

On the Kashaev invariant of twist knots

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Let K denote the knot represented by the diagram with $n + 3$ crossings below. In this talk, we compute its *Kashaev invariant*, which is a special value of the *colored Jones polynomial*, and study its asymptotic behavior.



1. KASHAEV'S INVARIANT

Let N be a positive integer and

$$\mathcal{N} = \{0, 1, \dots, N - 1\}.$$

Then, for $i, j, k, l \in \mathcal{N}$, we define θ_{kl}^{ij} by

$$\theta_{kl}^{ij} = \begin{cases} 1 & \text{if } [i - j] + [j - l] + [l - k - 1] + [k - i] = N - 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $[m] \in \mathcal{N}$ denotes the residue of m modulo N . Furthermore, for $x \in \mathbb{C}$, we define $(x)_m$ by

$$(x)_m = (1 - x)(1 - x^2) \cdots (1 - x^{[m]}).$$

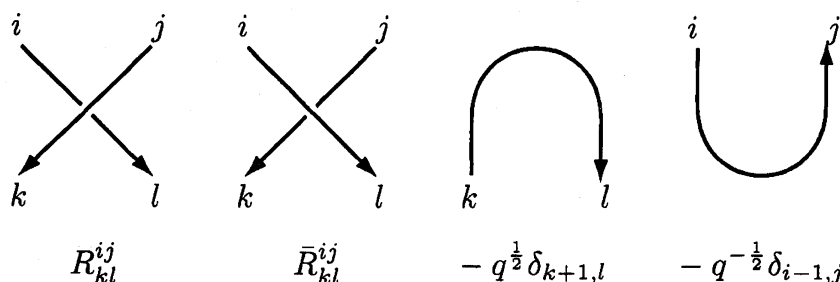
In what follows, we put

$$q = \exp \frac{2\pi\sqrt{-1}}{N}.$$

Then, the Kashaev invariant $\langle K \rangle_N$ of K is obtained by contracting the tensors associated to the following critical points, where

$$R_{kl}^{ij} = \frac{Nq^{-\frac{1}{2}+i-k}}{(q)_{[i-j]}(\bar{q})_{[j-l]}(q)_{[l-k-1]}(\bar{q})_{[k-i]}} \cdot \theta_{kl}^{ij},$$

$$\bar{R}_{kl}^{ij} = \frac{Nq^{\frac{1}{2}+j-l}}{(\bar{q})_{[i-j]}(q)_{[j-l]}(\bar{q})_{[l-k-1]}(q)_{[k-i]}} \cdot \theta_{kl}^{ij}.$$



Proposition. The Kashaev invariant $\langle K \rangle_N$ of K is given by

$$N^{n+1} \sum_{0 \leq i_1 \leq \dots \leq i_n < N} \frac{1}{(q)_{i_1}(\bar{q})_{i_n}} \prod_{\nu=1}^{n-1} \frac{1}{(\bar{q})_{i_\nu} (q)_{i_{\nu+1}-i_\nu} (\bar{q})_{N-1-i_{\nu+1}}}.$$

Example. Suppose $n = 3$. Then,

$$\begin{aligned} \langle K \rangle_N &= \sum_{c \leq d \leq b} \frac{Nq^{-\frac{1}{2}+b+1}}{(q)_{b-d}(\bar{q})_d(\bar{q})_{N-1-b}} \cdot \frac{Nq^{-\frac{1}{2}-c}}{(\bar{q})_{N-1-d}(q)_{d-c}(\bar{q})_c} \\ &\quad \times \left(\sum_{a=c}^{N-1} \frac{Nq^{\frac{1}{2}}}{(\bar{q})_{N-1-a}(q)_a} \cdot \frac{Nq^{\frac{1}{2}+c}}{(\bar{q})_{a-c}(q)_c(q)_{N-1-a}} \right) \\ &\quad \times \left(\sum_{e=b}^{N-1} \frac{Nq^{-\frac{1}{2}}}{(q)_e(\bar{q})_{N-1-e}} \cdot \frac{Nq^{-\frac{1}{2}-b}}{(q)_{N-1-e}(q)_{e-b}(\bar{q})_b} \right). \end{aligned}$$

