

Quasiconformal extension of univalent functions and Becker's theorem

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Abstract

This is a research for a subclass of univalent holomorphic functions on the unit disc normalized by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, which can be extended to k -quasiconformal mappings on the disc $\{z \mid |z| < R\}$ where $R > 1$. Such a subclass is denoted by $S(k, R)$. In this note, the class $S(k, R)$ is introduced through the observation of Becker's theorem which ensures a k -quasiconformal extendibility of univalent holomorphic functions on the disc to the Riemann sphere with Löwner chains.

1 Motivation

Let $D = \{z \mid |z| < 1\}$ and

$S = \{f \mid f \text{ is holomorphic and univalent on } D, f(0) = f'(0) - 1 = 0\},$

$S(k) = \{f \mid f \in S, f \text{ can be extended to a } k\text{-quasiconformal mapping on } \widehat{\mathbb{C}}\},$

$S_0(k) = \{f \mid f \in S(k), \text{ the extended mappings fix } \infty\},$

respectively, where $k \in [0, 1)$. The class $S(k)$ has been studied by numerous authors in connection with the theory of Teichmüller spaces. In those investigations, an interesting method for quasiconformal extension of univalent functions was obtained by Becker ([1], see also [5]) which relies on the *Löwner chains* described by the *Löwner equation*

$$\frac{\partial f(z, t)}{\partial t} = zp(z, t) \frac{\partial f(z, t)}{\partial z} \quad (1)$$

for $z \in D$ and $t \in [0, \infty)$. This equation determines an expanding flow. Here, the function $f(z, t) = e^t z + \sum_{n=2}^{\infty} a_n(t) z^n$ is holomorphic in $|z| < 1$ for each $t \in [0, \infty)$, absolutely continuous in $t \in [0, \infty)$ for each $|z| < r_0$ and satisfies the inequality $|f(z, t)| \leq K_0 e^t$ ($|z| < r_0, t \leq 0$) for some positive constants K_0 and r_0 . Also a function $p(z, t)$ is measurable on $D \times [0, \infty)$, holomorphic in $|z| < 1$, and satisfies $\operatorname{Re} p(z, t) > 0$ and the partial differential equation (1) for

a.e. t .

Theorem 1 ([1]). *If $f(z, t)$ is a univalent solution to (1) with $p(z, t)$ satisfying the condition*

$$\left| \frac{p(z, t) - 1}{p(z, t) + 1} \right| \leq k < 1 \quad (2)$$

then, for each $t \geq 0$, the function $f_t(z) = f(z, t)$ maps $\mathbb{D}$ onto a Jordan domain bounded by a k -quasiconformal image of $\partial\mathbb{D}$, and the map $\hat{f}(z)$ defined by

$$\hat{f}(re^{i\theta}) = \begin{cases} f(re^{i\theta}, 0) & r \leq 1 \\ f(e^{i\theta}, \log r) & r > 1 \end{cases}$$

is a k -quasiconformal extension of $f(z, 0)$ onto $\widehat{\mathbb{C}}$ with $\hat{f}(\infty) = \infty$ (thus $\hat{f}(z) \in S_0(k)$).

Observe that $p(\mathbb{D}, t)$ must be contained in the disc $|z - (1 + k^2)/(1 - k^2)| \leq 2k/(1 - k^2)$ for all $t \in [0, \infty)$ so that we can apply Theorem 1 to the Löwner chains (Fig.1). This strong assumption can be weakened by restricting the range of the parameter t . In fact, the following is true;

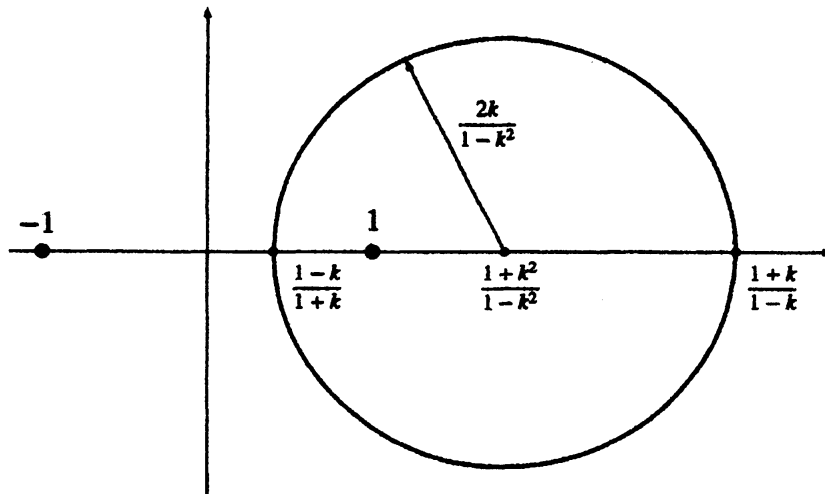


Figure 1 : $p(z, t)$ must be in this circle for all $z \in \mathbb{D}$ and $t \in [0, \infty)$.

