

ON INVERSES FOR DIFFERENTIAL OPERATORS

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ABSTRACT. We propose geometrically invariant formulas for fundamental solutions and heat kernels of subelliptic partial differential operators.

1. The fundamental solution

The differential operator

$$(1.1) \quad \Delta_\lambda = \frac{1}{2}(X_1^2 + X_2^2) - \frac{1}{2}i\lambda[X_1, X_2],$$

with

$$(1.2) \quad \begin{cases} X_1 = \frac{\partial}{\partial x_1} + 2kx_2|x|^{2k-2}\frac{\partial}{\partial u}, \\ X_2 = \frac{\partial}{\partial x_2} - 2kx_1|x|^{2k-2}\frac{\partial}{\partial u}, \end{cases}$$

and $|x|^2 = x_1^2 + x_2^2$, $[X_1, X_2] = X_1X_2 - X_2X_1$ is not elliptic since it is the sum of squares of only 2 linearly independent vector fields in 3 dimensions, but consecutive Lie brackets of X_1 and X_2 do generate the full tangent space at every point of $\mathbb{R}^3$, so Δ_λ is subelliptic. The fundamental solution K_λ is the distribution solution of

$$(1.3) \quad \Delta_\lambda K_\lambda(x, u; x^{(0)}, u^{(0)}) = \delta(x - x^{(0)})\delta(u - u^{(0)}),$$

parametrized by $(x^{(0)}, u^{(0)}) \in \mathbb{R}^3 \times \mathbb{R}$. We set

$$(1.4) \quad K_\lambda(x, u; x^{(0)}, u^{(0)}) = \int_{\mathbb{R}} \frac{v_\lambda(x, u; x^{(0)}, u^{(0)}, \tau)}{g(x, u; x^{(0)}, u^{(0)}, \tau)} E(x, u; x^{(0)}, u^{(0)}, \tau) d\tau,$$

where g is a solution of the Hamilton-Jacobi equation

$$(1.5) \quad \frac{\partial g}{\partial \tau} + \frac{1}{2}(X_1 g)^2 + \frac{1}{2}(X_2 g)^2 = 0.$$

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g is given by a modified action integral of a complex Hamiltonian problem.

The associated energy

$$(1.6) \quad E = -\frac{\partial g}{\partial \tau}$$

is the first invariant of the motion, and the volume element v_λ is the solution of a second order transport equation. Let

$$(1.7) \quad H = \frac{1}{2}(\xi_1 + 2kx_2|x|^{2k-2}\theta)^2 + \frac{1}{2}(\xi_2 - 2kx_1|x|^{2k-2}\theta)^2$$

denote the Hamiltonian where ξ_1 , ξ_2 and θ represent the momenta of x_1 , x_2 , and u , respectively. The complex bicharacteristics are solutions of the Hamiltonian system of differential equations

$$(1.8) \quad \begin{cases} \dot{x}_j = H_{\xi_j}, & \dot{\xi}_j = -H_{x_j}, & j = 1, 2, \\ \dot{u} = H_\theta, & \dot{\theta} = -H_u \end{cases}$$

with the somewhat unusual boundary condition

$$(1.9) \quad \begin{cases} x_1(0) = x_1^{(0)}, & x_1(\tau) = x_1, \\ x_2(0) = x_2^{(0)}, & x_2(\tau) = x_2, \end{cases}$$

$$(1.10) \quad u(\tau) = u,$$

$$(1.11) \quad \theta(0) = -i.$$

Then the energy E is

$$E = \frac{1}{2}\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2,$$

and the modified action g is given by

$$g = -iu(0) + \int_0^\tau \left[\sum_{j=1}^2 \xi_j(s)\dot{x}_j(s) + \theta(s)\dot{u}(s) - H(x(s), u(s), \xi(s), \theta(s)) \right] ds.$$

We note that the "missing direction" u must be treated separately.