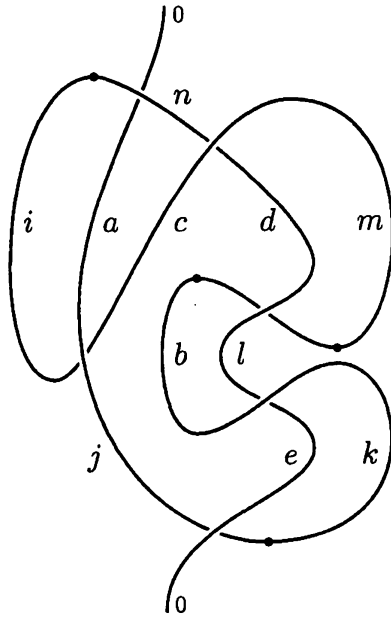
from the picture below. This is further equal to

$$\sum_{c \leq d \leq b} \frac{Nq^{-\frac{1}{2}+b+1}}{(q)_{b-d}(\bar{q})_d(\bar{q})_{N-1-b}} \cdot \frac{Nq^{-\frac{1}{2}-c}}{(\bar{q})_{N-1-d}(q)_{d-c}(\bar{q})_c} \cdot \frac{N^2 q^{c-b}}{(q)_c(\bar{q})_b},$$

where we used the following lemma.

Lemma. $(q)_m(\bar{q})_{N-1-m} = N$ and

$$\sum_{i \leq m \leq j} \frac{1}{(q)_{j-m}(\bar{q})_{m-i}} = 1.$$



$$\begin{aligned} 6(N-1) &= ([i+1] + [j-i-1] + [-j]) \\ &\quad + ([k] + [l-k] + [m-l] + [n-m] + [-n]) \\ &\quad + ([a-i-1] + [i-a]) \\ &\quad + ([n-a-1] + [c-n] + [a-c]) \\ &\quad + ([d-c-1] + [b+1-d] + [e-b-1] + [j-e] + [c-j]) \\ &\quad + ([m-d] + [d-m-1]) \\ &\quad + ([l-b-1] + [b-l]) \\ &\quad + ([k-e] + [e-k-1]) \\ &\geq ([i+1] + [j-i-1] + [-j]) \\ &\quad + ([k] + [l-k] + [m-l] + [n-m] + [-n]) \\ &\quad + 6(N-1) \longrightarrow i = -1, j = k = l = m = n = 0, \\ &\quad 0 \leq a < N, \\ &\quad 0 \leq c \leq a < N, \\ &\quad 0 \leq c < d \leq b+1 \leq e \leq N, \\ &\quad 1 \leq d \leq N, \\ &\quad 0 \leq b < N, \\ &\quad 1 \leq e \leq N \end{aligned}$$

2. QUANTUM DILOGARITHMS

Let

$$\psi_N(z) = \exp \int_{-\infty}^{\infty} \frac{e^{\sqrt{N}(2z+1)t} dt}{4t \sinh(t/\sqrt{N}) \sinh(\sqrt{N}t)},$$

and

$$p_k = \frac{2k+1}{2N}.$$

Then, the sets of poles and zeros of ψ_N are given by $\{p_k \mid k \geq N\}$ and $\{p_k \mid k < 0\}$ respectively, and

$$\begin{aligned} \frac{1}{(q)_k} &= \frac{\psi_N(p_k)}{\psi_N(p_0)}, & \frac{1}{(\bar{q})_k} &= \frac{\psi_N(1-p_0)}{\psi_N(1-p_k)}, \\ \frac{\psi_N(p_0)}{\sqrt{N}} &= \exp \frac{N}{2\pi\sqrt{-1}} \left(\frac{\pi^2}{6} - \frac{\pi^2}{2N} + \frac{\pi^2}{6N^2} \right) = \sqrt{N} \psi_N(1-p_0). \end{aligned}$$

By using these quantum dilogarithms, we can write

$$\begin{aligned} \langle K \rangle_N &= N^{n+1} \sum_{k_1 \leq \dots \leq k_n} \frac{\psi_N(p_{k_1})}{\psi_N(p_0)} \frac{\psi_N(1-p_0)}{\psi_N(1-p_{k_n})} \\ &\quad \times \prod_{\nu=1}^{n-1} \frac{\psi_N(1-p_0)}{\psi_N(1-p_{k_\nu})} \frac{\psi_N(p_{k_{\nu+1}-k_\nu})}{\psi_N(p_0)} \frac{\psi_N(1-p_0)}{\psi_N(1-p_{k_{\nu+1}})}. \end{aligned}$$

