

On Quasi-Hadamard Product of Certain p -Valent Functions
with Negative Coefficients

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1. INTRODUCTION

In [1], Kumar showed some results for the quasi-Hadamard product of certain univalent functions with negative coefficients.

In the present note, we show that Kumar's results [1] are generalized to the case of certain p -valent functions with negative coefficients.

Let $A(p)$ be the class of analytic and p -valent function $f(z)$ of the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (a_{p+n} \geq 0, p \in \mathbb{N})$$

in the unit disk $U = \{z: |z| < 1\}$.

Let $T^*(p, \alpha)$ and $C(p, \alpha)$ denote the subclasses of $A(p)$ which satisfy $\operatorname{Re} \left[\frac{zf'(z)}{f(z)} \right] > \alpha$ and $\operatorname{Re} \left[1 + \frac{zf''(z)}{f'(z)} \right] > \alpha$, for $0 \leq \alpha < p$, respectively.

Clearly the function in $T^*(p, \alpha)$ and $C(p, \alpha)$ are p -valent starlike function and p -valent convex function of order α , respectively.

For these classes, Owa has obtained the following results in [2].

LEMMA 1. A function $f(z)$ is in the class $T^*(p, \alpha)$ if and only if

$$\sum_{n=1}^{\infty} (p + n - \alpha) \leq p - \alpha.$$

The result is sharp.

LEMMA 2. A function $f(z)$ is in the class $C(p, \alpha)$ if and only if

$$\sum_{n=1}^{\infty} (p + n)(p + n - \alpha) a_{p+n} \leq p(p - \alpha).$$

The result is sharp.

Let $A_0(p)$ denote the class of analytic and p -valent function $f(z)$ of the form

$$f(z) = a_p z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (a_p > 0, a_{p+n} \geq 0, p \in \mathbb{N})$$

in the unit disk U .

Furthermore, let $T_0^*(p, \alpha)$ and $C_0(p, \alpha)$ be the subclasses of $A_0(p)$ as follows:

$$T_0^*(p, \alpha) = \left\{ f(z) \in A_0(p) : \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (0 \leq \alpha < p) \right\}$$

and

$$C_0(p, \alpha) = \left\{ f(z) \in A_0(p) : \operatorname{Re} \left[1 + \frac{zf''(z)}{f'(z)} \right] > \alpha \quad (0 \leq \alpha < p) \right\}.$$

For these classes, by Lemma 1 and Lemma 2, we easily obtain the following theorems, respectively.

THEOREM 1. A function $f(z)$ in the class $T_0^*(p, \alpha)$ if and only if

$$\sum_{n=1}^{\infty} (p + n - \alpha) a_{p+n} \leq (p - \alpha) a_p.$$

THEOREM 2. A function $f(z)$ is in the class $C_0(p, \alpha)$ if and only if

$$\sum_{n=1}^{\infty} (p + n)(p + n - \alpha) a_{p+n} \leq p(p - \alpha) a_p.$$

We now introduce the subclass $S_0(k, p, \alpha)$ of the class $A_0(p)$ as follows.

A function $f(z)$ belongs to the class $S_0(k, p, \alpha)$ if and only if

$$\sum_{n=1}^{\infty} \left(\frac{p+n}{p} \right)^k (p + n - \alpha) a_{p+n} \leq (p - \alpha) a_p,$$

where k is any real number.

Evidently, $S_0(0, p, \alpha) \equiv T_0^*(p, \alpha)$ and $S_0(1, p, \alpha) \equiv C_0(p, \alpha)$.

From now on, let the functions of the class $A_0(p)$ be the following forms:

$$f_i(z) = a_{p,i} z^p - \sum_{n=1}^{\infty} a_{p+n,i} z^{p+n} \quad (a_{p,i} > 0, a_{p+n,i} \geq 0)$$

and

$$g_j(z) = b_{p,j} z^p - \sum_{n=1}^{\infty} b_{p+n,j} z^{p+n} \quad (b_{p,j} > 0, b_{p+n,j} \geq 0),$$

respectively.

Let us define the quasi-Hadamard product $f_i * g_j(z)$ of the functions $f_i(z)$ and $g_j(z)$ by

$$f_i(z) * g_j(z) = a_{p,i} b_{p,j} z^p - \sum_{n=1}^{\infty} a_{p+n,i} b_{p+n,j} z^{p+n}.$$

2. RESULTS

Consequently, we have the following theorems. we can prove these theorems by using the same way as Kumar [1].

THEOREM 3. Let the functions $f_i(z)$ belong to the classes $T_0^*(p, \alpha_i)$ for each $i = 1, 2, 3, \dots, m$, respectively. Then the quasi-Hadamard product $f_1 * f_2 * f_3 * \dots * f_m(z)$ belongs to the class $S_0(m-1, p, \beta)$, where $\beta = \max\{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_m\}$.

THEOREM 4. Let the functions $f_i(z)$ belong to the classes $C_0(p, \alpha_i)$ for each $i = 1, 2, 3, \dots, m$, respectively. Then the quasi-Hadamard product $f_1 * f_2 * f_3 * \dots * f_m(z)$ belongs to the class $S_0(2m-1, p, \beta)$, where $\beta = \max\{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_m\}$.

THEOREM 5. Let the functions $f_i(z)$ belong to the classes $T_0^*(p, \alpha_i)$ for each $i = 1, 2, 3, \dots, m$ and for each $j = 1, 2, \dots, q$, let the functions $g_j(z)$ belong to the classes $C_0(p, \beta_j)$, respectively. Then the quasi-Hadamard product $f_1 * f_2 * f_3 * \dots * f_m * g_1 * g_2 * g_3 * \dots * g_q(z)$ belongs to the class $S_0(m+2q-1, p, \gamma)$, where $\gamma = \max\{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_m, \beta_1, \beta_2, \beta_3, \dots, \beta_q\}$.

THEOREM 6. Let the functions $f_i(z)$ belong to the class $C_0(p, \alpha)$ for each $i = 1, 2, 3, \dots, m$ and let $0 \leq \alpha \leq r_0$, where r_0 is a root of the equation $(p+1)^m(p-mr) - p(p-r)^m = 0$ in the interval $(0, \frac{p}{m})$. Then the quasi-Hadamard product $f_1 * f_2 * f_3 * \dots * f_m(z)$ belongs to the class $S_0(m-1, p, m\alpha)$.

REMARK. If we put $p = 1$ in these theorems, we have the Kumar's results [1].

REFERENCES

- [1] V. Kumar, Quasi-Hadamard product of certain univalent functions, J. Math. Anal. Appl. 126, 70-77(1987).
- [2] S. Owa, On certain classes of p-valent functions with negative coefficients, SIMON STEVIN, A Quart. J. Pure Appl. Math. 59, No4 385-402(1985).