x}(x, i) < 0$ on some interval $(-\infty, x_1)$ with $x_1 < 0$, then by (8)

$$- \frac{\partial v}{\partial x}(x, i) \leq \frac{1}{a} Av(x, i) + C(1 + |x|^\kappa) \quad (25)$$

Multiplying (25) by π_i and summing up, we get by (13)

$$- \sum_i \pi_i \frac{\partial v}{\partial x}(x, i) \leq C(1 + |x|^\kappa).$$

Integrating over (x, x_1) , we have

$$\sum_i \pi_i(v(x, i) - v(x_1, i)) \leq C(1 + |x|^{k+1}).$$

This relation can be obtained in the case that $\frac{\partial v}{\partial x} \geq 0$ on some interval (x_2, ∞) with $x_2 > 0$. Therefore we can obtain the desired result by $\pi > 0$.

Next, we consider the equation

$$\frac{dx^*(t)}{dt} = p^*(x^*(t), z(t)) - z(t), \quad x^*(0) = x, \quad z(0) = i, \quad P - a.s., \quad (26)$$

where

$$p^*(x, i) = \begin{cases} k & \text{if } \frac{\partial v}{\partial x}(x, i) < 0 \\ i & \text{if } \frac{\partial v}{\partial x}(x, i) = 0 \\ 0 & \text{if } \frac{\partial v}{\partial x}(x, i) > 0. \end{cases} \quad (27)$$

Lemma 3.3 *Equation (26) admits a unique solution $x^*(t)$, which satisfies*

$$\sup_t \|x^*(t)\|_{L^\infty} < \infty.$$

Proof. Since $p^*(x, i)$ is nonincreasing in x , the differential equation (26) has a unique solution by [6, Theorem 6.2].

To complete the proof, let $\bar{x} = \sup\{x \in R^1 : p^*(x, i) \geq i \text{ for some } i \in Z\}$. Obviously, $\bar{x}$ is finite, because $p^*(x, i)$ is nonnegative. Similarly, let $\tilde{x} = \inf\{x \in R^1 : p^*(x, i) \leq i \text{ for some } i \in Z\}$. Suppose that $\tilde{x}$ is not finite. Then there exists $i \in Z$ such that $\frac{\partial v}{\partial x}(x, i) \geq -2i$ for all $x \in R^1$. On the other hand, by Lemma 3.1, $\frac{\partial v}{\partial x}(x, i) \rightarrow -\infty$ as $x \rightarrow -\infty$. This is a contradiction.

Now, if $x^*(t) > \bar{x}$ (resp. $x^*(t) < \tilde{x}$), then $\frac{dx^*}{dt}(t) < 0$ (resp. $\frac{dx^*}{dt}(t) > 0$). Hence the interval $[\tilde{x}, \bar{x}]$ is an attracting set for (26). Thus the boundedness of $x^*(t)$ is immediate.

Lemma 3.4 *The constant solution λ of equation (1) satisfies*

$$\lambda = \inf_{p(\cdot) \in \mathcal{P}(x, i)} \limsup_{\alpha \rightarrow 0} \alpha E \left[\int_0^\infty e^{-\alpha t} h(x(t)) dt \mid x(0) = x, z(0) = i \right]. \quad (28)$$

Proof. For the convex solution $v(\cdot, i) \in C^1(R^1)$, let us apply an elementary rule and Dynkin's formula to the first and the second variables of $v(x(t), z(t))$ respectively. Then we have the relation:

$$\begin{aligned} E[e^{-\alpha t} v(x(t), z(t)) \mid x(0) = x, z(0) = i] \\ &= v(x, i) - \alpha E \left[\int_0^t e^{-\alpha s} v(x(s), z(s)) ds \mid x(0) = x, z(0) = i \right] \\ &\quad + E \left[\int_0^t e^{-\alpha s} \frac{\partial v}{\partial x}(x(s), z(s)) dx(s) \mid x(0) = x, z(0) = i \right] \\ &\quad + E \left[\int_0^t e^{-\alpha s} A v(x(s), z(s)) ds \mid x(0) = x, z(0) = i \right] \end{aligned} \quad (29)$$