The volume element v_λ is the solution of the following second order "transport" equation:

$$(1.12) \quad (T + \Delta_\lambda g) \frac{\partial v_\lambda}{\partial \tau} + E \Delta_\lambda v_\lambda = 0,$$

see [1], p. 92, where

$$(1.13) \quad T = \frac{\partial}{\partial \tau} + \sum_{j=1}^2 (X_j g) X_j$$

is differentiation along the bicharacteristic. Formula (1.4) has a simple geometric interpretation. The operator Δ_λ has a characteristic variety in $T^*\mathbb{R}_3$ given by $H = 0$. Over every point $(x, u) \in \mathbb{R}_3$, $H = 0$ is a line parametrized by $\theta \in (-\infty, \infty)$,

$$(1.14) \quad \xi_1 = -2kx_2|x|^{2k-2}\theta, \quad \xi_2 = 2kx_1|x|^{2k-2}\theta.$$

Consequently, K_λ may be thought of as the $(\text{action})^{-1}$ summed over the characteristic variety with measure Ev_λ . We note that when Δ_λ is elliptic, its characteristic variety is the zero section, so we do get simply $(\text{action})^{-1}$, the Newton potential, as expected; τg behaves like the square of a distance function, even though it is complex.

K_λ has been worked out explicitly in [1]. Let $(x^{(0)}, u^{(0)})$ and (x, u) denote 2 arbitrary points of $\mathbb{R}^3$. We obtain 2 invariants of the motion, the energy E and the angular momentum Ω . Then

$$g = -i(u - u^{(0)}) + \left(1 - \frac{1}{k}\right)E\tau + \frac{1}{2k}\text{sgn}\tau \left[\sqrt{2E|x|^2 + W(|x|^2)^2} - \sqrt{2E|x^{(0)}|^2 + W(|x^{(0)}|^2)^2} \right],$$

where we use the principal branch of the square roots, and

$$(1.15) \quad W(u) = 2ku^k - \Omega, \quad \Omega = \Omega(x, u; x^{(0)}, u^{(0)}, \tau).$$

Theorem 1.1. *The fundamental solution $K_\lambda(x, u; x^{(0)}, u^{(0)})$ has the following invariant representation:*

$$(1.16) \quad K_\lambda = \int_{\mathbb{R}} v_\lambda \frac{Ed\tau}{g} = - \int_{g_-}^{g_+} v_\lambda \frac{dg}{g},$$

where the second order transport equation for v_λ may be reduced to an Euler-Poisson-Darboux equation and solved explicitly as a function of E and Ω .

Namely,

$$v_\lambda = \frac{c_\lambda}{k} (A_+ - g)^{-\frac{1-\lambda}{2}} (A_- + g)^{-\frac{1+\lambda}{2}} F_\lambda(q_+, q_-),$$

with

$$c_\lambda = \frac{-e^{-i\pi\frac{1-\lambda}{2}}}{4\pi^2\Gamma(\frac{1-\lambda}{2})\Gamma(\frac{1+\lambda}{2})},$$

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$$A_{\pm} = \frac{1}{k}\Omega_{\pm} + g_{\pm}, \quad \Omega_{\pm} = \lim_{\tau \rightarrow \pm\infty} \Omega, \quad g_{\pm} = \lim_{\tau \rightarrow \pm\infty} g,$$

and

$$q_{\pm} = \frac{2^{1/k}(x_1 \pm ix_2)(x_1^{(0)} \mp ix_2^{(0)})}{(A_{\pm} \mp g)^{1/k}},$$

and $F_{\lambda}(q_+, q_-)$ is a hypergeometric function of 2 variables

$$F_{\lambda}(q_+, q_-) = \frac{1}{\Gamma(\frac{1-\lambda}{2})\Gamma(\frac{1+\lambda}{2})} \int_0^1 \int_0^1 \frac{ds_+ ds_-}{s_+ s_-} \left\{ \left(\frac{s_+}{1-s_+} \right)^{\frac{1-\lambda}{2}} \left(\frac{s_-}{1-s_-} \right)^{\frac{1+\lambda}{2}} \frac{1 - q_+ q_- (s_+ s_-)^{1/k}}{(1 - q_+ s_+^{1/k})(1 - q_- s_-^{1/k})(1 - (q_+ q_-)^k s_+ s_-)} \right\}.$$

2. The heat kernel

$P_t(x, u; x^{(0)}, u^{(0)})$, $t > 0$, the heat kernel, is the solution of the following initial value problem:

$$(2.1) \quad \frac{\partial P_t}{\partial t} - \Delta_{\lambda} P_t = 0, \quad t > 0,$$

$$(2.2) \quad \lim_{t \rightarrow 0} P_t(x, u; x^{(0)}, u^{(0)}) = \delta(x - x^{(0)})\delta(u - u^{(0)}).$$

