

MODULI OF PARABOLIC CONNECTIONS ON CURVES AND THE RIEMANN-HILBERT CORRESPONDENCE

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Abstract

Let $(C, \mathbf{t})$ ($\mathbf{t} = (t_1, \dots, t_n)$) be an n -pointed smooth projective curve of genus g and take an element $\boldsymbol{\lambda} = (\lambda_j^{(i)}) \in \mathbf{C}^{nr}$ such that $-\sum_{i,j} \lambda_j^{(i)} = d \in \mathbf{Z}$. For a weight $\boldsymbol{\alpha}$, let $M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda})$ be the moduli space of $\boldsymbol{\alpha}$ -stable $(\mathbf{t}, \boldsymbol{\lambda})$ -parabolic connections on C and let $RP_r(C, \mathbf{t})_{\mathbf{a}}$ be the moduli space of representations of the fundamental group $\pi_1(C \setminus \{t_1, \dots, t_n\}, *)$ with the local monodromy data $\mathbf{a}$ for a certain $\mathbf{a} \in \mathbf{C}^{nr}$. Then we prove that the morphism $\mathbf{RH} : M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}) \rightarrow RP_r(C, \mathbf{t})_{\mathbf{a}}$ determined by the Riemann-Hilbert correspondence is a proper surjective bimeromorphic morphism. As a corollary, we prove the geometric Painlevé property of the isomonodromic deformation defined on the moduli space of parabolic connections.

1. Introduction

Let C be a smooth projective curve over $\mathbf{C}$ and $t_1, \dots, t_n$ be distinct points of C . For an algebraic vector bundle E on C and a logarithmic connection $\nabla : E \rightarrow E \otimes \Omega_C^1(t_1 + \dots + t_n)$, $\ker \nabla^{an}|_{C \setminus \{t_1, \dots, t_n\}}$ becomes a local system on $C \setminus \{t_1, \dots, t_n\}$ and corresponds to a representation of the fundamental group $\pi_1(C \setminus \{t_1, \dots, t_n\}, *)$, where ∇^{an} is the analytic connection corresponding to ∇ . The correspondence $(E, \nabla) \mapsto \ker \nabla^{an}|_{C \setminus \{t_1, \dots, t_n\}}$ is said to be the Riemann-Hilbert correspondence. If $l \subset E|_{t_i}$ is a subspace satisfying $\text{res}_{t_i}(\nabla)(l) \subset l$, then ∇ induces a connection $\nabla' : E' \rightarrow E' \otimes \Omega_C^1(t_1 + \dots + t_n)$, where $E' := \ker(E \rightarrow (E|_{t_i}/l))$ and $\text{res}_{t_i}(\nabla)$ means the residue of ∇ at t_i . We say (E', ∇') is the elementary transform of (E, ∇) along t_i by l . Note that $\ker \nabla^{an}|_{C \setminus \{t_1, \dots, t_n\}} \cong \ker(\nabla')^{an}|_{C \setminus \{t_1, \dots, t_n\}}$.

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We put

$$\Lambda_r^{(n)}(d) := \left\{ (\lambda_j^{(i)})_{0 \leq j \leq r-1}^{1 \leq i \leq n} \in \mathbf{C}^{nr} \left| d + \sum_{i,j} \lambda_j^{(i)} = 0 \right. \right\}$$

for integers d, r, n with $r > 0, n > 0$. We write $\mathbf{t} = (t_1, \dots, t_n)$ and take an element $\boldsymbol{\lambda} \in \Lambda_r^{(n)}(d)$.

Definition 1.1. We say $(E, \nabla, \{l_*^{(i)}\}_{1 \leq i \leq n})$ is a $(\mathbf{t}, \boldsymbol{\lambda})$ -parabolic connection of rank r if

- (1) E is a rank r algebraic vector bundle on C ,
- (2) $\nabla : E \rightarrow E \otimes \Omega_C^1(t_1 + \dots + t_n)$ is a connection, and
- (3) for each t_i , $l_*^{(i)}$ is a filtration $E|_{t_i} = l_0^{(i)} \supset l_1^{(i)} \supset \dots \supset l_{r-1}^{(i)} \supset l_r^{(i)} = 0$ such that $\dim(l_j^{(i)}/l_{j+1}^{(i)}) = 1$ and $(\text{res}_{t_i}(\nabla) - \lambda_j^{(i)} \text{id}_{E|_{t_i}})(l_j^{(i)}) \subset l_{j+1}^{(i)}$ for $j = 0, \dots, r-1$.

The filtration $l_*^{(i)}$ ($1 \leq i \leq n$) is said to be a parabolic structure on the vector bundle E . For a parabolic connection $(E, \nabla, \{l_j^{(i)}\})$, the elementary transform of (E, ∇) along t_i by $l_j^{(i)}$ determines another parabolic connection. Then an elementary transform gives a transformation $\text{Elm}_j^{(i)}$ on the set of isomorphism classes of parabolic connections; see section 3, (3) for the precise definition of $\text{Elm}_j^{(i)}$. Then the Riemann-Hilbert correspondence gives a bijection RH between the set

$$(\dagger) \quad \left\{ (E, \nabla, \{l_j^{(i)}\}) : \text{parabolic connection of rank } r \right\} / \sim$$

and the set

$$(\dagger\dagger) \quad \{ \rho : \pi_1(C \setminus \{t_1, \dots, t_n\}, *) \rightarrow GL_r(\mathbf{C}) : \text{representation} \} / \cong,$$

where $\sim$ is the equivalence relation generated by $x \sim \text{Elm}_{l_j^{(i)}}(x)$, $(E, \nabla, \{l_j^{(i)}\}) \sim (E, \nabla, \{l_j^{(i)}\}) \otimes \mathcal{O}_C(t_k)$ and $(E, \nabla, \{l_j^{(i)}\}) \sim (E, \nabla, \{l_j'^{(i)}\})$ and $/ \cong$ means the isomorphism classes. The bijectivity of RH immediately follows from the theory of Deligne [4]. Indeed take any representation $\rho : \pi_1(C \setminus \{t_1, \dots, t_n\}, *) \rightarrow GL_r(\mathbf{C})$. Then ρ corresponds to a locally constant sheaf $\mathbf{V}$ on $C \setminus \{t_1, \dots, t_n\}$. By [4, II, Proposition 5.4], there is a unique pair (E, ∇) of a vector bundle E and a connection $\nabla : E \rightarrow E \otimes \Omega_C^1(t_1 + \dots + t_n)$ such that all the eigenvalues of $\text{res}_{t_i}(\nabla)$ lie in $\{\lambda \in \mathbf{C} | 0 \leq \text{Re}(\lambda) < 1\}$ and $\ker \nabla^{an}|_{C \setminus \{t_1, \dots, t_n\}} \cong \mathbf{V}$. We can take a parabolic structure $\{l_j^{(i)}\}$ on E compatible with the connection ∇ . Then $(E, \nabla, \{l_j^{(i)}\})$ becomes a parabolic connection and $\text{RH}([(E, \nabla, \{l_j^{(i)}\})]) = \rho$. So RH is surjective. Let $(E, \nabla, \{l_j^{(i)}\})$ be a $(\mathbf{t}, \boldsymbol{\lambda})$ -parabolic connection and $(E', \nabla', \{l_j'^{(i)}\})$ be a $(\mathbf{t}, \boldsymbol{\lambda}')$ -parabolic connection such that $\text{RH}([(E, \nabla, \{l_j^{(i)}\})])$

$= \text{RH}([(E', \nabla', \{(l')_j^{(i)}\})])$, namely $\ker \nabla^{an}|_{C \setminus \{t_1, \dots, t_n\}} \cong \ker (\nabla')^{an}|_{C \setminus \{t_1, \dots, t_n\}}$. By Proposition 3.1, there is a $(\mathbf{t}, \boldsymbol{\mu})$ -parabolic connection $(E_1, \nabla_1, \{(l_1)_j^{(i)}\})$ and a $(\mathbf{t}, \boldsymbol{\mu}')$ -parabolic connection $(E'_1, \nabla'_1, \{(l'_1)_j^{(i)}\})$ such that $(E, \nabla, \{(l)_j^{(i)}\}) \sim (E_1, \nabla_1, \{(l_1)_j^{(i)}\})$, $(E', \nabla', \{(l')_j^{(i)}\}) \sim (E'_1, \nabla'_1, \{(l'_1)_j^{(i)}\})$ and $0 \leq \text{Re}(\mu_j^{(i)}) < 1$, $0 \leq \text{Re}((\mu')_j^{(i)}) < 1$ for any i, j . Since $\ker \nabla_1^{an}|_{C \setminus \{t_1, \dots, t_n\}} \cong \ker (\nabla'_1)^{an}|_{C \setminus \{t_1, \dots, t_n\}}$, we can see by [4, II, Proposition 5.4], that $(E_1, \nabla_1) \cong (E'_1, \nabla'_1)$. So we have $(E, \nabla, \{(l)_j^{(i)}\}) \sim (E_1, \nabla_1, \{(l_1)_j^{(i)}\}) \sim (E'_1, \nabla'_1, \{(l'_1)_j^{(i)}\}) \sim (E', \nabla', \{(l')_j^{(i)}\})$. Thus we obtain the injectivity of RH.

Unfortunately, we cannot expect an appropriate algebraic structure on the moduli space of the elements $[(E, \nabla, \{(l)_j^{(i)}\})]$ in the set $(\dagger)$; so, it is natural to consider the moduli space of isomorphism classes of parabolic connections in the moduli theoretic description of the Riemann-Hilbert correspondence. However, we must consider a stability when we construct the moduli of parabolic connections as an appropriate space, so we set $M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda})$ as the moduli space of α -stable parabolic connections; see Theorem 2.1 and Definition 2.1 for the precise definition of $M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda})$. Next we consider the moduli space $RP_r(C, \mathbf{t})_{\mathbf{a}}$ of certain equivalence classes of representations of the fundamental group $\pi_1(C \setminus \{t_1, \dots, t_n\}, *)$. Here two representations are equivalent if their semisimplifications are isomorphic. (There is not an appropriate moduli space of the isomorphism classes of the representations of the fundamental group.) For a construction of a good moduli space containing all the representations, we must consider such an equivalence relation; see section 2 for the precise definition of $RP_r(C, \mathbf{t})_{\mathbf{a}}$. The most crucial part of the main result (Theorem 2.2) is that the morphism $\mathbf{RH} : M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}) \rightarrow RP_r(C, \mathbf{t})_{\mathbf{a}}$ determined by the Riemann-Hilbert correspondence is a proper surjective bimeromorphic morphism. This theorem was proved in [7] for $C = \mathbf{P}^1$ and $r = 2$. However there were certain difficulties to generalize this fact to the case of general C and r .