Corollary 2. *If $f(z, t)$ is a univalent solution to (1) and there exists $t_0 > 0$ such that for all $t \in [0, t_0]$ $p(z, t)$ satisfies the condition*

$$\left| \frac{p(z, t) - 1}{p(z, t) + 1} \right| \leq k < 1, \quad (3)$$

then the map $\hat{f}(z)$ is a k -quasiconformal extension of $f(z, 0)$ defined on $\{z \mid |z| < e^{t_0}\}$ with $\hat{f} \neq \infty$.

Now we shall introduce the classes $S(k, R)$ and $S_0(k, R)$; namely

$$S(k, R) = \{f \mid f \in S, f \text{ can be extended to a } k\text{-quasiconformal mapping } \hat{f} \text{ on } \{|z| < R\}\}$$

and

$$S_0(k, R) = \{f \mid f \in S(k, R), \text{ the extended mapping } \hat{f} \text{ doesn't take } \infty \text{ on } \{|z| < R\}\}$$

respectively, where $R > 1$.

2 Properties of the class $S(k, R)$

The class $S(k, R)$ was studied by some authors in another context. We shall give some known results for the classes $S(k, R)$ and $\Sigma(k, r)$, where $\Sigma(k, r)$ is a family of univalent holomorphic functions on $\{z \in \widehat{\mathbb{C}} - \overline{\mathbb{D}}\}$ which can be extended to a k -quasiconformal mapping on $\{|z| > r\}$, $r < 1$.

McLeavey [8] (see also [9]) first considered the subclass of Σ with $K(|z|)$ -quasiconformal extensions into the interior of $\mathbb{D}$ where $K(|z|)$ is a piecewise continuous function of bounded variation on $[r, 1]$, $0 \leq r < 1$. She obtained for this class the analogs of the classical Grunsky and Goluzin inequalities and sharp estimates for the coefficients b_0 and b_1 of $\Sigma(k, r)$ and a_2 of $S_0(k, R)$ with extremal function as follow;

Theorem 3 ([8]). *If $g \in \Sigma(k, r)$ and $g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}$ for $|z| > 1$, then*

$$|b_1| \leq \frac{k + r^2}{1 + kr^2}.$$

Equality occurs if and only if

$$g(z) = \begin{cases} z + b_0 + \left(\frac{k + r^2}{1 + kr^2} \right) \frac{e^{i\alpha}}{z} & |z| \geq 1 \\ \left(\frac{1}{1 + kr^2} \right) \left(z + \frac{r^2 e^{i\alpha}}{z} + k e^{i\alpha} \bar{z} + \frac{kr^2}{\bar{z}} \right) & r < |z| \leq 1. \end{cases}$$

Corollary 4 ([8]). Suppose $f \in S(k, R)$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ for $z \in D$. Then

$$|a_3 - a_2^2| \leq \frac{1 + kR^2}{k + R^2}.$$

If, in addition, extend mappings do not take ∞ on $\{|z| < R\}$, then

$$|a_2| \leq 2 \frac{1 + kR}{k + R}. \quad (4)$$

Kühnau [6] also proved similar results of those through introducing the class $\Sigma(Q_1, \dots, Q_n)$ of $K(|z|)$ -quasiconformal mapping of the plane which are conformal on $\{z; |z| > 1\}$ with a development $f(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}$ and which have piecewise bounded dilatation in $D : K(|z|) \leq Q_i$ ($Q_i \geq 1$) in $R_i < |z| < R_{i-1}$ ($i = 1, \dots, n$), with $R_0 = 1$, $R_n = 0$. Schober [9] mentioned above results in his book, Chap.14. He also gave some more results, for instance, generalized Gronwall's area theorem for $\Sigma(k, r)$;

Theorem 5 ([9]). If $g \in \Sigma(k, r)$ and $g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}$ for $|z| > 1$, then

$$\sum_{m=1}^{\infty} m |b_m|^2 \leq \left(\frac{k + r^2}{1 + kr^2} \right)^2. \quad (5)$$

Under the more general case, Lehto [7] showed a majorant principle for a holomorphic functional as follow;

Let A be a domain in $\widehat{\mathbb{C}}$ which is bounded by a quasicircle, B be a domain whose closure $\overline{B} \subset A$, and $\mathcal{F}_k$, $0 \leq k < 1$, be a family of functions which are k -quasiconformal on A and conformal on $\overline{B}$. Denote by $\mathcal{F}_1$ the family of all conformal mappings on B .

