

THE DISTRIBUTION OF CLASS NUMBERS OF PURE NUMBER FIELDS

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Much is known about the statistical distribution of class numbers of binary quadratic forms and quadratic fields. Let $d \equiv 0, 1 \pmod{4}$ and d not a perfect square. Define $h(d)$ as the number of equivalence classes of primitive binary quadratic forms with discriminant d (and positive definite in case $d < 0$). For $d > 0$, let $\epsilon_d := (u_d + v_d\sqrt{d})/2$, where (u_d, v_d) is the fundamental solution of Pell's equation $u^2 - dv^2 = 4$. If d is a fundamental discriminant then $h(d)$ is also the class number of $\mathbb{Q}(\sqrt{d})$ in the narrow sense.

Gauß [5] conjectured and Mertens [9] and Siegel [11] later proved that

$$\sum_{0 < d \leq x} h(d) \log \epsilon_d \sim \frac{\pi^2}{18\zeta(3)} x^{3/2}, \quad \sum_{0 > d \geq -x} h(d) \sim \frac{\pi}{18\zeta(3)} x^{3/2}.$$

Chowla and Erdős [3] proved that there is a continuous distribution function F such that for all $z \in \mathbb{R}$,

$$\lim_{x \rightarrow \infty} \frac{1}{x/2} \# \left\{ 0 < d \leq x \mid \frac{h(d) \log \epsilon_d}{d^{1/2}} \leq e^z \right\} = F(z),$$

$$\lim_{x \rightarrow \infty} \frac{1}{x/2} \# \left\{ 0 > d \geq -x \mid \frac{h(d)\pi}{|d|^{1/2}} \leq e^z \right\} = F(z).$$

Elliott [4] showed that $F \in C^\infty(\mathbb{R})$ and it has the characteristic function

$$\Psi(t) = \prod_p \left(\frac{1}{p} + \frac{1}{2} \left(1 - \frac{1}{p} \right) \left(1 - \frac{1}{p} \right)^{-it} + \frac{1}{2} \left(1 - \frac{1}{p} \right) \left(1 + \frac{1}{p} \right)^{-it} \right), \quad t \in \mathbb{R}.$$

Barban [1] proved that for $q \in \mathbb{N}$, the q -th moment β_q of $F(\log z)$ exists and that

$$\lim_{x \rightarrow \infty} \frac{1}{x/2} \sum_{0 < d \leq x} \left(\frac{h(d) \log \epsilon_d}{d^{1/2}} \right)^q = \beta_q = \sum_{n \geq 1} \frac{\varphi(n) d_q(n^2)}{2n^3},$$

$$\lim_{x \rightarrow \infty} \frac{1}{x/2} \sum_{0 > d \geq -x} \left(\frac{h(d)\pi}{|d|^{1/2}} \right)^q = \beta_q,$$

where φ is Euler's totient function and $d_q(n)$ is the number of ways one can write n as a product of q positive integers. For all these results, error term estimates can be given (see [2], [6], [10], [12], [13]).

It seems that for number fields of higher degree, no analogous results are known. The Brauer-Siegel Theorem (see, e.g., [8], Chapter XVI) gives a rough idea of the size of the class number times the regulator: Let k range over a sequence of number fields which are galois over $\mathbb{Q}$ such that $n/\log d \rightarrow 0$, where $n := [k : \mathbb{Q}]$ is the degree and $d = d_{k/\mathbb{Q}}$ is the

MANFRED PETER

absolute discriminant of k . Let h_k be the class number of k and R_k its regulator. Then

$$\frac{\log(h_k R_k)}{\log d^{1/2}} \rightarrow 1.$$

When looking for more precise information on the value distribution of

$$\frac{h_k R_k}{d^{1/2}},$$

we run into the problem of how to effectively parametrize number fields. This problem is avoided in the present paper by choosing a special class of number fields: Let l be a fixed rational prime and

$$S_l := \{m \in \mathbb{N} \setminus \{1\} \mid m \text{ is } l\text{-power-free}\}.$$

For $m \in S_l$, define the pure number field $k_m := \mathbb{Q}(\sqrt[l]{m})$ where the radical is chosen in $\mathbb{R}^+$. Let $r(m) := \text{res}_{s=1} \zeta_{k_m}(s)$ where ζ_{k_m} is the Dedekind zeta function of k_m . Then

$$r(m) = \frac{h_{k_m} R_{k_m}}{d_{k_m}^{1/2}} c(l), \quad c(l) = \begin{cases} 2, & l = 2, \\ (2\pi)^{(l-1)/2}, & l \geq 3, \end{cases}$$

and $d_{k_m} \asymp K(m)^{l-1}$, where $K(m)$ is the squarefree kernel of m . For $m \in \mathbb{N} \setminus S_l$, define $r(m) := 0$.