One of the most important motivations to consider such a theorem is the search for a geometric description of the differential equation determined by the isomonodromic deformation; see section 8 for the precise definition of the isomonodromic deformation. This differential equation is said to be the Schlesinger equation for $C = \mathbf{P}^1$, the Garnier equation for $C = \mathbf{P}^1$ and $r = 2$, and the Painlevé VI equation for $C = \mathbf{P}^1$, $r = 2$ and $n = 4$. M. Jimbo, T. Miwa and K. Ueno give in [12] and [13] an explicit description of the Schlesinger equation. In this paper, we will construct a space M with a morphism $\pi : M \rightarrow T$ such that there is a differential equation on M with respect to a time variable $t \in T$ which is determined by the isomonodromic deformation. If we fix a point $t_0 \in T$, $\pi^{-1}(t_0)$ should become a space of initial

conditions of the differential equation determined by the isomonodromic deformation. Take any point $x \in \pi^{-1}(t_0)$ and consider the analytic continuation γ starting at x and $\pi\gamma$ coming back to the initial point t_0 such that γ is a solution of the differential equation determined by the isomonodromic deformation. Then γ should come back to a point of $\pi^{-1}(t_0)$. We will generalize this property to the geometric Painlevé property. Roughly speaking, the geometric Painlevé property is the property of the space where the differential equation is defined and any analytic continuation of a solution of the differential equation stays in the space; see Definition 2.4 for the precise definition of the geometric Painlevé property. In fact we will take M as the moduli space $M_{C/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)$ of α -stable parabolic connections and $\pi : M_{C/T}^{\alpha}(\tilde{\mathbf{t}}, r, d) \rightarrow T$ as the structure morphism. For example let us assume that $C = \mathbf{P}^1$ and take an α -stable parabolic connection $x = (E, \nabla, \{l_j^{(i)}\})$ such that $E \cong \mathcal{O}_{\mathbf{P}^1}^{\oplus r}$. Let $\gamma : [0, 1] \rightarrow M_{\mathbf{P} \times T/T}^{\alpha}(\tilde{\mathbf{t}}, r, 0)$ be a path starting at x and $\pi\gamma$ ends at $\pi(x)$ such that γ is a solution of the differential equation determined by the isomonodromic deformation. Then the ending point $\gamma(1)$ corresponds to an α -stable parabolic connection $(E', \nabla', \{(l')_j^{(i)}\})$, but E' may not be trivial. So we recognize that it is not enough to consider only trivial vector bundles with a connection. We should also consider non-trivial vector bundles with a connection.

Our aim here is to construct a space where the isomonodromic deformation is defined and has the geometric Painlevé property over every value of λ . (It is not difficult to construct such a space over generic λ , but it is difficult to construct over special λ .) In fact, that space is nothing but the moduli space of α -stable parabolic connections and the result is given in Theorem 2.3, which is essentially a corollary of Theorem 2.2. Here note that the properness of the morphism **RH** is very essential in the proof of Theorem 2.3. The geometric Painlevé property immediately implies the usual analytic Painlevé property. So we can say that Theorem 2.3 gives a most clear proof of the Painlevé property of the isomonodromic deformation. As is well known, the solutions of the Painlevé equation have the Painlevé property, which is in some sense the property characterizing the Painlevé equation (and there were many proofs of the Painlevé property). So the usual analytic Painlevé property plays an important role in the theory of the Painlevé equation, but we can see from the definition that the “geometric Painlevé property” is much more important from the viewpoint of the description of the geometric picture of the isomonodromic deformation.

To prove the main results, this paper consists of several sections. In section 4, we prove the existence of the moduli space of stable parabolic connections.

An algebraic moduli space of parabolic connections was essentially considered by D. Arinkin and S. Lysenko in [1], [2] and [3] and they showed that the moduli space of parabolic connections on $\mathbf{P}^1$ of rank 2 with $n = 4$ for generic λ is isomorphic to the space of initial conditions of the Painlevé VI equation constructed by K. Okamoto [16]. For special λ , we should consider a certain stability condition to construct an appropriate moduli space of parabolic connections. In the case of $C = \mathbf{P}^1$ and $r = 2$, K. Iwasaki, M.-H. Saito and the author already considered in [7] the moduli space of stable parabolic connections and they proved in [8] that the moduli space of stable parabolic connections on $\mathbf{P}^1$ of rank 2 with $n = 4$ is isomorphic to the space of initial conditions of the Painlevé VI equation constructed by K. Okamoto for all λ . An analytic construction of the moduli space of stable parabolic connections for general C and $r = 2$ was given by H. Nakajima in [14]. However, in our aim, the algebraic construction of the moduli space is necessary. The morphism $\mathbf{RH}$ determined by the Riemann-Hilbert correspondence is quite transcendental, which is explicitly shown in the case of $C = \mathbf{P}^1$, $r = 2$ and $n = 4$ in [8]. This statement makes sense only if we construct the moduli spaces $M_C^\alpha(\mathbf{t}, \lambda)$ and $RP_r(C, \mathbf{t})_{\mathbf{a}}$ algebraically. Moduli of logarithmic connections without parabolic structure are constructed in [15]. For the proof of Theorem 2.1, we do not use the method in [17]. We construct $M_{C/T}^\alpha(\tilde{\mathbf{t}}, r, d)$ as a subscheme of the moduli space of parabolic Λ_D^1 -triples constructed in [7]. We also use this embedding in the proof of the properness of $\mathbf{RH}$ in Theorem 2.2.

In section 5, we prove that the moduli space of stable parabolic connections is an irreducible variety. For the proof, we need certain complicated calculations and this part is a new difficulty, which did not appear in [7].

In section 6, we consider the morphism $\mathbf{RH}$ determined by the Riemann-Hilbert correspondence and prove the surjectivity and properness of $\mathbf{RH}$. This is the essential part of the proof of Theorem 2.2. Notice that for generic λ , any $(\mathbf{t}, \lambda)$ -parabolic connection is irreducible and so all $(\mathbf{t}, \lambda)$ -parabolic connections are stable. Moreover we can easily see that the moduli space $M_C^\alpha(\mathbf{t}, \lambda)$ of parabolic connections for generic λ is analytically isomorphic via the Riemann-Hilbert correspondence to the moduli space $RP_r(C, \mathbf{t})_{\mathbf{a}}$ of representations of the fundamental group $\pi_1(C \setminus \{t_1, \dots, t_n\}, *)$. However, for special λ , several parabolic connections may not be stable and the stability condition depends on α . So we can see that the surjectivity of $\mathbf{RH}$ is not trivial at all for special λ , because for a point $[\rho] \in RP_r(C, \mathbf{t})_{\mathbf{a}}$, we must find a parabolic connection corresponding to $[\rho]$ which is α -stable. Moreover, the moduli space $M_C^\alpha(\mathbf{t}, \lambda)$ of α -stable parabolic connections is smooth, but the moduli space $RP_r(C, \mathbf{t})_{\mathbf{a}}$ of representations of the fundamental group becomes

singular. So the morphism $\mathbf{RH} : M_{\mathcal{C}}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda}) \rightarrow RP_r(C, \mathbf{t})_{\mathbf{a}}$ is more complicated in the case of special $\boldsymbol{\lambda}$ than the case of generic $\boldsymbol{\lambda}$. We first prove the surjectivity of $\mathbf{RH}$ in Proposition 6.1. In this proof, we use the Langton's type theorem in the case of parabolic connections. This idea was already used in [7]. Secondly, we prove in Proposition 6.2 that every fiber of $\mathbf{RH}$ is compact by using an embedding of $M_{\mathcal{C}}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})$ into a certain compact moduli space. In this proof, we cannot use the idea given in [7], so we develop a new technique again. Finally we obtain the properness of $\mathbf{RH}$ by the lemma given by A. Fujiki.

In section 7, we construct a canonical symplectic form on the moduli space $M_{\mathcal{C}}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})$ of α -stable parabolic connections. Combined with the fact that $\mathbf{RH}$ gives an analytic resolution of singularities of $RP_r(C, \mathbf{t})_{\mathbf{a}}$, we can say that $RP_r(C, \mathbf{t})_{\mathbf{a}}$ has symplectic singularities (for special $\mathbf{a}$) and $\mathbf{RH}$ gives a symplectic resolution of singularities. For the case of $r = 2$, H. Nakajima constructed the moduli space $M_{\mathcal{C}}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})$ as a hyper-Kähler manifold and it obviously has a holomorphic symplectic structure. Such a construction for general r is also an important problem, although we do not treat it here.

In section 8, we first give in Proposition 8.1 an algebraic construction of the differential equation on $M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)$ determined by the isomonodromic deformation. Finally we complete the proof of Theorem 2.3.

2. Main results

Let C be a smooth projective curve of genus g . We put

$$T_n := \left\{ (t_1, \dots, t_n) \in \overbrace{C \times \cdots \times C}^n \mid t_i \neq t_j \text{ for } i \neq j \right\}$$

for a positive integer n . For integers d, r with $r > 0$, recall the definition of $\Lambda_r^{(n)}(d)$ given in the introduction and the definition of $(\mathbf{t}, \boldsymbol{\lambda})$ -parabolic connection for $(\mathbf{t}, \boldsymbol{\lambda}) \in T_n \times \Lambda_r^{(n)}(d)$ in Definition 1.1.

Remark 2.1. Take a $(\mathbf{t}, \boldsymbol{\lambda})$ -parabolic connection $(E, \nabla, \{l_j^{(i)}\})$. By condition (3) of Definition 1.1, we have

$$\deg E = \deg(\det(E)) = - \sum_{i=1}^n \operatorname{res}_{t_i}(\nabla_{\det E}) = - \sum_{i=1}^n \sum_{j=0}^{r-1} \lambda_j^{(i)} = d.$$

Take rational numbers

$$0 < \alpha_1^{(i)} < \alpha_2^{(i)} < \cdots < \alpha_r^{(i)} < 1$$

for $i = 1, \dots, n$ satisfying $\alpha_j^{(i)} \neq \alpha_{j'}^{(i')}$ for $(i, j) \neq (i', j')$.

Definition 2.1. A parabolic connection $(E, \nabla, \{l_*^{(i)}\}_{1 \leq i \leq n})$ is α -stable (resp. α -semistable) if for any proper non-zero subbundle $F \subset E$ satisfying $\nabla(F) \subset F \otimes \Omega_C^1(t_1 + \cdots + t_n)$, the inequality

$$\frac{\deg F + \sum_{i=1}^n \sum_{j=1}^r \alpha_j^{(i)} \dim((F|_{t_i} \cap l_{j-1}^{(i)})/(F|_{t_i} \cap l_j^{(i)}))}{\operatorname{rank} F} < \frac{\deg E + \sum_{i=1}^n \sum_{j=1}^r \alpha_j^{(i)} \dim(l_{j-1}^{(i)}/l_j^{(i)})}{\operatorname{rank} E} \quad (\text{resp. } \leq)$$

holds.

Remark 2.2. Assume that $\alpha = (\alpha_j^{(i)})$ satisfies the condition: For any integer r' with $0 < r' < \operatorname{rank}(E)$ and for any $(\epsilon_j^{(i)})_{j=1, \dots, \operatorname{rank}(E)}^{i=1, \dots, n}$ with $\epsilon_j^{(i)} \in \{0, 1\}$ and $\sum_{j=1}^r \epsilon_j^{(i)} = r'$ for any i ,

$$\sum_{i=1}^n \sum_{j=1}^{\operatorname{rank}(E)} \alpha_j^{(i)} (r' - \operatorname{rank}(E) \epsilon_j^{(i)}) \notin \mathbf{Z}.$$

Then a parabolic connection $(E, \nabla, \{l_j^{(i)}\})$ is α -stable if and only if $(E, \nabla, \{l_j^{(i)}\})$ is α -semistable.

