

Critically $(k, \bar{k})$ -connected graphs

Kiyoshi Ando
Nippon Ika University
Kosugi Nakahara-ku Kawasaki Japan

Yoko Usami
Ochanomizu University
Otsuka Bunkyo-ku Tokyo Japan

Abstract

A fragment of a connected simple graph G is a subset A of $V(G)$ consisted of components of $G-S$ such that $V(G)-A-S \neq \emptyset$ where S is a minimum cut of G . A minimal fragment of G is said to be an end of G . The complement of G is denoted by $\bar{G}$. A simple graph G is said to be critically $(k, \bar{k})$ -connected if $\kappa(G-x) = \kappa(G)-1$ or $\kappa(\bar{G}-x) = \kappa(\bar{G})-1$ for each x of $V(G)$ where $\kappa(G)$ means the vertex connectivity of G . We proved the followings:

Let G be a critically $(k, \bar{k})$ -connected graph ($k \geq \bar{k} \geq 2$). We denote by η and $\bar{\eta}$ the number of ends of G and the minimum order of the ends of G , respectively. Similarly $\bar{\eta}$ and $\bar{\alpha}$ denote those of $\bar{G}$. Suppose $2\alpha > k$ and $2\bar{\alpha} > \bar{k}$. Then

(A) If there is no minimum cut of G containing all the ends of $\bar{G}$ then

$$\eta = 2, 3 \text{ or } 4 \quad \text{and} \quad \frac{\eta}{\eta-1} \left\lfloor \frac{k}{\alpha} \right\rfloor \geq \bar{\eta} \geq \frac{\eta(k+1)}{2(2\bar{\alpha}-1)}.$$

Furthermore, if $\eta = 2$, then $|G| \leq 2k + \bar{\eta}\bar{k}$, if $\eta = 3$, then $|G| \leq \frac{9}{4}k + \bar{\eta}\bar{k} - \frac{11}{4}$ and if $\eta = 4$, then $|G| \leq 2k + \bar{\eta}\bar{k} - 6$.

(B) If there is a minimum cut of G containing all the ends of $\bar{G}$ then

$$\eta(G) = 2 \text{ or } 3 \quad \text{and} \quad \left\lceil \frac{\eta\alpha}{\bar{k}} \right\rceil \leq \bar{\eta} \leq \left\lfloor \frac{k}{\bar{\alpha}} \right\rfloor.$$

§1 Introduction and main results

In this paper we consider only finite simple graphs. We denote by $V(G)$ the vertex set of a graph G . Let G be a connected graph. We call a set S of $V(G)$ a cut of G if $G-S$ is disconnected, and S is said to be a minimum cut of G if $|S| \leq |S'|$ for any cut S' of G . The order of a minimum cut of G is called the vertex connectivity of G and denoted by $\kappa(G)$. The minimum degree of the vertices of G is denoted by $\delta(G)$. The complement of

a graph G is denoted by $\bar{G}$. As usual for a real number r we denote by $\lceil r \rceil$ and $\lfloor r \rfloor$ the integers such that $r \leq \lceil r \rceil < r+1$ and $r-1 < \lfloor r \rfloor \leq r$.

A non-empty subset A of $V(G)$ is called a *fragment* of G if A is consisted of components of $G-S$ and $V(G)-A-S \neq \emptyset$ for some minimum cut S of G . A minimal fragment of G is called an *end* of G . We denote by $\eta(G)$ the number of ends of G .

A graph G is said to be *critically k -connected* if $\kappa(G) = k$ and $\kappa(G-x) = k-1$ for each vertex x of $V(G)$. G. Chartrand, A. Kaugars and D. R. Lick [2] have shown that if G is a critically k -connected graph, $k \geq 2$, then $\delta(G) \geq \frac{3k-1}{2}$ and this bound is sharp.

In [1] we introduced critically (k,k) -connectedness of graphs. More generally we define here that a graph is said to be *critically $(k, \bar{k})$ -connected* if $\kappa(G) = k$, $\kappa(\bar{G}) = \bar{k}$ and $\kappa(G-x) = k-1$ or $\kappa(\bar{G}-x) = \bar{k}-1$ for each vertex x of $V(G)$. In [1] we proved the following theorem concerning critically (k,k) -connected graphs.