If we put

$$\Psi_N(z_1, \dots, z_n) = e^{\frac{N(n-1)}{2\pi\sqrt{-1}} \left(\frac{\pi^2}{6} - \frac{\pi^2}{2N} + \frac{\pi^2}{6N^2} \right)} \frac{\psi_N(z_1)}{\psi_N(1-z_n)} \prod_{\nu=1}^{n-1} \frac{\psi_N(z_{\nu+1} - z_\nu + p_0)}{\psi_N(1-z_\nu) \psi_N(1-z_{\nu+1})},$$

we have

$$\langle K \rangle_N = N^{\frac{3-n}{2}} \sum_{k_1=0}^{N-1} \dots \sum_{k_n=0}^{N-1} \Psi_N(p_{k_1}, \dots, p_{k_n})$$

because $\psi_N(p_{k_{\nu+1}-k_\nu}) = 0$ if $k_{\nu+1} < k_\nu$.

3. INTEGRALS

Let $Q_\nu = e^{2\pi\sqrt{-1}z_\nu}$ and

$$C = \left\{ x + y\sqrt{-1} \mid (x - \frac{1}{2})^2 + y^2 = (\frac{1}{2})^2 \right\}.$$

Then, by the residue theorem,

$$\langle K \rangle_N = (-1)^n N^{\frac{n+3}{2}} \int_C \frac{dz_1}{1+Q_1^N} \dots \int_C \frac{dz_n}{1+Q_n^N} \Psi_N(z_1, \dots, z_n).$$

Let

$$A = \{z \in C \mid \operatorname{Im} z \geq 0\}, \quad B = \{z \in C \mid \operatorname{Im} z \leq 0\}.$$

Then, we have

$$\int_C \frac{dz_\nu}{1 + Q_\nu^N} = \int_0^1 (Q_\nu^N - 1 + Q_\nu^{-N}) dz_\nu + \int_A \frac{Q_\nu^{2N}}{1 + Q_\nu^N} dz_\nu - \int_B \frac{Q_\nu^{-2N}}{1 + Q_\nu^{-N}} dz_\nu$$

because

$$\begin{aligned} \int_A \frac{dz_\nu}{1 + Q_\nu^N} &= \int_1^0 (1 - Q_\nu^N) dz_\nu + \int_A \frac{Q_\nu^{2N}}{1 + Q_\nu^N} dz_\nu, \\ \int_B \frac{dz_\nu}{1 + Q_\nu^N} &= \int_B \frac{Q_\nu^{-N} dz_\nu}{1 + Q_\nu^{-N}} = \int_0^1 Q_\nu^{-N} dz_\nu - \int_B \frac{Q_\nu^{-2N}}{1 + Q_\nu^{-N}} dz_\nu. \end{aligned}$$

In what follows, for $z \in \mathbb{C}$, we put

$$x_z = \operatorname{Re} 2\pi z, \quad y_z = \operatorname{Im} 2\pi z, \quad \omega_z = -\arg(1 - e^{2\pi\sqrt{-1}z}).$$

Lemma.

$$\lim_{y_{z_\nu} \rightarrow \pm\infty} \Psi_N(z_1, \dots, z_n) Q_\nu^{\pm N} = 0.$$

Proposition.

$$\langle K \rangle_N \sim (-1)^n N^{\frac{n+3}{2}} \int_0^1 dz_1 \cdots \int_0^1 dz_n \Psi_N(z_1, \dots, z_n).$$

4. ASYMPTOTICS

Define

$$\mathcal{L}(z) = \operatorname{Li}_2(e^{2\pi\sqrt{-1}z}) + \beta_z(2\pi z - \frac{1}{2}\beta_z - \pi),$$

where Li_2 denotes Euler's dilogarithm function and

$$\beta_z = \begin{cases} 2\pi[\operatorname{Re} z] & \text{if } \operatorname{Im} z < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have

$$\psi_N(z) \sim e^{\frac{N}{2\pi\sqrt{-1}}\{\mathcal{L}(z) + O(N^{-2})\}}.$$

Furthermore, we put

$$V(z) = \Lambda(x_z) + \Lambda(\omega_z) - \Lambda(x_z + \omega_z),$$

where Λ denotes Lobachevsky's function. Then, we have

$$\operatorname{Im} \mathcal{L}(z) = y_z(\beta_z - \omega_z) + V(z), \quad \frac{\partial}{\partial y_z} \operatorname{Im} \mathcal{L}(z) = \beta_z - \omega_z.$$

Define $H(z_1, \dots, z_n)$ by

$$\mathcal{L}(z_1) - \mathcal{L}(z_n) + \frac{(n-1)\pi^2}{6} + \sum_{\nu=1}^{n-1} \{\mathcal{L}(z_{\nu+1} - z_\nu) - \mathcal{L}(1 - z_\nu) - \mathcal{L}(z_{\nu+1})\}.$$

For simplicity, we put $f(z_1, \dots, z_n) = \text{Im } H(z_1, \dots, z_n)$.