Let T be a smooth algebraic scheme which is a smooth covering of the moduli stack of n -pointed smooth projective curves of genus g over $\mathbf{C}$ and take a universal family $(\mathcal{C}, \tilde{t}_1, \dots, \tilde{t}_n)$ over T .

Theorem 2.1. *There exists a relative fine moduli scheme*

$$M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d) \rightarrow T \times \Lambda_r^{(n)}(d)$$

of α -stable parabolic connections of rank r and degree d , which is smooth and quasi-projective. The fiber $M_{\mathcal{C}_x}^\alpha(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda})$ over $(x, \boldsymbol{\lambda}) \in T \times \Lambda_r^{(n)}(d)$ is the moduli space of α -stable $(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda})$ -parabolic connections whose dimension is $2r^2(g-1) + nr(r-1) + 2$ if it is non-empty.

Definition 2.2. Take an element $\boldsymbol{\lambda} \in \Lambda_r^{(n)}(d)$. We call $\boldsymbol{\lambda}$ special if

- (1) $\lambda_j^{(i)} - \lambda_k^{(i)} \in \mathbf{Z}$ for some i and $j \neq k$, or
- (2) there exists an integer s with $0 < s < r$ and a subset $\{j_1^i, \dots, j_s^i\} \subset \{0, \dots, r-1\}$ for each $1 \leq i \leq n$ such that

$$\sum_{i=1}^n \sum_{k=1}^s \lambda_{j_k^i}^{(i)} \in \mathbf{Z}.$$

We call $\boldsymbol{\lambda}$ resonant if it satisfies condition (1) above. Let $(E, \nabla, \{l_j^{(i)}\})$ be a $(\tilde{\mathbf{t}}, \boldsymbol{\lambda})$ -parabolic connection such that $\boldsymbol{\lambda}$ does not satisfy condition (2) above. Then $(E, \nabla, \{l_j^{(i)}\})$ is irreducible. Here we say a $(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda})$ -connection $(E, \nabla_E, \{l_j^{(i)}\})$ is reducible if there is a non-trivial subbundle $0 \neq F \subsetneq E$

such that $\nabla_E(F) \subset F \otimes \Omega_{\mathcal{C}_x}^1((\tilde{t}_1)_x + \cdots + (\tilde{t}_n)_x)$. We say $(E, \nabla_E, \{l_j^{(i)}\})$ is irreducible if it is not reducible. We call $\lambda \in \Lambda_r^{(n)}(d)$ generic if it is not special.

Fix a point $x \in T$. Then the fundamental group $\pi_1(\mathcal{C}_x \setminus \{(\tilde{t}_1)_x, \dots, (\tilde{t}_n)_x\}, *)$ is generated by cycles $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ and loops γ_i around t_i for $1 \leq i \leq n$ whose relation is given by

$$\prod_{j=1}^g \alpha_j^{-1} \beta_j^{-1} \alpha_j \beta_j \prod_{i=1}^n \gamma_i = 1.$$

So the fundamental group is isomorphic to a free group generated by $2g+n-1$ free generators. Then the space

$$\mathrm{Hom}(\pi_1(\mathcal{C}_x \setminus \{(\tilde{t}_1)_x, \dots, (\tilde{t}_n)_x\}, *), GL_r(\mathbf{C}))$$

of representations of the fundamental group becomes an affine variety isomorphic to $GL_r(\mathbf{C})^{2g+n-1}$ and $GL_r(\mathbf{C})$ acts on this space by the adjoint action. We define

$$RP_r(\mathcal{C}_x, \tilde{\mathbf{t}}_x) := \mathrm{Hom}(\pi_1(\mathcal{C}_x \setminus \{(\tilde{t}_1)_x, \dots, (\tilde{t}_n)_x\}, *), GL_r(\mathbf{C})) // GL_r(\mathbf{C})$$

as the categorical quotient by the adjoint action of $GL_r(\mathbf{C})$; that is, $RP_r(\mathcal{C}_x, \tilde{\mathbf{t}}_x)$ is the affine scheme whose affine coordinate ring is the invariant ring with respect to the adjoint action of $GL_r(\mathbf{C})$ on the affine coordinate ring of $\mathrm{Hom}(\pi_1(\mathcal{C}_x \setminus \{(\tilde{t}_1)_x, \dots, (\tilde{t}_n)_x\}, *), GL_r(\mathbf{C}))$. If we put

$$\mathcal{A}_r^{(n)} := \left\{ \mathbf{a} = (a_j^{(i)})_{0 \leq j \leq r-1}^{1 \leq i \leq n} \in \mathbf{C}^{nr} \mid a_0^{(1)} a_0^{(2)} \cdots a_0^{(n)} = (-1)^{rn} \right\},$$

then we can define a morphism

$$RP_r((\mathcal{C})_x, \tilde{\mathbf{t}}_x) \longrightarrow \mathcal{A}_r^{(n)} \\ [\rho] \mapsto \mathbf{a} = (a_j^{(i)})$$

by the relation

$$\det(XI_r - \rho(\gamma_i)) = X^r + a_{r-1}^{(i)} X^{r-1} + \cdots + a_0^{(i)},$$

where X is an indeterminate and I_r is the identity matrix of size r . Note that for any $GL_r(\mathbf{C})$ -representation ρ of $\pi_1(\mathcal{C}_x \setminus \{(\tilde{t}_1)_x, \dots, (\tilde{t}_n)_x\}, *)$,

$$\begin{aligned} & \det(\rho(\gamma_1)) \cdots \det(\rho(\gamma_n)) \\ &= \det(\rho(\beta_g))^{-1} \det(\rho(\alpha_g))^{-1} \det(\beta_g) \det(\alpha_g) \cdots \det(\rho(\beta_1))^{-1} \\ & \quad \times \det(\rho(\alpha_1))^{-1} \det(\beta_1) \det(\alpha_1) = 1 \end{aligned}$$

and so the equation $a_0^{(1)} a_0^{(2)} \cdots a_0^{(n)} = (-1)^{rn}$ should be satisfied.

We can define a relative moduli space $RP_r(\mathcal{C}, \tilde{\mathbf{t}}) = \coprod_{x \in T} RP_r(\mathcal{C}_x, \tilde{\mathbf{t}}_x)$ of representations. Let $\tilde{T} \rightarrow T$ be the universal covering. Then $RP_r(\mathcal{C}, \tilde{\mathbf{t}}) \times_T \tilde{T}$ becomes a trivial fibration over $\tilde{T}$. So we can define a morphism

$$RP_r(\mathcal{C}, \tilde{\mathbf{t}}) \times_T \tilde{T} \longrightarrow \tilde{T} \times \mathcal{A}_r^{(n)}.$$

We define a morphism

$$(1) \quad rh : \Lambda_r^{(n)}(d) \ni \boldsymbol{\lambda} \mapsto \mathbf{a} \in \mathcal{A}_r^{(n)}$$

by

$$\prod_{j=0}^{r-1} \left(X - \exp(-2\pi\sqrt{-1}\lambda_j^{(i)}) \right) = X^r + a_{r-1}^{(i)} X^{r-1} + \cdots + a_0^{(i)}.$$

For each member $(E, \nabla, \{l_j^{(i)}\}) \in M_{\mathcal{C}_x}^{\boldsymbol{\alpha}}(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda})$, $\ker(\nabla^{an}|_{\mathcal{C}_x \setminus \{(\tilde{t}_1)_x, \dots, (\tilde{t}_n)_x\}})$ becomes a local system on $\mathcal{C}_x \setminus \{(\tilde{t}_1)_x, \dots, (\tilde{t}_n)_x\}$, where ∇^{an} means the analytic connection corresponding to ∇ . The local system $\ker(\nabla^{an}|_{\mathcal{C}_x \setminus \{(\tilde{t}_1)_x, \dots, (\tilde{t}_n)_x\}})$ corresponds to a representation of $\pi_1(\mathcal{C}_x \setminus \{(\tilde{t}_1)_x, \dots, (\tilde{t}_n)_x\}, *)$. So we can define a morphism

$$\mathbf{RH}_{(x, \boldsymbol{\lambda})} : M_{\mathcal{C}_x}^{\boldsymbol{\alpha}}(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda}) \longrightarrow RP_r(\mathcal{C}_x, \tilde{\mathbf{t}}_x)_{\mathbf{a}},$$

where $\mathbf{a} = rh(\boldsymbol{\lambda})$. $\{\mathbf{RH}_{(x, \boldsymbol{\lambda})}\}$ induces a morphism

$$(2) \quad \mathbf{RH} : M_{\mathcal{C}/T}^{\boldsymbol{\alpha}}(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T} \longrightarrow RP_r(\mathcal{C}, \tilde{\mathbf{t}}) \times_T \tilde{T}$$

which makes the diagram

$$\begin{array}{ccc} M_{\mathcal{C}/T}^{\boldsymbol{\alpha}}(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T} & \xrightarrow{\mathbf{RH}} & RP_r(\mathcal{C}, \tilde{\mathbf{t}}) \times_T \tilde{T} \\ \downarrow & & \downarrow \\ \tilde{T} \times \Lambda_r^{(n)}(d) & \xrightarrow{\text{id} \times rh} & \tilde{T} \times \mathcal{A}_r^{(n)} \end{array}$$

commute.

The following is the main theorem, whose proof is given in section 6.

Theorem 2.2. *Assume $\boldsymbol{\alpha}$ is so generic that $\boldsymbol{\alpha}$ -stable $\Leftrightarrow \boldsymbol{\alpha}$ -semistable. Moreover, we assume $rn - 2r - 2 > 0$ if $g = 0$, $n > 1$ if $g = 1$ and $n \geq 1$ if $g \geq 2$. Then the morphism*

$$\mathbf{RH} : M_{\mathcal{C}/T}^{\boldsymbol{\alpha}}(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T} \longrightarrow (RP_r(\mathcal{C}, \tilde{\mathbf{t}}) \times_T \tilde{T}) \times_{\mathcal{A}_r^{(n)}} \Lambda_r^{(n)}(d)$$

induced by (2) is a proper surjective bimeromorphic morphism. Combined with Proposition 7.1, we can say that for each $(x, \boldsymbol{\lambda}) \in T \times \Lambda_r^{(n)}(d)$,

- (1) $\mathbf{RH}_{(x, \boldsymbol{\lambda})} : M_{\mathcal{C}_x}^{\boldsymbol{\alpha}}(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda}) \longrightarrow RP_r(\mathcal{C}_x, \tilde{\mathbf{t}}_x)_{\mathbf{a}}$ is an analytic isomorphism of symplectic varieties for generic $\boldsymbol{\lambda}$ and

- (2) $\mathbf{RH}_{(x,\lambda)} : M_{\mathcal{C}_x}^\alpha(\tilde{\mathbf{t}}_x, \lambda) \longrightarrow RP_r(\mathcal{C}_x, \tilde{\mathbf{t}}_x)_{\mathbf{a}}$ gives an analytic symplectic resolution of singularities of $RP_r(\mathcal{C}_x, \tilde{\mathbf{t}}_x)_{\mathbf{a}}$ for special λ .