Theorem A ([1]) If G is a critically (k,k) -connected graph, $k \geq 2$, $\delta(G) \geq \frac{3k-1}{2}$ and $\delta(\bar{G}) \geq \frac{3k-1}{2}$, then $|G| \leq 4k$.

In fact in [1] we proved the following stronger assertion:

Theorem ([1]) Let G be a critically (k,k) -connected graph, $k \geq 2$. Let a (resp. $\bar{a}$) be the order of minimum end of G (resp. $\bar{G}$). If $2a > k$ and $2\bar{a} > k$, then $|G| \leq 4k$ and $(\eta(G), \eta(\bar{G})) = (2, 2)$.

In this paper we will study critically $(k, \bar{k})$ -connected graphs and we will show more general results described as follows:

Main Theorem. Let G be a critically $(k, \bar{k})$ -connected graph ($k \geq \bar{k} \geq 2$). We denote by η and a the number of ends of G and the minimum order of the ends of G , respectively. Similarly $\bar{\eta}$ and $\bar{a}$ denote those of $\bar{G}$. Suppose $2a > k$ and $2\bar{a} > \bar{k}$. Then

(1) (A) If there is no minimum cut of G containing all the ends of $\bar{G}$ then

$$\eta = 2, 3 \text{ or } 4 \quad \text{and} \quad \frac{\eta}{\eta-1} \left\lfloor \frac{k}{\bar{a}} \right\rfloor \geq \bar{\eta} \geq \frac{\eta(k+1)}{2(2\bar{a}-1)}.$$

In particular $\eta \leq \bar{\eta}$.

(B) If there is a minimum cut of G containing all the ends of $\bar{G}$ then

$$\eta = 2 \text{ or } 3 \quad \text{and} \quad \left\lceil \frac{\eta a}{k} \right\rceil \leq \bar{\eta} \leq \left\lfloor \frac{k}{\bar{a}} \right\rfloor.$$

(2) In case (A) in (1) we have

$$\text{If } \eta = 2 \text{ then } |G| \leq 2k + \bar{\eta} \bar{k}.$$

$$\text{If } \eta = 3 \text{ then } |G| \leq \frac{9}{4}k + \bar{\eta} \bar{k} - \frac{9}{4}.$$

$$\text{If } \eta = 4 \text{ then } |G| \leq 2k + \bar{\eta} \bar{k} - 6.$$

In case (B) in (1) there is no upper bound of the order of G for each k and $\bar{k}$.

§2 Preliminaries

In this section we introduce some more notation and present preliminary lemmas which we will use in the following two sections to prove our main results. Let G be a connected graph. We denote by $\mathcal{C}(G)$ the family of all minimum cuts of a graph G , and set $C(G) = \bigcup_{S \in \mathcal{C}(G)} S$. We denote by $G[A]$ the subgraph of G induced by $A \subset V(G)$. Let $N_G(x)$ be the set of the vertices adjacent to x in G . For $A \subset V(G)$, we put $N_G(A) = \bigcup_{x \in A} N_G(x) - A$ and $N_G[A] = N_G(A) \cup A$. Recall a fragment of G is a non-empty subset A of $V(G)$ such that (i) $N_G(A)$ is a minimum cut of G and (ii) $G - N_G[A]$ is non-empty, and

that an end of G is a minimal fragment of G . We call a minimum fragment of G an *atom* and we denote by a_G the order of an atom of G . If there is no danger of ambiguity, we abbreviate $\eta(G)$, $\eta(\bar{G})$, a_G and $a_{\bar{G}}$ to η , $\bar{\eta}$, a and $\bar{a}$, respectively.

The following lemma expresses the essential relation between a graph G and its complement $\bar{G}$, so that we call it "Complement lemma".