Theorem. *There is a distribution function $F \in C^\infty(\mathbb{R})$ such that for all $z \in \mathbb{R}$,*

$$\lim_{x \rightarrow \infty} \frac{\#\{m \in S_l \mid m \leq x, r(m) \leq e^z\}}{\#\{m \in S_l \mid m \leq x\}} = F(z).$$

Furthermore,

$$\lim_{x \rightarrow \infty} \frac{1}{\#\{m \in S_l \mid m \leq x\}} \sum_{m \in S_l: m \leq x} r(m)^q = \int_{\mathbb{R}^+} z^q dF(\log z)$$

for all $q \in \mathbb{N}$. The characteristic function $\Psi(t)$ of F is an Euler product whose factors depend on $t \in \mathbb{R}$.

In order to give an idea of the proof let us first review the method for the well-known case $l = 2$. For $m > 1$ squarefree, Dirichlet's class number formula gives

$$\zeta_{\mathbb{Q}(\sqrt{m})}(s) = \zeta(s) L(s, \chi_d),$$

where

$$d = \begin{cases} m, & m \equiv 1 \pmod{4}, \\ 4m, & m \equiv 2, 3 \pmod{4}, \end{cases}$$

is the discriminant of $\mathbb{Q}(\sqrt{m})$ and χ_d is the Jacobi character for the modulus $|d|$. Therefore

$$r(m) = L(1, \chi_d) = \sum_{n \geq 1} \frac{\chi_d(n)}{n} = \prod_p \left(1 - \frac{\chi_d(p)}{p^s}\right)^{-1} \Big|_{s=1}.$$

The idea of proof is as follows: For $q \geq 1$, the function r is approximated in the q -th mean by functions R_P , $P \in \mathbb{N}$, such that

$$\|r - R_P\|_q \rightarrow 0 \text{ as } P \rightarrow \infty. \quad (1)$$

Here

$$\|f\|_q := \left(\limsup_{x \rightarrow \infty} \frac{1}{x} \sum_{m \leq x} |f(m)|^q \right)^{1/q} \in [0, \infty]$$

for $f : \mathbb{N} \rightarrow \mathbb{C}$. The functions R_P are partial products of the Euler product above, i.e.

$$R_P(m) := \prod_{p \leq P} \left(1 - \frac{\chi_d(p)}{p} \right)^{-1}.$$

They are periodic in m since for $p > 2$, we have

$$\chi_d(p) = \left(\frac{d}{p} \right) = \begin{cases} 1, & x^2 \equiv d \pmod{p} \text{ solvable, } p \nmid d, \\ -1, & x^2 \equiv d \pmod{p} \text{ unsolvable,} \\ 0 & p \mid d. \end{cases}$$

Since periodic functions have limit distributions a standard procedure shows the same for r . In fact the procedure in this last step is somewhat different since we also want to show the smoothness of F .

The approximation (1) could be done with character sum estimates. More suitable for generalizations is the following method which uses contour integration and zero density estimates. Let $\mathcal{K}$ be the rectangle with vertices $2 + iT$, $\gamma + iT$, $\gamma - iT$ and $2 - iT$, and $N, T \geq 1$ and $1/2 < \gamma < 1$ free parameters. The Residue Theorem gives

$$\frac{1}{2\pi i} \int_{\mathcal{K}} L(s, \chi_d) \Gamma(s-1) N^{s-1} ds = L(1, \chi_d).$$

Since the Γ -function decays exponentially in vertical strips of finite width the limit $T \rightarrow \infty$ together with Mellin's inversion formula gives

$$L(1, \chi_d) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} - \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} = \sum_{n \geq 1} \frac{\chi_d(n)}{n} e^{-n/N} - I(m, N).$$

If we assume the Generalized Lindelöf Hypothesis

$$L(s, \chi_d) \ll_{\epsilon} (d(1 + |\Im s|))^{\epsilon} \quad (2)$$

for $\gamma \leq \Re s \leq 1$ and $m > 1$ squarefree, we easily get the estimate

$$I(m, N) \ll_{\epsilon} d^{\epsilon} N^{\gamma-1}. \quad (3)$$

Here it is important that the exponent of d can be made arbitrarily small and the exponent of N is negative.