We notice that the minimum of

$$\min_{0 \leq p \leq k} p \frac{\partial v}{\partial x} = F\left(\frac{\partial v}{\partial x}\right)$$

is attained by $p^*(x, i)$. By (1), we have

$$\lambda \leq \frac{\partial v}{\partial x}(x, i)(p - i) + Av(x, i) + h(x), \quad (30)$$

and the equality holds for $p = p^*(x, i)$. Clearly, by (3)

$$|x(t)| \leq C(t + 1) \quad \text{for all } p(\cdot) \in \mathcal{P}(x, i).$$

By Lemma 3.2

$$\begin{aligned} E[e^{-\alpha t} |v(x(t), z(t))| \mid x(0) = x, z(0) = i] \\ \leq CE[e^{-\alpha t}(1 + |x(t)|^{\kappa+1}) \mid x(0) = x, z(0) = i] \\ \leq Ce^{-\alpha t}(1 + (t + 1)^{\kappa+1}) \rightarrow 0 \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Hence, substituting (30) into (29), we get

$$\begin{aligned} \frac{\lambda}{\alpha} \leq & -v(x, i) + \alpha E\left[\int_0^\infty e^{-\alpha s} v(x(s), z(s)) ds \mid x(0) = x, z(0) = i\right] \\ & + E\left[\int_0^\infty e^{-\alpha s} h(x(s)) ds \mid x(0) = x, z(0) = i\right]. \end{aligned}$$

We note that by Lemma 3.2 and 3.3

$$\begin{aligned} \alpha^2 E\left[\int_0^\infty e^{-\alpha s} |v(x(s), z(s))| ds \mid x(0) = x, z(0) = i\right] \\ \leq \alpha^2 C \int_0^\infty e^{-\alpha s} (1 + |x^*(s)|^{\kappa+1}) ds \rightarrow 0 \quad \text{as } \alpha \rightarrow 0. \end{aligned}$$

Thus we deduce

$$\lambda \leq \inf_{p(\cdot) \in \mathcal{P}(x, i)} \limsup_{\alpha \rightarrow 0} \alpha E\left[\int_0^\infty e^{-\alpha s} h(x(s)) ds \mid x(0) = x, z(0) = i\right],$$

and the equality holds for $p(t) = p^*(x^*(t), z(t))$ of (27).

4 An application to ergodic control

We shall study the ergodic control problem to minimize the cost:

$$J(p(\cdot) : x, i) = \limsup_{T \rightarrow \infty} \frac{1}{T} E\left[\int_0^T h(x(t)) dt \mid x(0) = x, z(0) = i\right]$$

over all $p(\cdot) \in U$ subject to

$$\frac{dx(t)}{dt} = p(t) - z(t), \quad x(0) = x, \quad z(0) = i, \quad \text{P-a.s.},$$

where U is the set of all nonnegative progressively measurable processes $p(t)$ such that

$p(t)$ is adapted to $\mathcal{F}_t$,

$$0 \leq p(t) \leq k,$$

$$\sup_t E[|x(t)|^{\kappa+1} \mid x(0) = x, z(0) = i] < \infty \quad \text{for } \kappa \text{ in (8).}$$

Theorem 4.1 *We assume (7), (8), (9) and (22). Then the optimal control $p^*(t)$ is given by*

$$p^*(t) = p^*(x^*(t), z(t)),$$

and the value by

$$J(p^*(\cdot) : x, i) = \lambda,$$

where $p^*(x^*(t), z(t))$ is as in (27).