Lemma (Complement Lemma) *Let G be a graph and let A, B be subsets of $V(G)$. If B is not contained in $N_G[A]$, then $N_{\bar{G}}[B]$ contains A .*

Proof. Let x be a vertex of B not contained in $N_G[A]$. It is immediate that $N_{\bar{G}}(x) \supset A$, since $x \notin N_G[A]$. ■

In the above lemma, if $A \cap B = \emptyset$, then we can replace the closed neighbourhoods $N_G[A]$ and $N_{\bar{G}}[B]$ with the open neighbourhoods $N_G(A)$ and $N_{\bar{G}}(B)$, respectively.

Therefore the next lemma (we also call it "Complement lemma") is an immediate consequence of the above lemma. This lemma will play a fundamental roll in our argument through this paper.

Lemma (Complement Lemma) *Let G be a graph and let B be a subset of $V(G)$. If $\mathcal{A}$ is a family of subsets of $V(G)$ such that $B \cap A = \emptyset$ for each A in $\mathcal{A}$ and $N_{\bar{G}}(B) \not\subset \bigcup_{A \in \mathcal{A}} A$, then there is a subset A in $\mathcal{A}$ such that $N_G(A) \supset B$.* ■

Lemma 1 *Let G be a connected graph and let W be a subset of $V(G)$ such*

that (i) $|W| > \kappa(G)$, (ii) for any minimum cut S of G , $W-S$ is contained in a component of $G-S$. We denote by $\mathcal{A}$ the family of the maximal fragments of G each of which has no intersection with W . Then

(1) $A \cap B = \emptyset$ for any two distinct elements A, B in $\mathcal{A}$,

(2) $\mathcal{A} \neq \emptyset$ and any minimum cut of G is contained in $\bigcup_{A \in \mathcal{A}} N_G[A]$.

proof. To prove (1) suppose not, i.e. suppose that there are two distinct fragments A_1 and A_2 in $\mathcal{A}$ such that $A_1 \cap A_2 \neq \emptyset$. Let $\tilde{A}_i = V(G) - N_G[A_i]$ for $i = 1, 2$. Then since $A_1 \cap A_2 \neq \emptyset$ $|N_G(A_1 \cap A_2)| \geq \kappa(G)$. Consequently $\tilde{A}_1 \cap \tilde{A}_2 \neq \emptyset$, since $|N_G[\tilde{A}_1] \cap N_G[\tilde{A}_2]| \geq |W| > \kappa(G)$. Therefore $|N_G(\tilde{A}_1 \cap \tilde{A}_2)| = \kappa(G)$, which implies $A_1 \cup A_2$ is also a fragment of G , contradicting the maximality of A_1 and A_2 .

To prove (2) let S be any minimum cut of G and let H_S be the component of $G-S$ containing $W-S$. Then the fragment $A = V(G) - N_G[V(H_S)]$ has no intersection with W , so there is an element A' in $\mathcal{A}$ containing A such that $S = N_G(A) \subset N_G[A']$. ■

We remark that in the above Lemma 1 if $W \subset C(G)$, then $W \subset \bigcup_{A \in \mathcal{A}} N_G(A)$, in

particular, $|W| \leq \eta(G) \kappa(G)$.

As a slight extension of a result of Mader[4], we can easily show the followings which will be the firm bases of our arguments. (cf. Theorem 1 and Lemma 1 in [1])

Lemma 2 Let G be a critically $(k, \bar{k})$ -connected graph and let $\{X_1, X_2, \dots, X_\eta\}$ and $\{Y_1, Y_2, \dots, Y_{\bar{\eta}}\}$ be the set of all the ends of G and that of $\bar{G}$, respectively.

Set $X = \bigcup_{i=1}^{\eta} X_i$ and $Y = \bigcup_{j=1}^{\bar{\eta}} Y_j$. Suppose $2a > k$ and $2\bar{a} > \bar{k}$. Then

(i) $X \cap C(G) = \emptyset$ and $Y \cap C(\bar{G}) = \emptyset$.