Here we put $\mathbf{a} = rh(\lambda)$.

Remark 2.3. (1) If we take α sufficiently generic, then α satisfies the condition of Remark 2.2. So we have α -stable $\Leftrightarrow \alpha$ -semistable.

(2) In section 7, we will show that $M_{\mathcal{C}_x}^\alpha(\tilde{\mathbf{t}}_x, \lambda)$ and $RP_r(\mathcal{C}_x, \tilde{\mathbf{t}}_x)_{\mathbf{a}}$ become symplectic varieties.

(3) For the case of $r = 1$, $(\mathbf{RH})_{(x,\lambda)}$ is an isomorphism for any $(x, \lambda) \in T \times \Lambda_1^{(n)}(d)$ and $RP_1(\mathcal{C}_x, \tilde{\mathbf{t}}_x)_{\mathbf{a}}$ is smooth for any $\mathbf{a}$.

(4) In Theorem 2.2 we consider the case $n \geq 1$. As is stated in [18, Proposition 7.8], the Riemann-Hilbert correspondence for the case of $n = 0$ gives an analytic isomorphism between the moduli space of integrable connections and the moduli space of representations of the fundamental group, but in that case the moduli space of integrable connections may be singular.

Since $RP_r(\mathcal{C}, \tilde{\mathbf{t}}) \times_T \tilde{T} \rightarrow \tilde{T}$ is a trivial fibration, we can consider the set of constant sections

$$\mathcal{F}_R := \left\{ \sigma : \tilde{T} \rightarrow RP_r(\mathcal{C}, \tilde{\mathbf{t}}) \times_T \tilde{T} \right\}.$$

As is stated in Remark 6.2,

$$RP_r(\mathcal{C}_x, \tilde{\mathbf{t}}_x)_{\mathbf{a}}^{\text{sing}} = \left\{ [\rho] \in RP_r(\mathcal{C}_x, \tilde{\mathbf{t}}_x)_{\mathbf{a}} \left| \begin{array}{l} \rho \text{ is reducible, or} \\ \dim \left(\ker \left(\rho(\gamma_i) - \exp(-2\pi\sqrt{-1}\lambda_j^{(i)}) I_r \right) \right) \right. \\ \left. \geq 2 \text{ for some } i, j \right. \end{array} \right\}$$

is just the singular locus of $RP_r(\mathcal{C}_x, \tilde{\mathbf{t}}_x)_{\mathbf{a}}$. If we put

$$RP_r(\mathcal{C}_x, \tilde{\mathbf{t}}_x)_{\mathbf{a}}^\# := RP_r(\mathcal{C}_x, \tilde{\mathbf{t}}_x)_{\mathbf{a}} \setminus RP_r(\mathcal{C}_x, \tilde{\mathbf{t}}_x)_{\mathbf{a}}^{\text{sing}},$$

then $\mathbf{RH}|_{M_{\mathcal{C}_x}^\alpha(\tilde{\mathbf{t}}_x, \lambda)^\#} : M_{\mathcal{C}_x}^\alpha(\tilde{\mathbf{t}}_x, \lambda)^\# \rightarrow RP_r(\mathcal{C}_x, \tilde{\mathbf{t}}_x)_{\mathbf{a}}^\#$ becomes an isomorphism, where we put $M_{\mathcal{C}_x}^\alpha(\tilde{\mathbf{t}}_x, \lambda)^\# := \mathbf{RH}_{(x,\lambda)}^{-1}(RP_r(\mathcal{C}_x, \tilde{\mathbf{t}}_x)_{\mathbf{a}}^\#)$. Put

$$(RP_r(\mathcal{C}, \tilde{\mathbf{t}}) \times_T \tilde{T})^\# = \coprod_{(x, \mathbf{a}) \in \tilde{T} \times \mathcal{A}_r^{(n)}} RP_r(\mathcal{C}_x, \tilde{\mathbf{t}}_x)_{\mathbf{a}}^\#.$$

Then the restriction $\mathcal{F}_R^\#$ of $\mathcal{F}_R$ to $(RP_r(\mathcal{C}, \tilde{\mathbf{t}}) \times_T \tilde{T})^\#$ gives a foliation on $(RP_r(\mathcal{C}, \tilde{\mathbf{t}}) \times_T \tilde{T})^\#$. The pull-back $\tilde{\mathcal{F}}_M^\# := \mathbf{RH}^{-1}(\mathcal{F}_R^\#)$ determines a foliation on $(M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T})^\#$ where $(M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T})^\# = \mathbf{RH}^{-1}((RP_r(\mathcal{C}, \tilde{\mathbf{t}}) \times_T \tilde{T})^\#)$. This foliation corresponds to a subbundle of the analytic tangent bundle $\Theta_{(M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T})}^{an}$ determined by a splitting

$$D^\# : \tilde{\pi}^*(\Theta_{\tilde{T}}^{an}) \rightarrow \Theta_{(M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T})}^{an}$$

of the analytic tangent map

$$\Theta_{(M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T})^{\sharp}}^{an} \longrightarrow \tilde{\pi}^* \Theta_{\tilde{T}}^{an} \rightarrow 0,$$

where $\tilde{\pi} : (M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T})^{\sharp} \rightarrow \tilde{T}$ is the projection. We will show in Proposition 8.1 that this splitting $D^{\sharp}$ is in fact induced by a splitting

$$D : \pi^*(\Theta_T) \rightarrow \Theta_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)}$$

of the algebraic tangent bundle, where $\pi : M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d) \rightarrow T$ is the projection.

In order to explain the concept of the geometric Painlevé property, let us review here several notions introduced by K. Iwasaki in [9].

Definition 2.3 (see [9]). A time-dependent dynamical system $(M, \mathcal{F})$ is a smooth fibration $\pi : M \rightarrow T$ of complex manifolds together with a complex foliation $\mathcal{F}$ on M that is transverse to each fiber $M_t = \pi^{-1}(t)$, $t \in T$. The total space M is referred to as the phase space, while the base space T is called the space of time-variables. Moreover, the fiber M_t is called the space of initial conditions at time t .

Definition 2.4 (see [9]). A time-dependent dynamical system $(M, \mathcal{F})$ is said to have the geometric Painlevé property if for any path γ in T and any point $p \in M_t$, where t is the initial point of γ , there exists a unique $\mathcal{F}$ -horizontal lift $\tilde{\gamma}_p$ of γ with initial point p . Here a curve in M is said to be $\mathcal{F}$ -horizontal if it lies in a leaf of $\mathcal{F}$.

$D^{\sharp}(\tilde{\pi}^* \Theta_{\tilde{T}}^{an})$ obviously satisfies the integrability condition

$$[D^{\sharp}(\tilde{\pi}^* \Theta_{\tilde{T}}^{an}), D^{\sharp}(\tilde{\pi}^* \Theta_{\tilde{T}}^{an})] \subset D^{\sharp}(\tilde{\pi}^* \Theta_{\tilde{T}}^{an})$$

because it corresponds to the foliation $\tilde{\mathcal{F}}_M^{\sharp}$. Since

$$\text{codim}_{(M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T})^{\sharp}} \left((M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T}) \setminus (M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T})^{\sharp} \right) \geq 2,$$

we can see that $D(\pi^*(\Theta_T))$ also satisfies the integrability condition

$$[D(\pi^*(\Theta_T)), D(\pi^*(\Theta_T))] \subset D(\pi^*(\Theta_T)).$$

Then the foliation $\tilde{\mathcal{F}}_M^{\sharp}$ extends to a foliation $\tilde{\mathcal{F}}_M$ on $M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T}$. We can see that $\tilde{\mathcal{F}}_M$ descends to a foliation $\mathcal{F}_M$ on $M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)$ which corresponds to the subbundle $D(\pi^*(\Theta_T)) \subset \Theta_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)}$. By construction, we can see that $(M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d), \mathcal{F}_M)$ becomes a time-dependent dynamical system with base space T . As a corollary of Theorem 2.2, we can prove the following theorem whose proof is given in section 8.

Theorem 2.3. *Under the assumption of Theorem 2.2, the time-dependent dynamical system $(M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d), \mathcal{F}_M)$ determined by the differential system $D(\pi^* \Theta_T)$ has the geometric Painlevé property.*

3. Elementary transform of parabolic connections

For a parabolic connection $(E, \nabla, \{l_j^{(i)}\})$ on $(C, \mathbf{t})$, put

$$E' := \ker(E \longrightarrow E|_{t_p}/l_q^{(p)}).$$

Then ∇ induces a connection $\nabla' : E' \rightarrow E' \otimes \Omega_C^1(t_1 + \cdots + t_n)$ such that the diagram

$$\begin{array}{ccc} E' & \xrightarrow{\nabla'} & E' \otimes \Omega_C^1(t_1 + \cdots + t_n) \\ \downarrow & & \downarrow \\ E & \xrightarrow{\nabla} & E \otimes \Omega_C^1(t_1 + \cdots + t_n) \end{array}$$

commutes. Let $E'|_{t_p} = (l')_0^{(p)} \supset (l')_1^{(p)} \supset \cdots \supset (l')_{r-q}^{(p)}$ be the inverse image of the filtration $l_q^{(p)} \supset \cdots \supset l_r^{(p)} = 0$ by the linear map $E'|_{t_p} \rightarrow E|_{t_p}$ and $(l')_{r-q}^{(p)} \supset \cdots \supset (l')_r^{(p)} = 0$ be the image of the filtration $E(-t_p)|_{t_p} = l_0^{(p)} \otimes \mathcal{O}(-t_p) \supset \cdots \supset l_q^{(p)} \otimes \mathcal{O}(-t_p)$ by the linear map $E(-t_p)|_{t_p} \rightarrow E'|_{t_p}$. Then $(l')_*^{(p)}$ becomes a filtration of $E'|_{t_p}$. For $i \neq p$, we put $(l')_j^{(i)} = l_j^{(i)}$ for any j . Then $(E', \nabla', \{(l')_j^{(i)}\})$ becomes a parabolic connection on $(C, \mathbf{t})$. We call $(E', \nabla', \{(l')_j^{(i)}\})$ the elementary transform of $(E, \nabla, \{l_j^{(i)}\})$ along t_p by $l_q^{(p)}$.

Definition 3.1. We define a functor $\mathcal{M}_{C/T}(\tilde{\mathbf{t}}, r, d)$ from the category of locally Noetherian schemes over T to the category of sets by

$$\mathcal{M}_{C/T}(\tilde{\mathbf{t}}, r, d)(S) = \left\{ (E, \nabla, \{l_j^{(i)}\}) \right\} / \sim$$

for a locally Noetherian scheme S over T , where

- (1) E is a vector bundle on $\mathcal{C}_S$ of rank r ,
- (2) $\nabla : E \rightarrow E \otimes \Omega_{\mathcal{C}_S/S}^1((\tilde{t}_1)_S + \cdots + (\tilde{t}_n)_S)$ is a relative connection, that is, ∇ is a morphism of sheaves such that $\nabla(fa) = a \otimes df + f\nabla(a)$ for $f \in \mathcal{O}_{\mathcal{C}_S}$ and $a \in E$,
- (3) $E|_{(\tilde{t}_i)_S} = l_0^{(i)} \supset \cdots \supset l_r^{(i)} = 0$ is a filtration by subbundles on $(\tilde{t}_i)_S$ such that $\text{res}_{(\tilde{t}_i)_S}(\nabla)(l_j^{(i)}) \subset l_j^{(i)}$ for $i = 1, \dots, n$ and $j = 0, 1, \dots, r-1$, and
- (4) for any $s \in S$, $E \otimes k(s)$ is of degree d and $\dim \left((l_j^{(i)}/l_{j+1}^{(i)}) \otimes k(s) \right) = 1$ for any i, j .