Proposition.

$$\Psi_N(z_1, \dots, z_n) \sim e^{\frac{N}{2\pi\sqrt{-1}}\{H(z_1, \dots, z_n) + O(N^{-1})\}}.$$

Example. Suppose $n = 3$. Then, $f(z_1, z_2, z_3)$ is equal to

$$\begin{aligned} & V(z_1, z_2, z_3) + y_{z_1}(-\omega_{z_1} - \omega_{1-z_1} - \beta_{z_2-z_1} + \omega_{z_2-z_1}) \\ & + y_{z_2}(\beta_{z_2-z_1} - \omega_{z_2-z_1} + \pi - x_{z_2} - \beta_{z_3-z_2} + \omega_{z_3-z_2}) \\ & + y_{z_3}(\beta_{z_3-z_2} - \omega_{z_3-z_2} + \pi - x_{z_3}), \end{aligned}$$

where $V(z_1, \dots, z_n)$ is defined by

$$V(z_1) - V(z_n) + \sum_{\nu=1}^{n-1} \{V(z_{\nu+1} - z_\nu) - V(1 - z_\nu) - V(z_{\nu+1})\}.$$

How does $f(z_1, z_2, z_3)$ behave when $y_{z_1}^2 + y_{z_2}^2 + y_{z_3}^2 \rightarrow \infty$? Since

$$\lim_{y_z \rightarrow \infty} (\beta_z - \omega_z) = 0, \quad \lim_{y_z \rightarrow -\infty} (\beta_z - \omega_z) = x_z - \pi,$$

we check its behavior along the following 3 lines;

$$y_{z_2-z_1} = y_{z_3-z_2} = 0, \quad y_{z_1} = y_{z_3-z_2} = 0, \quad y_{z_1} = y_{z_2-z_1} = 0$$

with $x_{z_1}, x_{z_2}, x_{z_3}$ fixed. For simplicity, we put

$$\lambda(y) = \frac{1}{2}(y - |y|).$$

Then, $f(z_1, z_2, z_3)$ is approximated by

$$\lambda(y_{z_1})(x_{z_1} - x_{z_2} - x_{z_3} + \pi) + \lambda(-y_{z_1})(x_{z_1} + x_{z_2} + x_{z_3} - 3\pi)$$

when $y_{z_2-z_1} = y_{z_3-z_2} = 0$, by

$$\lambda(y_{z_2})(-x_{z_1} - x_{z_3} + \pi) + \lambda(-y_{z_2})(x_{z_2} + x_{z_3} - 2\pi)$$

when $y_{z_1} = y_{z_3-z_2} = 0$, and by

$$\lambda(y_{z_3})(-x_{z_2}) + \lambda(-y_{z_3})(x_{z_3} - \pi)$$

when $y_{z_1} = y_{z_2-z_1} = 0$. Therefore, we can observe

$$\lim_{y_{z_1}^2 + y_{z_2}^2 + y_{z_3}^2 \rightarrow \infty} f(z_1, z_2, z_3) = \infty$$

if $x_{z_1}, x_{z_2}, x_{z_3}$ satisfy the following conditions.

$$\begin{aligned} \pi &< -x_{z_1} + x_{z_2} + x_{z_3} < x_{z_1} + x_{z_2} + x_{z_3} < 3\pi, \\ \pi &< x_{z_1} + x_{z_3} < x_{z_2} + x_{z_3} < 2\pi, \quad x_{z_3} < \pi. \end{aligned}$$

This region will play an important role in the following argument.