(ii) Let A and B be any two distinct elements of $\{X_1, X_2, \dots, X_\eta, Y_1, \dots, Y_{\bar{\eta}}\}$. Then $A \cap B = \emptyset$. ■

§4 A proof of Main Theorem (1)

Throughout this section and the next section suppose G is a critically $(k, \bar{k})$ -connected graph such that $2a > k$, $2\bar{a} > \bar{k}$ and $k \geq \bar{k}$. Let $\{X_1, X_2, \dots, X_\eta\}$ and $\{Y_1, Y_2, \dots, Y_{\bar{\eta}}\}$ be the set of all the ends of G and $\bar{G}$, respectively, and put $X = \bigcup_{i=1}^{\eta} X_i$ and $Y = \bigcup_{j=1}^{\bar{\eta}} Y_j$.

proof of (A) To prove the former part of (A) it suffices to show the following two inequalities: $\eta(k+1) \leq 2\bar{\eta}(2\bar{a}-1)$ and $\bar{\eta} + \left\lfloor \frac{k}{a} \right\rfloor \eta \geq \eta \bar{\eta}$. If they hold, then $\bar{a}\bar{\eta} + 2(2\bar{a}-1)\bar{\eta} - \eta \geq \bar{a}\eta\bar{\eta}$, which implies $\eta = 2, 3$ or 4 . By the assumption of (A) $N_G(X_i) \not\supset Y$ for each i thus $X \subset N_{\bar{G}}(Y)$ and this implies the first inequality, since $\eta k \leq \eta(2a-1) \leq 2|X| - \eta \leq 2|N_{\bar{G}}(Y)| - \eta \leq 2\bar{\eta}\bar{k} - \eta \leq 2\bar{\eta}(2\bar{a}-1) - \eta$. To show the second inequality note that $N_G(X_i)$ contains at most $\left\lfloor \frac{k}{a} \right\rfloor$ ends of $\bar{G}$ and $N_{\bar{G}}(Y_j)$ contains at most one end of G , since $2a > k \geq \bar{k}$. Furthermore, Complement lemma assures us that for any of pairs (i, j) either $N_G(X_i) \supset Y_j$ or $X_i \subset N_{\bar{G}}(Y_j)$. Thus $\bar{\eta} + \left\lfloor \frac{k}{a} \right\rfloor \eta \geq \eta \bar{\eta}$. Next we prove the latter part of (A). For each end Y_j of $\bar{G}$, $N_{\bar{G}}(Y_j)$ can contain at most one end of G . On

the other hand, the assumption of (A) implies each X_i contained in $N_{\bar{G}}(Y_j)$ for some j , so $\bar{\eta} \geq \eta$.

proof of (B). To prove (B) it suffices to show the following three inequalities: $\eta \leq 3$, $\eta a \leq \bar{\eta} \bar{k}$ and $k \geq \bar{\eta} \bar{a}$. The last one is immediate consequence of the assumption of (B). To show the former two inequalities put $H = \bar{G}[\bigcup_{i=1}^{\eta} X_i]$. Then as a consequence of Lemma 2 H has the complete η -partite graph with vertex classes $X_1, X_2, \dots, X_{\eta}$ as its spanning subgraph. Therefore if $\eta \geq 3$ then $\kappa(H) \geq 2a > k \geq \bar{k}$, so by the remark after Lemma 1 $\eta a \leq |H| \leq \bar{\eta} \bar{k}$. In particular, if $\eta \geq 3$ then $a\bar{a}\eta \leq \bar{a}\bar{\eta}\bar{k} \leq k\bar{k}$ so $\eta \leq \frac{k\bar{k}}{a\bar{a}} < 4$, thus the first inequality holds. In the case that $\eta = 2$, we may suppose $a > \bar{k}$, since otherwise $\eta a = 2a \leq 2\bar{k} \leq \bar{\eta} \bar{k}$. If $a > \bar{k}$ then $\kappa(H) \geq a > \bar{k}$ and again by the same remark $\eta a \leq \bar{\eta} \bar{k}$. ■

§4 A proof of Main Theorem (2)

At first we introduce two new families of subsets of $V(G)$, $\mathcal{A}$ and $\mathcal{B}$, which will hold the key of our proof. Recall X is the union of all the ends of G and Y is that of $\bar{G}$. Let $\mathcal{A}$ be the family of the maximal fragments of G each of which has no intersection with Y . Similarly $\mathcal{B}$ stands for the family of the maximal fragments of $\bar{G}$ each of which has no intersection with X . To prove Main Theorem (2) we need the following two lemmas which express remarkable properties of $\mathcal{A}$ and $\mathcal{B}$. Throughout this section assume that there is no minimum cut of G containing Y . We remark that there is no minimum cut of $\bar{G}$ containing X , since $\eta a \geq 2a > k \geq \bar{k}$.