We introduce four different normalizations to cover a large number of cases appearing in applications. Let z_1, z_2, z_3 be distinct points of B and $\alpha_1, \alpha_2, \alpha_3, \beta$ are complex numbers, the

α 's are different from each other and $\beta \neq 0$. The families $\mathcal{F}_k$ and $\mathcal{F}_1$ are called normalized if all the functions f of A contained in $\mathcal{F}_k$ or $\mathcal{F}_1$ have one of the following conditions;

1. $f(z_i) = \alpha_i$, $i = 1, 2, 3$,
2. $f(z_i) = \alpha_i$, $i = 1, 2$, and $f(z) \neq \infty$ in A ,
3. $f(z_1) = \alpha_1$, $f'(z_1) = \beta$ and $f(z) \neq \infty$ in A ,
4. If $\infty \in B$, then $f(z) - z \rightarrow 0$ as $z \rightarrow \infty$.

We shall suppose here $\mathcal{F}_k$ and $\mathcal{F}_1$ are normalized. Remark that normalized $\mathcal{F}_k$ and $\mathcal{F}_1$ are closed normal families.

Let Ψ be a holomorphic functional defined on the family $\mathcal{F}_k$ or $\mathcal{F}_1$, i.e. $\Psi(f) = \omega(f(z_0), f'(z_1), \dots, f^{(n)}(z_n))$, where ω is a complex-valued holomorphic function of the variables $f^{(i)}(z_i)$, $i = 1, 2, \dots$, each $f^{(i)}(z_i)$ being the value at fixed point $z_i \in B$.

Set

$$M(k) = \sup_{f \in \mathcal{F}_k} |\Psi(f)|, \quad 0 \leq k \leq 1.$$

Since $\mathcal{F}_k$ is a closed normal family, there exists an extremal function maximizing $|\Psi(f)|$ in $\mathcal{F}_k$.

Theorem 6 ([7]). *For a holomorphic functional in $\mathcal{F}_k$,*

$$M(k) \leq M(1) \frac{k + \frac{M(1)}{M(0)}}{1 + k \frac{M(1)}{M(0)}}. \quad (6)$$

This result contains some coefficient estimates as corollaries; for the class $\Sigma(k, r)$

$$\max_{\Sigma(k, r)} |b_n| \leq \frac{k + r^{n+1}}{1 + kr^{n+1}} \max_{\Sigma} |b_n|, \quad n = 1, 2, \dots,$$

which imply

$$|b_1| \leq \frac{k + r^2}{1 + kr^2} \quad \text{and} \quad |b_2| \leq \frac{2}{3} \frac{k + r^3}{1 + kr^3}.$$

The Grunsky type inequalities for $\Sigma(k, r)$ also follow easily from the general inequality (6).

For $f \in \Sigma$, let A_{mn} , $m, n = 1, 2, \dots$, be the numbers determined by

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m, n=1}^{\infty} A_{mn} z^{-m} \zeta^{-n}.$$

Then for any complex numbers $x_1, x_2, \dots, x_N$,

$$\left| \sum_{m,n}^N A_{mn} x_m x_n \right| \leq \frac{k+r^2}{1+kr^2} \sum_n^N \frac{|x_n|^2}{n} \quad (7)$$

and

$$\sum_n^N n \left| \sum_m^N A_{mn} x_m \right|^2 \leq \left(\frac{k+r^2}{1+kr^2} \right)^2 \sum_{n=1}^N \frac{|x_n|^2}{n}. \quad (8)$$

Remark that (7) and (8) is not sharp (see [8]).

Deiermann treats several similar problems of those in [2] and [3] with the method of extremal length. Recently, Krushkal gives a short mention for those reserches in his survey [4], Chap.6.3.