Here $(E, \nabla, \{l_j^{(i)}\}) \sim (E', \nabla', \{(l')_j^{(i)}\})$ if there is a line bundle $\mathcal{L}$ on S such that $(E, \nabla, \{l_j^{(i)}\}) \cong (E', \nabla', \{(l')_j^{(i)}\}) \otimes \mathcal{L}$.

Remark 3.1. Take any member $(E, \nabla, \{l_j^{(i)}\}) \in \mathcal{M}_{C/T}(\tilde{\mathbf{t}}, r, d)(S)$. Then $\text{res}_{(\tilde{t}_i)_S}(\nabla)$ induces a morphism $\lambda_j^{(i)} : l_j^{(i)}/l_{j+1}^{(i)} \rightarrow l_j^{(i)}/l_{j+1}^{(i)}$ and $\boldsymbol{\lambda} = (\lambda_j^{(i)}) \in$

$\Lambda_r^{(n)}(d)(S)$. So we obtain a canonical morphism

$$\tilde{\lambda} : \mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d) \longrightarrow h_{\Lambda_r^{(n)}(d)}.$$

Since there is a canonical morphism $\mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d) \rightarrow h_T$, we obtain a morphism

$$(\tilde{x}, \tilde{\lambda}) : \mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d) \longrightarrow h_T \times h_{\Lambda_r^{(n)}(d)} = h_{T \times \Lambda_r^{(n)}(d)}.$$

Definition 3.2. For $\mathbf{C}$ -valued points $\lambda \in \Lambda_r^{(n)}(d)(\mathbf{C})$, $x \in T(\mathbf{C})$, (x, λ) can be considered as a morphism $h_{\text{Spec } \mathbf{C}} \rightarrow h_T \times h_{\Lambda_r^{(n)}(d)}$. Then we put

$$\mathcal{M}_{\mathcal{C}_x}(\tilde{\mathbf{t}}_x, \lambda) = \mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d) \times_{h_{T \times \Lambda_r^{(n)}(d)}} h_{\text{Spec } \mathbf{C}}.$$

$\mathcal{M}_{\mathcal{C}_x}(\tilde{\mathbf{t}}_x, \lambda)$ can be considered as a functor from the category of locally Noetherian schemes over $\mathbf{C}$ to the category of sets.

For a locally Noetherian scheme S over T , take any member $(E, \nabla, \{\iota_j^{(i)}\}) \in \mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)(S)$. For (p, q) with $1 \leq p \leq n$ and $0 \leq q \leq r$, we put

$$E' := \ker \left(E \longrightarrow (E|_{(\tilde{t}_p)_S} / l_q^{(p)}) \right).$$

Then ∇ induces a relative connection

$$\nabla' : E' \longrightarrow E' \otimes \Omega_{\mathcal{C}_S/S}^1((\tilde{t}_1)_S + \cdots + (\tilde{t}_n)_S)$$

such that the diagram

$$\begin{array}{ccc} E' & \xrightarrow{\nabla'} & E' \otimes \Omega_{\mathcal{C}_S/S}^1((\tilde{t}_1)_S + \cdots + (\tilde{t}_n)_S) \\ \downarrow & & \downarrow \\ E & \xrightarrow{\nabla} & E \otimes \Omega_{\mathcal{C}_S/S}^1((\tilde{t}_1)_S + \cdots + (\tilde{t}_n)_S) \end{array}$$

is commutative. Note that the sequence

$$E(-(\tilde{t}_p)_S)|_{(\tilde{t}_p)_S} \xrightarrow{\pi} E'|_{(\tilde{t}_p)_S} \xrightarrow{\iota} E|_{(\tilde{t}_p)_S}$$

is an exact sequence. $\text{Im } \iota = l_q^{(p)}$ is a subbundle of $E|_{(\tilde{t}_p)_S}$ and $\text{Im } \pi = \ker \iota$ is a subbundle of $E'|_{(\tilde{t}_p)_S}$. Moreover $\ker \pi = l_q^{(p)} \otimes \mathcal{O}_{\mathcal{C}_S}(-(\tilde{t}_p)_S)$ is a subbundle of $E(-(\tilde{t}_p)_S)|_{(\tilde{t}_p)_S}$. Let $E'|_{(\tilde{t}_p)_S} = (l')_0^{(p)} \supset \cdots \supset (l')_{r-q}^{(p)}$ be the inverse image by $\iota : E'|_{(\tilde{t}_p)_S} \rightarrow E|_{(\tilde{t}_p)_S}$ of the filtration $l_q^{(p)} \supset \cdots \supset l_r^{(p)}$. Then each $(l')_j^{(p)}$ is a subbundle of $E'|_{(\tilde{t}_p)_S}$ for $0 \leq j \leq r - q$. Let $(l')_{r-q}^{(p)} \supset \cdots \supset (l')_r^{(p)}$ be the image by $\pi : E(-(\tilde{t}_p)_S)|_{(\tilde{t}_p)_S} \rightarrow E'|_{(\tilde{t}_p)_S}$ of the filtration $l_0^{(p)} \otimes \mathcal{O}_{\mathcal{C}_S}(-(\tilde{t}_p)_S) \supset \cdots \supset l_q^{(p)} \otimes \mathcal{O}_{\mathcal{C}_S}(-(\tilde{t}_p)_S)$. For each j with $0 \leq j \leq q$, $l_j^{(p)} \otimes \mathcal{O}_{\mathcal{C}_S}(-(\tilde{t}_p)_S) / l_q^{(p)} \otimes \mathcal{O}_{\mathcal{C}_S}(-(\tilde{t}_p)_S) \xrightarrow{\sim} (l')_{r-q+j}^{(p)}$ is a subbundle of $E(-(\tilde{t}_p)_S)|_{(\tilde{t}_p)_S} / \ker \pi \cong \text{Im } \pi = \ker \iota$. So $(l')_{r-q+j}^{(p)}$ is a subbundle of $E'|_{(\tilde{t}_p)_S}$ for any j with $0 \leq j \leq q$.

For $i \neq p$ with $1 \leq i \leq n$, we put $(l')_j^{(i)} := l_j^{(i)}$ for $0 \leq j \leq r$. Then $(E', \nabla', \{(l')_j^{(i)}\})$ becomes a member of $\mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d-q)(S)$. We say $(E', \nabla', \{(l')_j^{(i)}\})$ is the elementary transform of $(E, \nabla, \{l_j^{(i)}\})$ along $(\tilde{t}_p)_S$ by $l_q^{(p)}$. The elementary transform induces a morphism

$$(3) \quad \text{Elm}_q^{(p)} : \mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d) \longrightarrow \mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d-q)$$

$$(4) \quad (E, \nabla, \{l_j^{(i)}\}) \mapsto (E', \nabla', \{(l')_j^{(i)}\})$$

of functors. For any member $(E, \nabla, \{l_j^{(i)}\}) \in \mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)(S)$, we have

$$\begin{aligned} & \text{Elm}_{r-q}^{(p)} \circ \text{Elm}_q^{(p)}(E, \nabla, \{l_j^{(i)}\}) \\ &= \text{Elm}_{r-q}^{(p)}(E', \nabla', \{(l')_j^{(i)}\}) \\ &= (E(-(\tilde{t}_p)_S), \nabla|_{E(-(\tilde{t}_p)_S)}, \{l_j^{(i)} \otimes \mathcal{O}_{\mathcal{C}_S}(-(\tilde{t}_p)_S)\}). \end{aligned}$$

If we put

$$(5) \quad b_p : \mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d-r) \xrightarrow{\sim} \mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)$$

$$(6) \quad (E, \nabla, \{l_j^{(i)}\}) \mapsto (E \otimes \mathcal{O}_{\mathcal{C}}(\tilde{t}_p), \nabla \otimes \text{id}_{\mathcal{O}_{\mathcal{C}}(\tilde{t}_p)} + \text{id}_E \otimes d_{\mathcal{O}_{\mathcal{C}}(\tilde{t}_p)}, \{l_j^{(i)} \otimes \mathcal{O}_{\mathcal{C}}(\tilde{t}_p)\}),$$

then we have $b_p \circ \text{Elm}_{r-q}^{(p)} \circ \text{Elm}_q^{(p)} = \text{id}_{\mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)}$. We can also check that

$$\begin{aligned} \text{Elm}_q^{(p)} \circ b_p \circ \text{Elm}_{r-q}^{(p)} &= b_p \circ \text{Elm}_q^{(p)} \circ \text{Elm}_{r-q}^{(p)} \\ &= \text{id}_{\mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d-q)}. \end{aligned}$$

Thus $\text{Elm}_q^{(p)}$ is an isomorphism:

$$\text{Elm}_q^{(p)} : \mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d) \xrightarrow{\sim} \mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d-q).$$

Take any $\mathbf{C}$ -valued points $x \in T(\mathbf{C})$, $\boldsymbol{\lambda} \in \Lambda_r^{(n)}(d)(\mathbf{C})$. We put

$$((\lambda')_0^{(p)}, \dots, (\lambda')_{r-1}^{(p)}) = (\lambda_q^{(p)}, \lambda_{q+1}^{(p)}, \dots, \lambda_{r-1}^{(p)}, \lambda_0^{(p)} + 1, \lambda_1^{(p)} + 1, \dots, \lambda_{q-1}^{(p)} + 1)$$

and $(\lambda')_j^{(i)} := \lambda_j^{(i)}$ for $i \neq p$ with $1 \leq i \leq n$ and $0 \leq j \leq r-1$. Then $\boldsymbol{\lambda}' := ((\lambda')_j^{(i)}) \in \Lambda_r^{(n)}(d-q)(\mathbf{C})$. For a member $(E, \nabla, \{l_j^{(i)}\}) \in \mathcal{M}_{\mathcal{C}_x}(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda})(S)$, $\text{Elm}_q^{(p)}(E, \nabla, \{l_j^{(i)}\}) \in \mathcal{M}_{\mathcal{C}_x}(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda}')(S)$. So $\text{Elm}_q^{(p)}$ induces an isomorphism

$$(7) \quad \text{Elm}_q^{(p)} : \mathcal{M}_{\mathcal{C}_x}(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda}) \xrightarrow{\sim} \mathcal{M}_{\mathcal{C}_x}(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda}').$$