We expect P_t to have the form

$$(2.3) \quad P_t = \int_{\mathbb{R}} \frac{e^{-f/t}}{t^2} w_{\lambda}(-f_{\tau} d\tau),$$

where $f = \tau g$ and $f_{\tau} = \partial f / \partial \tau$. So far (2.3) has been justified and worked out explicitly only when $k = 1$; in that case

$$f = -iu\tau + |x|^2 \tau \coth(2\tau),$$

$$w_{\lambda} = \frac{-2\tau e^{-2\tau\lambda}}{4\pi^2 f_{\tau} \sinh(2\tau)},$$

see [2]. We do expect (2.3) to hold in general and to that end state and prove

Theorem 2.1. (2.3) is a solution of (2.1) if w_{λ} is a solution of

$$(2.4) \quad \tau(T + \Delta_{\lambda} g) \frac{\partial w_{\lambda}}{\partial \tau} - f_{\tau} \Delta_{\lambda} w_{\lambda} = 0.$$

Furthermore, f_{τ} is a constant of motion.

Proof. To work out

$$\left(\Delta_\lambda - \frac{\partial}{\partial t}\right)P_t,$$

we start with

$$\Delta_\lambda e^{-f/t} = \left(\frac{H(\nabla f)}{t^2} - \frac{\Delta_\lambda f}{t}\right)e^{-f/t}.$$

Also,

$$\begin{aligned}\Delta_\lambda(e^{-f/t}W) &= \Delta_\lambda(e^{-f/t})W + \sum_{j=1}^2 X_j(e^{-f/t})X_jW + e^{-f/t}\Delta_\lambda W \\ &= \Delta_\lambda(e^{-f/t})W - \frac{e^{-f/t}}{t} \sum_{j=1}^2 (X_j f)(X_j W) + e^{-f/t}\Delta W,\end{aligned}$$

and

$$\frac{\partial}{\partial t} \frac{e^{-f/t}}{t^{\alpha+1}} = \frac{e^{-f/t}}{t^{\alpha+1}} \left(\frac{f}{t^2} - \frac{\alpha+1}{t} \right).$$

Consequently,

$$\begin{aligned}\left(\Delta_\lambda - \frac{\partial}{\partial t}\right) \frac{e^{-f/t}W}{t^{\alpha+1}} &= \frac{e^{-f/t}}{t^{\alpha+1}} \left[\frac{H(\nabla f) - f}{t^2} W \right. \\ &\quad \left. + \frac{(\alpha+1)W - (\Delta_\lambda f)W - \tau(Xg) \cdot (XW)}{t} + \Delta W \right].\end{aligned}$$

Using

$$(2.5) \quad \tau \frac{\partial f}{\partial \tau} + H(\nabla f) = f,$$

one has

$$\begin{aligned}(2.6) \quad e^{-f/t} [H(\nabla f) - f] W &= -e^{-f/t} \tau \frac{\partial f}{\partial \tau} W \\ &= t \tau \frac{\partial}{\partial \tau} (e^{-f/t}) W,\end{aligned}$$

and integrating by parts we obtain

$$\begin{aligned}(2.7) \quad \left(\Delta - \frac{\partial}{\partial t}\right)P_t &= \int_{\mathbb{R}} \frac{e^{-f/t}}{t^{\alpha+1}} \left[\frac{-\frac{\partial}{\partial \tau}(\tau W)}{t} \right. \\ &\quad \left. + \frac{(\alpha+1)W - (\Delta_\lambda f)W - \tau(Xg) \cdot (XW)}{t} \right. \\ &\quad \left. + \Delta_\lambda W \right] d\tau \\ &= \int_{\mathbb{R}} \frac{e^{-f/t}}{t^{\alpha+1}} \left[\frac{-\tau TW + (\alpha - \Delta_\lambda f)W}{t} + \Delta_\lambda W \right] d\tau.\end{aligned}$$

Next,

$$(2.8) \quad Tg = \frac{\partial g}{\partial \tau} + (Xg) \cdot (Xg) = -E + 2E = E,$$

so from (2.5) and from $TE = 0$,

$$(2.9) \quad \begin{aligned} f_\tau + \tau H(\nabla g) &= g, \\ f_\tau + \tau E &= g \\ Tf_\tau + E &= E, \\ Tf_\tau &= 0, \end{aligned}$$