3 Main Results

Now the more applications of Theorem 6 are given to $S_0(k, R)$ and $\Sigma(k, r)$ (again remark that these results are not sharp because (7) and (8) is not sharp) ;

Theorem 7.

$$\sup_{S_0(k,R)} |a_n| \leq n \frac{1 + kR^{n-1}}{k + R^{n-1}}.$$

Proof. Let us take $\mathcal{F}_k = S_0(k, R)$, then $\mathcal{F}_1$ is the well-known class $\mathcal{S}$. Choose $\Psi(f) = a_n$. Then $M(1) = n$, and $M(0) = n/R^{n-1}$ because $Rf(z/R) \in \mathcal{S}$ for arbitrary $f \in \mathcal{F}_0$. Hence the inequality (6) follow the theorem. $\square$

Theorem 8 (Generalized Goluzin inequality). If $g \in \Sigma(k, r)$ and $z_v \in \widehat{\mathbb{C}} - \mathbb{D}$, $\gamma_v \in \mathbb{C}$ ($v = 1, 2, \dots, n$), $n = 1, 2, \dots$, then

$$\left| \sum_{\mu} \sum_{\nu} \gamma_{\mu} \gamma_{\nu} \log \frac{g(z_{\mu}) - g(z_{\nu})}{z_{\mu} - z_{\nu}} \right| \leq \frac{k+r^2}{1+kr^2} \sum_{\mu} \sum_{\nu} \gamma_{\mu} \overline{\gamma_{\nu}} \log \frac{1}{1 - (z_{\mu} \overline{z_{\nu}})^{-1}}. \quad (9)$$

Proof. We shall apply the inequality (7) with $x_m = \sum_{v=1}^N \gamma_v z_v^{-m}$, $m = 1, 2, \dots$. In fact, we

have

$$\begin{aligned} \sum_{\mu} \sum_{\nu} \gamma_{\mu} \gamma_{\nu} \log \frac{g(z_{\mu}) - g(z_{\nu})}{z_{\mu} - z_{\nu}} &= - \sum_{m,n} \sum_{\mu,\nu} A_{mn} \gamma_{\mu} \gamma_{\nu} z_{\mu}^{-m} z_{\nu}^{-n} \\ &= - \sum_{m,n} A_{mn} x_m x_n. \end{aligned}$$

Hence (7) shows that the left-hand side of (9) is

$$\begin{aligned} &\leq \frac{1 + kr^2}{k + r^2} \sum_n \frac{1}{n} |x_n|^2 = \frac{1 + kr^2}{k + r^2} \sum_n \frac{1}{n} \sum_{\mu,\nu} \gamma_{\mu} \bar{\gamma}_{\nu} z_{\mu}^{-k} z_{\nu}^{-k} \\ &= \frac{1 + kr^2}{k + r^2} \sum_{\mu,\nu} \gamma_{\mu} \bar{\gamma}_{\nu} \log \frac{1}{1 - (z_{\mu} \bar{z}_{\nu})^{-1}}. \end{aligned}$$

This completes the proof of the theorem. $\square$

Theorem 9. For $f \in S_0(k, R)$ and $z \in \mathbb{D}$,

$$\left| \log \frac{zf'(z)}{f(z)} \right| \leq \frac{1 + kR}{k + R} \log \frac{1 + |z|}{1 - |z|}.$$

Proof. In (9) let $n = 2$, $\gamma_1 = 1$, $\gamma_2 = -1$, then

$$\left| \log \frac{g'(z)g'(\zeta)(z - \zeta)^2}{(g(z) - g(\zeta))^2} \right| \leq \frac{k + r^2}{1 + kr^2} \log \frac{|z\bar{\zeta} - 1|^2}{(|z|^2 - 1)(|\zeta|^2 - 1)} \quad (z, \zeta \in \widehat{\mathbb{C}} - \mathbb{D}). \quad (10)$$

We want to apply (9) to the function $f \in S_0(k, R)$. If we put

$$g(\zeta) = 1/\sqrt{f(\zeta^{-2})}, \quad (11)$$

then $g \in \Sigma(k, 1/\sqrt{R})$. Since g is odd function, it follows from (10) with $z = -\zeta$ that

$$\left| 2 \log \frac{\zeta g'(\zeta)}{g(\zeta)} \right| \leq \frac{k + (1/R)}{1 + k(1/R)} 2 \log \frac{|\zeta|^2 + 1}{|\zeta|^2 - 1} \quad (|\zeta| > 1).$$

If we choose $z = \zeta^{-2}$ and use (11) we obtain the desire inequality. $\square$

Corollary 10. For $f \in S_0(k, R)$ and $z \in \mathbb{D}$,

$$\left(\frac{1 - |z|}{1 + |z|} \right)^{(1+kR)/(k+R)} \leq \left| \frac{zf'(z)}{f(z)} \right| \leq \left(\frac{1 + |z|}{1 - |z|} \right)^{(1+kR)/(k+R)}.$$

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