Without any assumption this procedure can be imitated as follows: If $L(s, \chi_d)$ has no zeros in the rectangle

$$\{s \in \mathbb{C} \mid \Re s \geq \gamma - \epsilon, |\Im s| \leq (\log x)^2\}, \quad (4)$$

then the usual combination of the Borel-Carathéodory Theorem and Hadamard's Three Circles Theorem gives (2) for $\gamma \leq \Re s \leq 2$ and $|\Im s| \leq (\log x)^2/2$. Using the exponential decay of the Γ -function on $|\Im s| \geq (\log x)^2/2$ we again get (3). If there is a zero of $L(s, \chi_d)$ in the rectangle (4) all we can say is that

$$I(m, N) \ll d^{\epsilon} + N^{\epsilon}.$$

MANFRED PETER

Now zero density estimates can be used to show that the second case does not happen too often, i.e.

$$\#\{1 < m \leq x \mid m \text{ squarefree, } L(s, \chi_d) \text{ has a zero in the rectangle (4)}\} \ll_\epsilon x^{1-c(\gamma)+\epsilon}$$

with some constant $c(\gamma) > 0$.

In the q -th mean we have the approximation

$$\sum_{n \geq 1} \frac{\chi_d(n)}{n} e^{-n/N} \approx \sum_{n \leq N} \frac{\chi_d(n)}{n}.$$

Choosing N as a small power of x proves the statement (1).

In the general case $l \geq 2$ we have, for $\Re s > 1$,

$$\begin{aligned} \zeta_{k_m}(s) &= \prod_p \prod_{\mathfrak{p} \mid p} \left(1 - \frac{1}{p^{f(\mathfrak{p}/p)s}}\right)^{-1} \\ &= \prod_p \prod_{\mathfrak{p} \mid p: f(\mathfrak{p}/p) \geq 2} \left(1 - \frac{1}{p^{f(\mathfrak{p}/p)s}}\right)^{-1} \prod_p \left(1 - \frac{1}{p^s}\right)^{-\chi(m,p)} \zeta(s), \end{aligned}$$

where $f(\mathfrak{p}/p) := [\mathcal{O}_{k_m}/\mathfrak{p} : \mathbb{Z}/p\mathbb{Z}]$ is the residue class degree of $\mathfrak{p}$ and

$$\chi(m, p) := \#\{\mathfrak{p} \mid p \mid f(\mathfrak{p}/p) = 1\} - 1.$$

Thus

$$r(m) = \prod_p \prod_{\mathfrak{p} \mid p: f(\mathfrak{p}/p) \geq 2} \left(1 - \frac{1}{p^{f(\mathfrak{p}/p)}}\right)^{-1} \left(1 - \frac{1}{l}\right)^{-\chi(m,l)} \prod_{p \neq l} \left(1 - \frac{1}{p^s}\right)^{-\chi(m,p)} \Big|_{s=1}.$$

In order to get the almost periodicity of the partial products of this Euler product we exploit the relation between the splitting of rational primes p in k_m and the splitting of $X^l - m$ in $\mathbb{F}_p[X]$ and $\mathbb{Q}_p^{\text{unram}}[X]$. Here $\mathbb{Q}_p^{\text{unram}}$ is the maximal unramified extension of $\mathbb{Q}_p$. The following lemmas give the necessary information.

Lemma. For $p \neq l$, we have

$$\chi(m, p) = \#\{x \bmod p \mid x^l \equiv m \bmod p\} - 1.$$

In particular, the function $\chi(\cdot, p)$ is p -periodic and

$$\sum_{m \bmod p} \chi(m, p) = 0,$$

which serves as a substitute for the orthogonality relation for characters.

Lemma. Let p be a prime, $m \in S_l$ and $b \in \mathbb{N}_0$ such that $p^b \parallel m$. Then the factor

$$\prod_{\mathfrak{p} \mid p: f(\mathfrak{p}/p) \geq 2} \left(1 - \frac{1}{p^{f(\mathfrak{p}/p)}}\right)^{-1}$$

is constant on the residue class $m \bmod p^{b(l-1)+l \operatorname{ord}_p l+1}$.

Both lemmas are used to show the almost periodicity of R_P in the general case. In order to prove the approximation (1) we use the following zero density estimate of Kawada [7].

CLASS NUMBERS OF PURE NUMBER FIELDS

Theorem. *For sufficiently small $\eta > 0$, we have*

$$\sum_{m \in S_1: x < m \leq 2x} N(m; 1 - \eta, T) \ll (xT)^{1-\eta}, \quad x \geq T \geq 1,$$

where $N(\dots)$ is the number of zeros of $\zeta_{k_m}(s)\zeta(s)^{-1}$ in the rectangle $[1 - \eta, 1] \times [-T, T]$.

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