and f_τ is an invariant of the motion. Also,

$$(2.10) \quad Tf = g + \tau E.$$

Since $W = -f_\tau w_\lambda$, one has

$$\begin{aligned} & \left(\Delta_\lambda - \frac{\partial}{\partial t} \right) P_t \\ &= \int_{\mathbb{R}} \frac{e^{-f/\tau}}{t^{\alpha+1}} \left[\frac{f_\tau (\tau T w_\lambda + (\Delta_\lambda f - \alpha) w_\lambda)}{t} + \Delta W \right] d\tau \\ &= \int_{\mathbb{R}} \frac{d\tau}{t^{\alpha+1}} \left[-\frac{\partial}{\partial \tau} (e^{-f/t}) (\tau T w_\lambda + (\Delta_\lambda f - \alpha) w_\lambda) + e^{-f/t} \Delta_\lambda W \right] \\ &= \int_{\mathbb{R}} \frac{e^{-f/t}}{t^{\alpha+1}} \left[\frac{\partial}{\partial \tau} (\tau T w_\lambda + (\Delta_\lambda f - \alpha) w_\lambda) + \Delta_\lambda (-f_\tau w_\lambda) \right] d\tau \end{aligned}$$

after a second integration by parts. Next

$$\Delta_\lambda (-f_\tau w_\lambda) = -(\Delta_\lambda f_\tau) w_\lambda - (Xf) \cdot (Xw_\lambda) - f_\tau \Delta_\lambda w_\lambda,$$

so,

$$\begin{aligned}
& \frac{\partial}{\partial \tau} (\tau T w_\lambda + (\Delta_\lambda f - \alpha) w_\lambda) - \Delta_\lambda (f_\tau w_\lambda) \\
&= T w_\lambda + \tau \frac{\partial}{\partial \tau} (T w_\lambda) + (\Delta_\lambda f_\tau) w_\lambda + (\Delta_\lambda f - \alpha) \frac{\partial w_\lambda}{\partial \tau} \\
&\quad - (\Delta_\lambda f_\tau) w_\lambda - (X f_\tau) \cdot (X w_\lambda) - f_\tau \Delta_\lambda w_\lambda \\
&= \frac{\partial w_\lambda}{\partial \tau} + (X g) \cdot (X w_\lambda) + \tau T \frac{\partial w_\lambda}{\partial \tau} + \tau (X g_\tau) \cdot (X w_\lambda) \\
&\quad + (\Delta_\lambda f - \alpha) \frac{\partial w_\lambda}{\partial \tau} - (X f_\tau) \cdot (X w_\lambda) - f_\tau \Delta_\lambda w_\lambda \\
&= \tau T \frac{\partial w_\lambda}{\partial \tau} [\tau X g_\tau + X g - X f_\tau] \cdot (X w_\lambda) \\
&\quad + (\Delta_\lambda f + 1 - \alpha) \frac{\partial w_\lambda}{\partial \tau} - f_\tau \Delta_\lambda w_\lambda \\
&= \tau T \frac{\partial w_\lambda}{\partial \tau} + (\Delta_\lambda f + 1 - \alpha) \frac{\partial w_\lambda}{\partial \tau} - f_\tau \Delta_\lambda w_\lambda.
\end{aligned}$$

Setting $\alpha = 1$, the integrand in

$$\left(\Delta - \frac{\partial}{\partial t} \right) P_t$$

vanishes if

$$\tau (T + \Delta_\lambda f) \frac{\partial w_\lambda}{\partial \tau} - f_\tau \Delta_\lambda w_\lambda = 0$$

which yields (2.4) and we have derived Theorem 2.1. $\square$

In the proof of Theorem 2.1 we assumed that the non-integrated terms after the integrations-by-parts vanish at $\tau = \pm\infty$. These should be considered boundary conditions which may fix the required solution w_λ of (2.4) uniquely.

We note that (2.4) may be written in the following form:

$$(2.11) \quad \tau \left[(T + \Delta_\lambda g) \frac{\partial w_\lambda}{\partial \tau} + E \Delta_\lambda w_\lambda \right] = g \Delta_\lambda w_\lambda.$$

In view of (1.13) one may try to find w_λ as a power series expansion in g with first term v_λ .

We expect that formulas (1.4) and (2.3) apply to rather general subelliptic differential operators. Note that for the operators discussed here $\theta(s) =$ constant, which is not expected in general.

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REFERENCES

- [1] R. Beals, B. Gaveau, P.C. Greiner: *On a geometric formula for the fundamental solution of subelliptic Laplacians*, Math. Nachr. **181** (1996) 81–163.
- [2] R. Beals, B. Gaveau, P.C. Greiner: *Hamilton-Jacobi theory and the heat kernel on Heisenberg groups*, J. Math. Pures Appl. **79** (2000) 633–689.

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