Take a point $\boldsymbol{\lambda} \in \Lambda_r^{(n)}(d)(\mathbf{C})$. We can write

$$\prod_{j=0}^{r-1} (x - \lambda_j^{(p)}) = \prod_{k=1}^u (x - \mu_k^{(p)})^{m_k^{(p)}},$$

where $\operatorname{Re}(\mu_1^{(p)}) \geq \operatorname{Re}(\mu_2^{(p)}) \geq \cdots \geq \operatorname{Re}(\mu_u^{(p)})$, $\operatorname{Im}(\mu_j^{(p)}) > \operatorname{Im}(\mu_{j+1}^{(p)})$ if $\operatorname{Re}(\mu_j^{(p)}) = \operatorname{Re}(\mu_{j+1}^{(p)})$ and x is an indeterminate. For any member $(E, \nabla_E, \{l_j^{(i)}\}) \in \mathcal{M}_{\mathcal{C}_x}(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda})(S)$, we can find $j_0^{(k)}, j_1^{(k)}, \dots, j_{m_k}^{(k)}$ for $1 \leq k \leq u$ satisfying $0 \leq j_0^{(k)} < j_1^{(k)} < \cdots < j_{m_k}^{(k)} \leq r$ such that

$$\left(\ker \left(\operatorname{res}_{(\tilde{t}_p)_x}(\nabla_E) - \mu_k^{(p)} \cdot \operatorname{id} \right)^r \cap l_{j_m}^{(p)} \right) / \left(\ker \left(\operatorname{res}_{(\tilde{t}_p)_x}(\nabla_E) - \mu_k^{(p)} \cdot \operatorname{id} \right)^r \cap l_{j_{m+1}}^{(p)} \right) \xrightarrow{\sim} l_{j_m}^{(p)} / l_{j_{m+1}}^{(p)}$$

for $m = 0, \dots, m_k^{(p)} - 1$. Let $E|_{(\tilde{t}_p)_x} = (l')_0^{(p)} \supset (l')_1^{(p)} \supset \cdots \supset (l')_{r-1}^{(p)} \supset (l')_r^{(p)} = 0$ be the filtration by subbundles satisfying $\operatorname{rank}(l')_j^{(p)} / (l')_{j+1}^{(p)} = 1$ for $j = 0, \dots, r-1$ and

$$(l')_q^{(p)} = \left(\ker \left(\operatorname{res}_{(\tilde{t}_p)_x}(\nabla_E) - \mu_k^{(p)} \cdot \operatorname{id} \right)^r \cap l_{j_m}^{(p)} \right) + (l')_{q+1}^{(p)}$$

for $q = 0, \dots, r-1$, where $q = r - (m_k^{(p)} - m) - \sum_{k < k' \leq u} m_{k'}^{(p)}$, $1 \leq k \leq u$ and $0 \leq m < m_k^{(p)}$. We put

$$(\nu_0^{(p)}, \dots, \nu_{r-1}^{(p)}) := (\overbrace{\mu_1^{(p)}, \dots, \mu_1^{(p)}}^{m_1^{(p)}}, \dots, \overbrace{\mu_k^{(p)}, \dots, \mu_k^{(p)}}^{m_k^{(p)}}, \dots, \overbrace{\mu_u^{(p)}, \dots, \mu_u^{(p)}}^{m_u^{(p)}}),$$

$\nu_j^{(i)} = \lambda_j^{(i)}$ for $i \neq p$ and $\boldsymbol{\nu} := (\nu_j^{(i)})_{i,j}$. Then the correspondence $(E, \nabla_E, \{l_j^{(i)}\}) \mapsto (E, \nabla_E, \{(l')_j^{(i)}\})$ determines an isomorphism

$$(8) \quad a_p : \mathcal{M}_{\mathcal{C}_x}(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda}) \xrightarrow{\sim} \mathcal{M}_{\mathcal{C}_x}(\tilde{\mathbf{t}}_x, \boldsymbol{\nu})$$

of functors.

Looking at the change of $\boldsymbol{\lambda}$ under the transformations $\operatorname{Elm}_j^{(i)}$ and b_i , we can easily see the following proposition, which is useful in considering the Riemann-Hilbert correspondence in section 6.

Proposition 3.1. *Composing the isomorphisms a_i, b_i and $\operatorname{Elm}_j^{(i)}$ for $i = 1, \dots, n$, $j = 1, \dots, r$ given in (8), (7) and (5), we obtain an isomorphism*

$$(9) \quad \sigma : \mathcal{M}_{\mathcal{C}_x}(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda}) \xrightarrow{\sim} \mathcal{M}_{\mathcal{C}_x}(\tilde{\mathbf{t}}_x, \boldsymbol{\mu})$$

of functors, where $0 \leq \operatorname{Re}(\mu_j^{(i)}) < 1$ for any i, j .

As a corollary, we obtain the following:

Corollary 3.1. *Take $\boldsymbol{\lambda} \in \Lambda_r^{(n)}(d)$ and $\boldsymbol{\mu} \in \Lambda_r^{(n)}(d')$ which satisfy $rh(\boldsymbol{\lambda}) = rh(\boldsymbol{\mu}) \in \mathcal{A}_r^{(n)}$, where rh is the morphism defined in (1). Then we can obtain*

by composing $a_i, a_i^{-1}, b_i, b_i^{-1}$ and $\text{Elm}_j^{(i)}$ an isomorphism

$$\sigma : \mathcal{M}_{\mathcal{C}_x}(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda}) \xrightarrow{\sim} \mathcal{M}_{\mathcal{C}_x}(\tilde{\mathbf{t}}_x, \boldsymbol{\mu})$$

of functors.

Proof. Applying Proposition 3.1, we obtain an isomorphism

$$\sigma : \mathcal{M}_{\mathcal{C}_x}(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda}) \xrightarrow{\sim} \mathcal{M}_{\mathcal{C}_x}(\tilde{\mathbf{t}}_x, \boldsymbol{\nu})$$

of functors, where $0 \leq \text{Re}(\nu_j^{(i)}) < 1$ for any i, j . By composing a_i , we may also assume that $\text{Re}(\nu_0^{(i)}) \geq \text{Re}(\nu_1^{(i)}) \geq \cdots \geq \text{Re}(\nu_{r-1}^{(i)})$ for any i and $\text{Im}(\nu_j^{(i)}) \geq \text{Im}(\nu_{j+1}^{(i)})$ if $\text{Re}(\nu_j^{(i)}) = \text{Re}(\nu_{j+1}^{(i)})$. Applying Proposition 3.1 again, we have an isomorphism

$$\sigma' : \mathcal{M}_{\mathcal{C}_x}(\tilde{\mathbf{t}}_x, \boldsymbol{\mu}) \xrightarrow{\sim} \mathcal{M}_{\mathcal{C}_x}(\tilde{\mathbf{t}}_x, \boldsymbol{\nu}).$$

Then

$$(\sigma')^{-1} \circ \sigma : \mathcal{M}_{\mathcal{C}_x}(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda}) \xrightarrow{\sim} \mathcal{M}_{\mathcal{C}_x}(\tilde{\mathbf{t}}_x, \boldsymbol{\mu})$$

is the desired isomorphism. $\square$

4. Moduli of stable parabolic connections

Before proving Theorem 2.1, we recall the definition of parabolic Λ_D^1 -triple defined in [7]. Let D be an effective divisor on a curve C . We define Λ_D^1 as $\mathcal{O}_C \oplus \Omega_C^1(D)^\vee$ with the bimodule structure given by

$$\begin{aligned} f(a, v) &= (fa, fv) \quad (f, a \in \mathcal{O}_C, v \in \Omega_C^1(D)^\vee), \\ (a, v)f &= (fa + v(f), fv) \quad (f, a \in \mathcal{O}_C, v \in \Omega_C^1(D)^\vee). \end{aligned}$$

Definition 4.1. We say $(E_1, E_2, \Phi, F_*(E_1))$ is a parabolic Λ_D^1 -triple on C of rank r and degree d if

- (1) E_1 and E_2 are vector bundles on C of rank r and degree d ,
- (2) $\Phi : \Lambda_D^1 \otimes E_1 \rightarrow E_2$ is a left $\mathcal{O}_C$ -homomorphism, and
- (3) $E_1 = F_1(E_1) \supset F_2(E_1) \supset \cdots \supset F_l(E_1) \supset F_{l+1}(E_1) = E_1(-D)$ is a filtration by coherent subsheaves.

Note that to give a left $\mathcal{O}_C$ -homomorphism $\Phi : \Lambda_D^1 \otimes E_1 \rightarrow E_2$ is equivalent to giving an $\mathcal{O}_C$ -homomorphism $\phi : E_1 \rightarrow E_2$ and a morphism $\nabla : E_1 \rightarrow E_2 \otimes \Omega_C^1(D)$ such that $\nabla(fa) = \phi(a) \otimes df + f\nabla(a)$ for $f \in \mathcal{O}_C$ and $a \in E_1$. We also denote the parabolic Λ_D^1 -triple $(E_1, E_2, \Phi, F_*(E_1))$ by $(E_1, E_2, \phi, \nabla, F_*(E_1))$.

We take positive integers β_1, β_2, γ and rational numbers $0 < \alpha'_1 < \cdots < \alpha'_l < 1$. We assume $\gamma \gg 0$.

Definition 4.2. A parabolic Λ_D^1 -triple $(E_1, E_2, \Phi, F_*(E_1))$ is (α, β, γ) -stable (resp. (α, β, γ) -semistable) if for any subbundles $(F_1, F_2) \subset (E_1, E_2)$ satisfying $(0, 0) \neq (F_1, F_2) \neq (E_1, E_2)$ and $\Phi(\Lambda_{D(\mathbf{t})}^1 \otimes F_1) \subset F_2$, the inequality

$$\frac{\beta_1 \deg F_1(-D) + \beta_2(\deg F_2 - \gamma \operatorname{rank} F_2) + \beta_1 \sum_{j=1}^l \alpha'_j \operatorname{length}(\frac{F_j(E_1) \cap F_1}{F_{j+1}(E_1) \cap F_1})}{\beta_1 \operatorname{rank} F_1 + \beta_2 \operatorname{rank} F_2} < \frac{\beta_1 \deg E_1(-D) + \beta_2(\deg E_2 - \gamma \operatorname{rank} E_2) + \beta_1 \sum_{j=1}^l \alpha'_j \operatorname{length}(\frac{F_j(E_1)}{F_{j+1}(E_1)})}{\beta_1 \operatorname{rank} E_1 + \beta_2 \operatorname{rank} E_2}$$

(resp. $\leq$)

holds.