Proof. From the same formula as (29) it follows that

$$\begin{aligned} E[v(x(T), z(T)) \mid x(0) = x, z(0) = i] \\ = v(x, i) + E\left[\int_0^T \frac{\partial v}{\partial x}(x(s), z(s)) dx(s) \mid x(0) = x, z(0) = i\right] \\ + E\left[\int_0^T A v(x(s), z(s)) ds \mid x(0) = x, z(0) = i\right]. \end{aligned}$$

We recall (30) to obtain

$$\begin{aligned} E[v(x(T), z(T)) \mid x(0) = x, z(0) = i] \\ \geq v(x, i) + E\left[\int_0^T (\lambda - h(x(s))) ds \mid x(0) = x, z(0) = i\right], \end{aligned}$$

where the equality holds for $x = x^*$ and $p = p^*$ of (27). By Lemma 3.2 and the definition of U

$$\begin{aligned} \frac{1}{T} E[|v(x(T), z(T))| \mid x(0) = x, z(0) = i] \\ \leq \frac{C}{T} E[1 + |x(T)|^{\kappa+1} \mid x(0) = x, z(0) = i] \rightarrow 0 \quad \text{as } T \rightarrow \infty. \end{aligned}$$

Also, by Lemma 3.3, $p^*(t)$ belongs to U . Thus we deduce

$$\begin{aligned} J(p(\cdot) : x, i) &= \limsup_{T \rightarrow \infty} \frac{1}{T} E\left[\int_0^T h(x(s)) ds \mid x(0) = x, z(0) = i\right] \\ &\geq \lambda = J(p^*(\cdot) : x, i). \end{aligned}$$

The proof is complete.

5 An Example

In this section we present the example of an solution to the Bellman equation:

$$\lambda = F\left(\frac{\partial v}{\partial x}(x, i)\right) - i \frac{\partial v}{\partial x}(x, i) + Av(x, i) + h(x), \quad x \in R^1, i \in Z, \quad (31)$$

in the case that

$$h(x) = x^2, k = 3, \quad (32)$$

$$Z = \{1, 2\}, q_{12} = q_{21} = 1. \quad (33)$$

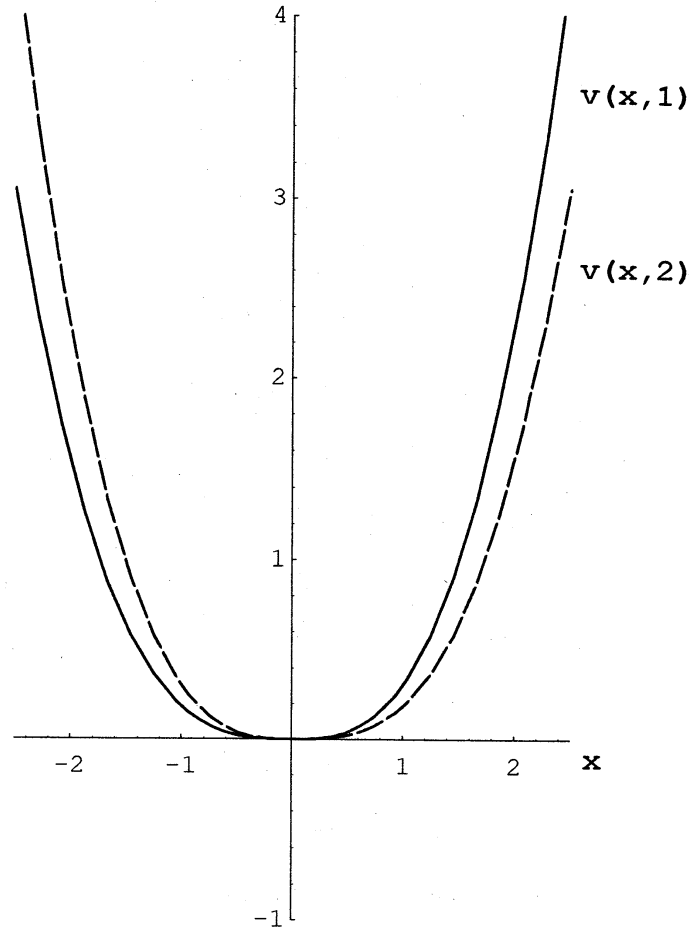


Figure Solution $v(x, i)$, $i=1, 2$, to the Bellman Equation (31)