Let Δ be the set of $(z_1, \dots, z_n) \in [0, 1]^n$ satisfying

$$\frac{1}{2}(n - \nu) < z_{\nu-1} + \sum_{k=\nu+1}^n z_k < \sum_{k=\nu}^n z_k < \frac{1}{2}(n - \nu + 1)$$

for $1 < \nu \leq n$ and

$$\frac{1}{2}(n - 2) < -z_1 + \sum_{k=2}^n z_k < \sum_{k=1}^n z_k < \frac{1}{2}n.$$

The main purpose of this note is to show

Proposition. *Let $(\zeta_1, \dots, \zeta_n)$ be the solution to*

$$\frac{\partial H}{\partial z_\nu} \equiv 0 \pmod{2\pi\sqrt{-1}}.$$

satisfying $(\operatorname{Re} \zeta_1, \dots, \operatorname{Re} \zeta_n) \in \Delta$. Then,

$$\int_{\Delta} \Psi_N(z_1, \dots, z_n) dz_1 \wedge \dots \wedge dz_n \sim N^{-\frac{n}{2}} e^{\frac{N}{2\pi\sqrt{-1}} H(\zeta_1, \dots, \zeta_n)}.$$

Note that $f(\zeta_1, \dots, \zeta_n)$ is equal to the complex volume of K .

Proof. Define $p: \mathbb{C}^n \rightarrow \mathbb{R}^n$ by

$$p(z_1, \dots, z_n) = (\operatorname{Re} z_1, \dots, \operatorname{Re} z_n).$$

Let Σ be the set of $(z_1, \dots, z_n) \in p^{-1}(\Delta)$ satisfying

$$\begin{aligned} y_{z_1} &= \log \frac{\sin \frac{1}{2}(n\pi - x_{z_1} + x_{z_2} + \dots + x_{z_n})}{\sin \frac{1}{2}(n\pi + x_{z_1} + x_{z_2} + \dots + x_{z_n})}, \\ y_{z_2} &= \log \frac{\sin(x_{z_1} + x_{z_3} + \dots + x_{z_n})}{\sin(x_{z_2} + x_{z_3} + \dots + x_{z_n})} + y_{z_1}, \\ &\dots \\ y_{z_n} &= \log \frac{\sin x_{z_{n-1}}}{\sin x_{z_n}} + y_{z_{n-1}}, \end{aligned}$$

which is the unique solution to

$$\frac{\partial f}{\partial y_{z_\nu}} = 0.$$

Then, $f|_{\Sigma}$ takes its unique maximum at $(\zeta_1, \dots, \zeta_n) \in \Sigma$. Let

$$E_{\pm} = f^{-1}((-\infty, f(\zeta_1, \dots, \zeta_n) \pm \varepsilon]) \cap p^{-1}(\Delta)$$

and I the set of $(z_1(t), \dots, z_n(t)) \in E_+ - E_-$ satisfying

$$\frac{dz_\nu}{dt} = \frac{\partial f}{\partial \bar{z}_\nu}, \quad \lim_{t \rightarrow -\infty} (z_1(t), \dots, z_n(t)) = (\zeta_1, \dots, \zeta_n).$$

Then, $\operatorname{Re} H(z_1, \dots, z_n)$ is constant on I because

$$\begin{aligned} \frac{d(\operatorname{Re} H)}{dt} &= \sum_{\nu=1}^n \operatorname{Re} \left\{ \frac{\partial H}{\partial z_\nu} \cdot \frac{dz_\nu}{dt} + \frac{\partial H}{\partial \bar{z}_\nu} \cdot \frac{d\bar{z}_\nu}{dt} \right\} \\ &= \sum_{\nu=1}^n \operatorname{Re} \left\{ \frac{\partial(\operatorname{Re} H + \sqrt{-1} f)}{\partial z_\nu} \cdot \frac{dz_\nu}{dt} \right\} \\ &= \sum_{\nu=1}^n \operatorname{Re} \left\{ 2\sqrt{-1} \cdot \frac{\partial f}{\partial z_\nu} \cdot \frac{\partial f}{\partial \bar{z}_\nu} \right\} = 0. \end{aligned}$$

Since $E_+ \simeq E_- \cup I \simeq \Sigma$, by the saddle point method, we have

$$\int_{\Delta} \Omega_N = \int_{\Sigma} \Omega_N \sim \int_I \Omega_N \sim N^{-\frac{n}{2}} e^{\frac{N}{2\pi\sqrt{-1}} H(\zeta_1, \dots, \zeta_n)},$$

where $\Omega_N = \Psi_N(z_1, \dots, z_n) dz_1 \wedge \dots \wedge dz_n$. $\square$