Theorem 4.1 (see [7], Theorem 5.1). *Let S be an algebraic scheme over $\mathbf{C}$, $\mathcal{C}$ be a flat family of smooth projective curves of genus g and D be an effective Cartier divisor on $\mathcal{C}$ flat over S . Then there exists a coarse moduli scheme $M_{\mathcal{C}/S}^{D, \alpha', \beta, \gamma}(r, d, \{d_i\}_{1 \leq i \leq nr})$ of (α', β, γ) -stable parabolic Λ_D^1 -triples $(E_1, E_2, \Phi, F_*(E_1))$ on $\mathcal{C}$ over S such that $r = \operatorname{rank} E_1 = \operatorname{rank} E_2$, $d = \deg E_1 = \deg E_2$ and $d_i = \operatorname{length}(E_1/F_{i+1}(E_1))$. If α' is generic, it is projective over S .*

Definition 4.3. We denote the pull-back of $\mathcal{C}$ and $\tilde{\mathbf{t}}$ by the morphism $T \times \Lambda_r^{(n)}(d) \rightarrow T$ by the same characters $\mathcal{C}$ and $\tilde{\mathbf{t}} = (\tilde{t}_1, \dots, \tilde{t}_n)$. Then $D(\tilde{\mathbf{t}}) := \tilde{t}_1 + \dots + \tilde{t}_n$ becomes an effective Cartier divisor on $\mathcal{C}$ flat over $T \times \Lambda_r^{(n)}(d)$. We also denote by $\tilde{\Lambda}$ the pull-back of the universal family on $\Lambda_r^{(n)}(d)$ by the morphism $T \times \Lambda_r^{(n)}(d) \rightarrow \Lambda_r^{(n)}(d)$. We define a functor $\mathcal{M}_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)$ of the category of locally Noetherian schemes to the category of sets by

$$\mathcal{M}_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)(S) := \left\{ (E, \nabla, \{l_j^{(i)}\}) \right\} / \sim,$$

for a locally Noetherian scheme S over $T \times \Lambda_r^{(n)}(d)$, where

- (1) E is a vector bundle on $\mathcal{C}_S$ of rank r ,
- (2) $\nabla : E \rightarrow E \otimes \Omega_{\mathcal{C}_S/S}^1(D(\tilde{\mathbf{t}})_S)$ is a relative connection,
- (3) $E|_{(\tilde{t}_i)_S} = l_0^{(i)} \supset l_1^{(i)} \supset \dots \supset l_{r-1}^{(i)} \supset l_r^{(i)} = 0$ is a filtration by subbundles such that $(\operatorname{res}_{(\tilde{t}_i)_S}(\nabla) - (\tilde{\lambda}_j^{(i)})_S)(l_j^{(i)}) \subset l_{j+1}^{(i)}$ for $0 \leq j \leq r-1$, $i = 1, \dots, n$, and
- (4) for any geometric point $s \in S$, $\dim(l_j^{(i)}/l_{j+1}^{(i)}) \otimes k(s) = 1$ for any i, j and $(E, \nabla, \{l_j^{(i)}\}) \otimes k(s)$ is α -stable.

Here $(E, \nabla, \{l_j^{(i)}\}) \sim (E', \nabla', \{l'_j{}^{(i)}\})$ if there exists a line bundle $\mathcal{L}$ on S and an isomorphism $\sigma : E \xrightarrow{\sim} E' \otimes \mathcal{L}$ such that $\sigma|_{l_j^{(i)}}(l_j^{(i)}) = l'_j{}^{(i)} \otimes \mathcal{L}$ for any i, j ,

and the diagram

$$\begin{array}{ccc} E & \xrightarrow{\nabla} & E \otimes \Omega_{\mathcal{C}/T}^1(D(\tilde{\mathbf{t}})) \\ \sigma \downarrow & & \sigma \otimes \text{id} \downarrow \\ E' \otimes \mathcal{L} & \xrightarrow{\nabla' \otimes \mathcal{L}} & E' \otimes \Omega_{\mathcal{C}/T}^1(D(\tilde{\mathbf{t}})) \otimes \mathcal{L} \end{array}$$

commutes.

Proof of Theorem 2.1. Fix a weight α which determines the stability of parabolic connections. We take positive integers β_1, β_2, γ and rational numbers $0 < \tilde{\alpha}_1^{(i)} < \tilde{\alpha}_2^{(i)} < \cdots < \tilde{\alpha}_r^{(i)} < 1$ satisfying $(\beta_1 + \beta_2)\alpha_j^{(i)} = \beta_1\tilde{\alpha}_j^{(i)}$ for any i, j . We assume $\gamma \gg 0$. We can take an increasing sequence $0 < \alpha'_1 < \cdots < \alpha'_{nr} < 1$ such that $\{\alpha'_i | 1 \leq i \leq nr\} = \{\tilde{\alpha}_j^{(i)} | 1 \leq i \leq n, 1 \leq j \leq r\}$. Take any member $(E, \nabla, \{l_j^{(i)}\}) \in \mathcal{M}_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)(S)$. For each $1 \leq p \leq rn$, there exists i, j satisfying $\tilde{\alpha}_j^{(i)} = \alpha'_p$. We put $F_1(E) := E$ and define inductively $F_{p+1}(E) := \ker(F_p(E) \rightarrow E|_{(\tilde{t}_i)_S}/l_j^{(i)})$ for $p = 1, \dots, rn$. We also put $d_p := \text{length}((E/F_{p+1}(E)) \otimes k(s))$ for $p = 1, \dots, rn$ and $s \in S$. Then $(E, \nabla, \{l_j^{(i)}\}) \mapsto (E, E, \text{id}_E, \nabla, F_*(E))$ determines a morphism

$$\iota : \mathcal{M}_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d) \rightarrow \overline{\mathcal{M}_{\mathcal{C}/T \times \Lambda_r^{(n)}(d)}^{D(\tilde{\mathbf{t}}), \alpha', \beta, \gamma}}(r, d, \{d_i\}_{1 \leq i \leq nr}),$$

where $\overline{\mathcal{M}_{\mathcal{C}/T}^{D(\tilde{\mathbf{t}}), \alpha', \beta, \gamma}}(r, d, \{d_i\}_{1 \leq i \leq nr})$ is the moduli functor of (α', β, γ) -stable parabolic $\Lambda_{D(\tilde{\mathbf{t}})}^1$ -triples whose coarse moduli scheme exists by Theorem 4.1. Note that a parabolic connection $(E, \nabla, \{l_j^{(i)}\})$ is α -stable if and only if the corresponding parabolic $\Lambda_{D(\tilde{\mathbf{t}})}^1$ -triple $(E, E, \text{id}, \nabla, F_*(E))$ is (α', β, γ) -stable since $\gamma \gg 0$. We can see that ι is representable by an immersion. So we can prove in the same way as [7, Theorem 2.1] that a certain subscheme $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$ of $\overline{M_{\mathcal{C}}^{D(\tilde{\mathbf{t}}), \alpha', \beta, \gamma}}(r, d, \{d_i\}_{1 \leq i \leq nr})$ is just the coarse moduli scheme of $\mathcal{M}_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$.

Applying a certain elementary transformation, we obtain an isomorphism

$$\sigma : \mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d) \xrightarrow{\sim} \mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d')$$

of functors, where d' and r are coprime. Then $\sigma(M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d))$ can be considered as the moduli scheme of parabolic connections of rank r and degree d' satisfying a certain stability condition. We can take a vector space V such that there exists a surjection $V \otimes \mathcal{O}_{\mathcal{C}_s}(-m) \rightarrow E$ such that $V \otimes k(s) \rightarrow H^0(E(m))$ is isomorphic and $h^i(E(m)) = 0$ for $i > 0$ for any member $(E, \nabla, \{l_j^{(i)}\}) \in \sigma(M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d))$ over $s \in T$. We put $P(n) := \chi(E(n))$. Let $\mathcal{E}$ be the universal family on $\mathcal{C} \times_{T \times \Lambda_r^{(n)}(d)} \text{Quot}_{V \otimes \mathcal{O}_{\mathcal{C}}(-m)/\mathcal{C}/T \times \Lambda_r^{(n)}(d)}^P$. Then we can construct

a scheme R over $\mathrm{Quot}_{V \otimes \mathcal{O}_{\mathcal{C}}(-m)/\mathcal{C}/T \times \Lambda_r^{(n)}(d)}^P$ which parametrizes connections $\nabla : \mathcal{E}_s \rightarrow \mathcal{E}_s \otimes \Omega_{\mathcal{C}_s}^1(D(\tilde{\mathbf{t}}_s))$ and parabolic structures $\mathcal{E}_s|_{(\tilde{t}_i)_s} = l_0^{(i)} \supset \cdots \supset l_{r-1}^{(i)} \supset l_r^{(i)} = 0$ such that $(\mathrm{res}_{(\tilde{t}_i)_s}(\nabla) - (\tilde{\lambda}_j^{(i)})_s)(l_j^{(i)}) \subset l_{j+1}^{(i)}$ for any i, j . Then $PGL(V)$ canonically acts on R and for some open subscheme R^s of R , there is a canonical morphism $R^s \rightarrow \sigma(M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d))$ which becomes a principal $PGL(V)$ -bundle. So we can prove in the same manner as [6, Theorem 4.6.5] that for a certain line bundle $\mathcal{L}$ on R^s , $\mathcal{E} \otimes \mathcal{L}$ descends to a vector bundle on $\mathcal{C} \times_{T \times \Lambda_r^{(n)}(d)} \sigma(M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d))$, since r and d' are coprime. We can easily see that the universal families $\tilde{\nabla}, \{\tilde{l}_j^{(i)}\}$ of connections and parabolic structures on $\mathcal{E} \otimes \mathcal{L}$ also descend. So we obtain a universal family for the moduli space $\sigma(M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d))$, and $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$ in fact becomes a fine moduli scheme.

Let $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ be a universal family on $\mathcal{C} \times_T M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$. We define a complex $\mathcal{F}^\bullet$ by

$$\begin{aligned} \mathcal{F}^0 &:= \left\{ s \in \mathrm{End}(\tilde{E}) \mid s|_{\tilde{t}_i \times M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)}(\tilde{l}_j^{(i)}) \subset \tilde{l}_j^{(i)} \text{ for any } i, j \right\}, \\ \mathcal{F}^1 &:= \left\{ s \in \mathrm{End}(\tilde{E}) \otimes \Omega_{\mathcal{C}/T}^1(D(\tilde{\mathbf{t}})) \mid \mathrm{res}_{\tilde{t}_i \times M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)}(s)(\tilde{l}_j^{(i)}) \subset \tilde{l}_{j+1}^{(i)} \text{ for any } i, j \right\}, \\ \nabla_{\mathcal{F}^\bullet} : \mathcal{F}^0 &\longrightarrow \mathcal{F}^1; \quad \nabla_{\mathcal{F}^\bullet}(s) = \tilde{\nabla} \circ s - s \circ \tilde{\nabla}. \end{aligned}$$