We remark that the matrix induced by A is given by

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix},$$

and the equilibrium distribution π is

$$\pi = \left(\frac{1}{2}, \frac{1}{2}\right).$$

Therefore the assumptions of Theorem 4.1 are fulfilled.

Now, recalling the form of optimal control p^* and solving the Bellman equation (31) with (32) and (33), we have

$$\begin{aligned} \lambda &= 0, \\ v(x, 1) &= \begin{cases} \frac{1}{81}(18x^3 + 18x^2 - 24x + 16 - 16e^{-\frac{3}{2}x}) & \text{if } x \geq 0 \\ -\frac{1}{81}(18x^3 + 9x^2 + 12x + 8 - 8e^{\frac{3}{2}x}) & \text{if } x < 0, \end{cases} \\ v(x, 2) &= \begin{cases} \frac{1}{81}(18x^3 - 9x^2 + 12x - 8 + 8e^{-\frac{3}{2}x}) & \text{if } x \geq 0 \\ -\frac{1}{81}(18x^3 - 18x^2 - 24x - 16 + 16e^{\frac{3}{2}x}) & \text{if } x < 0. \end{cases} \end{aligned}$$

Then the optimal control p^* is given by

$$p^*(x, i) = \begin{cases} 0 & \text{if } x > 0 \\ i & \text{if } x = 0 \\ 3 & \text{if } x < 0. \end{cases}$$

The solution $v(x, i)$ with (23) - (24), $i = 1, 2$ can be shown in Figure.

References

- [1] A. Bensoussan, S.P. Sethi, R. Vickson and N. Derzko, Stochastic production planning with production constraints, SIAM J. Control and Optim., 22 (1984), pp. 920-935.
- [2] A. Bensoussan and J. Frehse, On Bellman equations of ergodic control in R^n , J. reine angew. Math., 429 (1992), pp. 125-160.
- [3] E.B. Dynkin, Markov Processes, vols.I,II, Springer-Verlag, Berlin, 1965.
- [4] S.N. Ethier and T.G. Kurtz, Markov Processes: Characterization and Convergence, John Wiley & Sons, New York, 1986.
- [5] W.H. Fleming, S.P. Sethi and H.M. Soner, An optimal stochastic production planning problem with randomly fluctuating demand, SIAM J. Control and Optim., 25 (1987), pp. 1494-1502.

- [6] W.H. Fleming and H.M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer-Verlag, Berlin, 1993.
- [7] M.K. Ghosh, A. Arapostathis and S.I. Marcus, Optimal control of switching diffusions with application to flexible manufacturing systems, *SIAM J. Control and Optim.*, 31 (1993), pp. 1183-1204.
- [8] M.K. Ghosh, A. Arapostathis and S.I. Marcus, Optimal control of a hybrid system with pathwise average cost, *Proceedings of the 31st CDC*, (1992), pp. 1061-1066.
- [9] P. Hartman, *Ordinary Differential Equations*, Birkhäuser, Boston, 1982.
- [10] J. Lamperti, *Stochastic Processes*, Appl. Math. Sci. 23, Springer-Verlag, Berlin, 1977.
- [11] S.P. Sethi and Q. Zhang, *Hierarchical Decision Making in Stochastic Manufacturing Systems*, Birkhäuser, Boston, 1994.
- [12] S.P. Sethi, H.M. Soner, Q. Zhang and J. Jiang, Turnpike Sets and their Analysis in Stochastic Production Planning Problems, *Mathematics of Operations Research*. 17(1992), pp.932-950.