Lemma 3 Suppose $|G| > 2(k+\bar{k})$. Then

(1) Each of $\mathcal{A}$ and $\mathcal{B}$ is a family of mutually disjoint subsets of $V(G)$.

(2) $C(G) \subset \bigcup_{A \in \mathcal{A}} N_G[A]$ and $C(\bar{G}) \subset \bigcup_{B \in \mathcal{B}} N_{\bar{G}}[B]$.

(3) $|A| \leq \bar{k}$ for each $A \in \mathcal{A}$, and $|B| \leq k$ for each $B \in \mathcal{B}$.

proof. We give a proof for $\mathcal{A}$ (We can prove the result for $\mathcal{B}$ similarly).

Let S be any minimum cut of G and let A_1 and $A_2 \in \mathcal{A}$ such that $A_1 \cap A_2 \neq \emptyset$.

For $i = 1$ and 2 , let $\tilde{A}_i$ stand for $V(G) - N_G[A_i]$. Then according to the proof of Lemma 1 to prove (1) and (2) it suffices to show (i) $Y-S$ is contained in a component of $G-S$ and (ii) $\tilde{A}_1 \cap \tilde{A}_2 \neq \emptyset$.

(i) By the assumption that $S \not\supset Y$, there is an end Y_S of $\bar{G}$ not contained in S . Also there is an end of G , say X_1 , not contained in $N_{\bar{G}}(Y_S)$. By Complement lemma $N_G(X_1) \supset Y_S$, so $k \geq |Y_S|$ and $k+\bar{k} \geq |N_{\bar{G}}[Y_S]|$. Let $\tilde{Y}_S = V(G) - N_{\bar{G}}[Y_S]$. Then $|\tilde{Y}_S| > k$, since $|G| > 2(k+\bar{k})$. Hence $S \not\supset \tilde{Y}_S$.

Consequently the subgraph $G[Y_S \cup \tilde{Y}_S - S]$ of G is connected, for $G[Y_S \cup \tilde{Y}_S]$ includes the complete bipartite graph with vertex classes Y_S and $\tilde{Y}_S$. Let

H_S be the component of $G-S$ containing $Y_S \cup \tilde{Y}_S - S$. Note that any other end of $\bar{G}$ which is disjoint from Y_S is contained in $\tilde{Y}_S$. Then $Y-S \subset V(H_S)$.

(ii) Assume $\tilde{A}_1 \cap \tilde{A}_2 = \emptyset$. Then $|A_1 \cup A_2| > 2\bar{k}$. Thus, without loss of generality, we may assume $|A_1| > \bar{k}$. On the other hand, by the assumption that $N_G(A_1) \not\supset Y$ there is an end of $\bar{G}$, say Y_1 , not contained in $N_G(A_1)$, i.e. $Y_1 \not\subset N_G[A_1]$. However, since $Y_1 \cap A_1 = \emptyset$, as an immediate consequence of Complement lemma $N_{\bar{G}}(Y_1) \supset A_1$ contradicting the assumption $|A_1| > \bar{k}$.

To prove (3), for each A in $\mathcal{A}$ put $S = N_G(A)$. Then, by the choice of the component H_S , $H_S \cap Y \neq \emptyset$ so $A \cap V(H_S) = \emptyset$. Thus $A \subset N_{\bar{G}}(Y_S) \cup S$, this implies $|A| \leq \bar{k}$, so (3) holds. ■

Lemma 4 Suppose $|G| \geq 2(k+\bar{k})$. Then

- (1) For any $A \in \mathcal{A}$ and any $B \in \mathcal{B}$ $A \cap B = \emptyset$.
 (2) $\bigcup_{A \in \mathcal{A}} A \subset \bigcup_{B \in \mathcal{B}} N_{\bar{G}}(B)$ and $\bigcup_{B \in \mathcal{B}} B \subset \bigcup_{A \in \mathcal{A}} N_G(A)$.

proof. (1) Suppose not, i.e. there are $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $A \cap B \neq \emptyset$. Let $\tilde{A} = V(G) - N_G[A]$ then $N_G[\tilde{A}] \not\supset B$, since $A \cap B \neq \emptyset$. As a consequence of Complement lemma $\tilde{A} \subset N_{\bar{G}}[B]$ thus $V(G) \subset N_G[A] \cup N_{\bar{G}}[B]$. According to the previous lemma $|A| \leq \bar{k}$ and $|B| \leq k$, so $|V(G)| < 2(k+\bar{k})$ contradicting the assumption.

(2) We show only $\bigcup_{A \in \mathcal{A}} A \subset \bigcup_{B \in \mathcal{B}} N_{\bar{G}}(B)$. Recall Y is the union of all the ends of G . For each $A \in \mathcal{A}$ $N_G(A)$ can not contain whole Y and also $N_G(A) \not\supset \bigcup_{B \in \mathcal{B}} B$. According to (1) of this lemma it is an immediate consequence of Complement lemma that $\bigcup_{A \in \mathcal{A}} A \subset \bigcup_{B \in \mathcal{B}} N_{\bar{G}}(B)$. ■

By now we are all set to prove Main Theorem (2).

proof of Main Theorem (2) From the definition of critically $(k, \bar{k})$ -connected graph $V(G) = C(G) \cup C(\bar{G})$. Therefore as a consequence of Lemma 3 (2) $V(G) = \bigcup_{A \in \mathcal{A}} N_G[A] \cup \bigcup_{B \in \mathcal{B}} N_{\bar{G}}[B]$. Finally, by Lemma 4 (2) $V(G) = \bigcup_{A \in \mathcal{A}} N_G(A) \cup \bigcup_{B \in \mathcal{B}} N_{\bar{G}}(B)$, so that $|G| \leq \left| \bigcup_{A \in \mathcal{A}} N_G(A) \right| + \left| \bigcup_{B \in \mathcal{B}} N_{\bar{G}}(B) \right|$. To complete the proof it

remains to check the upper bound of $|\bigcup_{A \in \mathcal{A}} N_G(A)|$. We may suppose $\eta = 3$ or 4. We denote by $\#\mathcal{A}$ the number of fragments of $\mathcal{A}$. Because the family of ends of G is mutually disjoint the inequality $2a > \bar{k} \geq |A|$ implies each A of $\mathcal{A}$ contains exactly one end of G , so $\#\mathcal{A} = \eta$. For each B of $\mathcal{B}$, $N_{\bar{G}}(B)$ can contain at most one fragment A of $\mathcal{A}$, since $2a > \bar{k}$. Therefore by Complement lemma for each B there are $(\#\mathcal{A}-1)$ fragments of $\mathcal{A}$ such that $N_G(A)$ contains B . Consequently $|\bigcup_{A \in \mathcal{A}} N_G(A)| \leq \#\mathcal{A}k - (\#\mathcal{A}-2)|\bigcup_{B \in \mathcal{B}} B| \leq \eta k - \eta \bar{a}(\eta-2)$. From the first inequality in the proof of Main Theorem (1) (A) it follows $\eta \bar{a} \geq \frac{1}{4}(\eta k + 2\bar{\eta} + \eta)$, so finally $|\bigcup_{A \in \mathcal{A}} N_G(A)| \leq \frac{1}{4}\{(6-\eta)\eta k - (\eta+2\bar{\eta})(\eta-2)\}$ and this completes the proof. ■

References

- [1] K. Ando and Y. Usami, Critically (k,k) -connected graphs I, preprint
- [2] G. Chartrand, A. Kaugars and D. R. Lick, Critically n -connected graphs, Proceedings of the American Math. Soc. Vol 32, no. 1 (1972) 63-68
- [3] Y. O. Hamidoune, On critically h -connected simple graphs, Discrete Math. 32 (1980) 257-262
- [4] W. Mader, Ein Eigenschaft der Atome endlicher Graphen, Arch. Math. 22 (1971) 333-336
- [5] H. J. Veldman, Non- κ -critical vertices in graphs, Discrete Math. 44 (1983) 105-110