Let (A, m) be an Artinian local ring over $T \times \Lambda_r^{(n)}(d)$, I an ideal of A satisfying $mI = 0$. Take any member $(E, \nabla, \{l_j^{(i)}\}) \in M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)(A/I)$. We denote the image of $\mathrm{Spec} A/m \rightarrow M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$ by x . The structure morphism $\mathrm{Spec} A \rightarrow T \times \Lambda_r^{(n)}(d)$ corresponds to an A -valued point $(\mathbf{t}, \boldsymbol{\lambda}) \in T(A) \times (\Lambda_r^{(n)}(d))(A)$. Let $\mathcal{C} \times \mathrm{Spec} A = \bigcup_\alpha U_\alpha$ be an affine open covering such that $E|_{U_\alpha \otimes A/I} \cong \mathcal{O}_{U_\alpha \otimes A/I}^{\oplus r}$, $\#\{i | t_i \in U_\alpha\} \leq 1$ for any α and $\#\{\alpha | t_i \in U_\alpha\} \leq 1$ for any i . We can obviously take a vector bundle E_α on U_α such that there is an isomorphism $E_\alpha \otimes A/I \xrightarrow{\phi_\alpha} E|_{U_\alpha \otimes A/I}$. If $t_i \in U_\alpha$, we take a basis $e_1, e_2, \dots, e_r$ of E_α such that $l_{r-1}^{(i)} = \langle e_1 |_{t_i \otimes A/I} \rangle, l_{r-2}^{(i)} = \langle e_1 |_{t_i \otimes A/I}, e_2 |_{t_i \otimes A/I} \rangle, \dots, l_1^{(i)} = \langle e_1 |_{t_i \otimes A/I}, \dots, e_{r-1} |_{t_i \otimes A/I} \rangle$. The connection matrix of $\nabla|_{E|_{U_\alpha \otimes A/I}}$ with respect to the basis $e_1 \otimes A/I, \dots, e_r \otimes A/I$ can be given by a matrix $\Omega_\alpha \in M_r(\Omega_{\mathcal{C}/I}^1((t_i)_{A/I})|_{U_\alpha \otimes A/I})$ such that

$$\Omega_\alpha|_{t_i \otimes A/I} = \begin{pmatrix} \lambda_{r-1}^{(i)} \otimes A/I & \omega_{12} & \cdots & \omega_{1r} \\ 0 & \lambda_{r-2}^{(i)} \otimes A/I & \cdots & \omega_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_0^{(i)} \otimes A/I \end{pmatrix}.$$

We can take a lift $\tilde{\Omega}_\alpha \in M_r(\Omega_{\mathcal{C}_A/A}^1(t_i)|_{U_\alpha})$ of Ω_α such that

$$\tilde{\Omega}_\alpha|_{t_i} := \begin{pmatrix} \lambda_{r-1}^{(i)} & \tilde{\omega}_{12} & \cdots & \tilde{\omega}_{1r} \\ 0 & \lambda_{r-2}^{(i)} & \cdots & \tilde{\omega}_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_0^{(i)} \end{pmatrix},$$

which induces a connection $\nabla_\alpha : E_\alpha \rightarrow E_\alpha \otimes \Omega_{\mathcal{C}_A/A}^1((t_1 + t_2 + \cdots + t_n)_A)$. We define a parabolic structure $\{(l_\alpha)_j^{(i)}\}$ on E_α by $(l_\alpha)_{r-1}^{(i)} := \langle e_1|_{t_i} \rangle$, $(l_\alpha)_{r-2}^{(i)} := \langle e_1|_{t_i}, e_2|_{t_i} \rangle, \dots, (l_\alpha)_1^{(i)} := \langle e_1|_{t_i}, \dots, e_{r-1}|_{t_i} \rangle$. Then $(E_\alpha, \nabla_\alpha, \{(l_\alpha)_j^{(i)}\})$ become a lift $(E, \nabla, \{l_j^{(i)}\})|_{U_\alpha}$. If $t_i \notin U_\alpha$, we can easily take a lift $(E_\alpha, \nabla_\alpha, \{(l_\alpha)_j^{(i)}\})$ of $(E, \nabla, \{l_j^{(i)}\})|_{U_\alpha}$. We put $U_{\alpha\beta} := U_\alpha \cap U_\beta$ and $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$. Take a lift $\theta_{\beta\alpha} : E_\alpha|_{U_{\alpha\beta}} \xrightarrow{\sim} E_\beta|_{U_{\alpha\beta}}$ of the canonical isomorphism $E_\alpha|_{U_{\alpha\beta}} \otimes A/I \xrightarrow[\sim]{\phi_\alpha} E|_{U_{\alpha\beta} \otimes A/I} \xrightarrow[\sim]{\phi_\beta^{-1}} E_\beta|_{U_{\alpha\beta}} \otimes A/I$. We put

$$\begin{aligned} u_{\alpha\beta\gamma} &:= \phi_\alpha|_{U_{\alpha\beta\gamma}} \circ \left(\theta_{\gamma\alpha}^{-1}|_{U_{\alpha\beta\gamma}} \circ \theta_{\gamma\beta}|_{U_{\alpha\beta\gamma}} \circ \theta_{\beta\alpha}|_{U_{\alpha\beta\gamma}} - \text{id}_{E_\alpha|_{U_{\alpha\beta\gamma}}} \right) \circ \phi_\alpha^{-1}|_{U_{\alpha\beta\gamma}}, \\ v_{\alpha\beta} &:= \phi_\alpha \circ \left(\nabla_\alpha|_{U_{\alpha\beta}} - \theta_{\beta\alpha}^{-1} \circ \nabla_\beta \circ \theta_{\beta\alpha} \right) \circ \phi_\alpha^{-1}|_{U_{\alpha\beta\gamma}}. \end{aligned}$$

Then we have $\{u_{\alpha\beta\gamma}\} \in C^2(\{U_\alpha\}, \mathcal{F}^0 \otimes I)$ and $\{v_{\alpha\beta}\} \in C^1(\{U_\alpha\}, \mathcal{F}^1 \otimes I)$. We can easily see that

$$d\{v_{\alpha\beta}\} = -\nabla_{\mathcal{F}^\bullet}\{u_{\alpha\beta\gamma}\} \quad \text{and} \quad d\{u_{\alpha\beta\gamma}\} = 0.$$

So we can define an element

$$\omega(E, \nabla, \{l_j^{(i)}\}) := [(\{u_{\alpha\beta\gamma}\}, \{v_{\alpha\beta}\})] \in \check{\mathbf{H}}^2(\{U_\alpha\}, \mathcal{F}^\bullet \otimes I) = \mathbf{H}^2(\mathcal{F}^\bullet \otimes k(x)) \otimes I.$$

We can easily see that $(E, \nabla, \{l_j^{(i)}\})$ can be lifted to an A -valued point of $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$ if and only if $\omega(E, \nabla, \{l_j^{(i)}\}) = 0$ in the hypercohomology $\mathbf{H}^2(\mathcal{F}^\bullet \otimes k(x)) \otimes I$.

Consider the homomorphism

$$\begin{aligned} \text{Tr} : \mathbf{H}^2(\mathcal{F}^\bullet \otimes k(x)) \otimes I &\longrightarrow \mathbf{H}^2(\Omega_{\mathcal{C}_x}^\bullet) \otimes I \\ [\{u'_{\alpha\beta\gamma}\}, \{v'_{\alpha\beta}\}] &\mapsto [\{\text{Tr}(u'_{\alpha\beta\gamma})\}, \{\text{Tr}(v'_{\alpha\beta})\}]. \end{aligned}$$

Then we can see that $\text{Tr}(\omega(E, \nabla, \{l_j^{(i)}\}))$ is just the obstruction class for lifting $(\det E, \bigwedge^r \nabla)$ to a pair of a line bundle and a connection with the fixed residue $\det(\tilde{\lambda}_A)$ over A . Here $\det(\tilde{\lambda}_A) = (\sum_{j=0}^{r-1} (\tilde{\lambda}_j^{(i)})_A)^{1 \leq i \leq n}$. Since $\text{Pic}_{\mathcal{C}/T}^d$ is smooth over T , there is a line bundle L on $\mathcal{C}_A$ such that $L \otimes A/I \cong \det(E)$. We can construct a connection $\tilde{\nabla}^0 : L \rightarrow L \otimes \Omega_{\mathcal{C}_A/A}^1(\tilde{t}_1 + \cdots + \tilde{t}_n)$ because $H^1(\mathcal{E}nd(L) \otimes \Omega_{\mathcal{C}_A/A}^1((\tilde{t}_1 + \cdots + \tilde{t}_n)_A)) = 0$. We can find a member $u \in$

$H^0(\mathcal{E}nd(L) \otimes \Omega_{\mathcal{C}_A/A}^1((\tilde{t}_1 + \cdots + \tilde{t}_n)_A))$ such that $u \otimes A/I = \tilde{\nabla}^0 \otimes A/I - \bigwedge^r \nabla$. Then $(L, \tilde{\nabla}^0 - u)$ becomes a lift of $(\det(E), \bigwedge^r \nabla)$. If we put $\mu^{(i)} := \text{res}_{\tilde{t}_i \otimes A}(\tilde{\nabla}^0 - u) - \sum_j (\tilde{\lambda}_j^{(i)})_A$, then we have $\mu^{(i)} \in I$ for any i and $\sum_{i=1}^n \mu^{(i)} = 0$. Note that there is an exact sequence

$$\begin{aligned} H^0(I \otimes \mathcal{E}nd(L) \otimes \Omega_{\mathcal{C}_A/A}^1((\tilde{t}_1 + \cdots + \tilde{t}_n)_A)) \\ \longrightarrow \bigoplus_{i=1}^n I \otimes \mathcal{E}nd(L) \otimes \Omega_{\mathcal{C}_A/A}^1((\tilde{t}_1 + \cdots + \tilde{t}_n)_A)|_{(\tilde{t}_i)_A} \cong \bigoplus_{i=1}^n I \\ \xrightarrow{f} I \cong H^1(I \otimes \mathcal{E}nd(L) \otimes \Omega_{\mathcal{C}_A/A}^1) \longrightarrow 0, \end{aligned}$$

where f is given by $f((\nu_i)_{1 \leq i \leq n}) = \sum_{i=1}^n \nu_i$. Then we can find an element $\omega \in I \otimes H^0(\mathcal{E}nd(L) \otimes \Omega_{\mathcal{C}_A/A}^1((\tilde{t}_1 + \cdots + \tilde{t}_n)_A))$ such that $\text{res}_{\tilde{t}_i \otimes A}(\omega) = \mu^{(i)}$ for any i . Then $(L, \tilde{\nabla}^0 - u - \omega)$ is a lift of $(\det(E), \bigwedge^r \nabla)$ with the residue $(\det(\tilde{\lambda})_A)$. Thus we have $\text{Tr}(\omega(E, \nabla, \{l_j^{(i)}\})) = 0$. Since $(\mathcal{F}^1)^\vee \otimes \Omega_{\mathcal{C}_A/A}^1 \cong \mathcal{F}^0$ and $(\mathcal{F}^0)^\vee \otimes \Omega_{\mathcal{C}_A/A}^1 \cong \mathcal{F}^1$, we have

$$\begin{aligned} \mathbf{H}^2(\mathcal{F}^\bullet \otimes k(x)) \\ \cong \text{coker} \left(H^1(\mathcal{F}^0 \otimes k(x)) \rightarrow H^1(\mathcal{F}^1 \otimes k(x)) \right) \\ \cong \ker \left(H^0((\mathcal{F}^1)^\vee \otimes \Omega_{\mathcal{C}/T}^1 \otimes k(x)) \rightarrow H^0((\mathcal{F}^0)^\vee \otimes \Omega_{\mathcal{C}/T}^1 \otimes k(x)) \right)^\vee \\ \cong \ker \left(H^0(\mathcal{F}^0 \otimes k(x)) \rightarrow H^0(\mathcal{F}^1 \otimes k(x)) \right)^\vee \end{aligned}$$

and a commutative diagram

$$\begin{array}{ccc} \mathbf{H}^2(\mathcal{F}^\bullet \otimes k(x)) & \xrightarrow{\text{Tr}} & \mathbf{H}^2(\Omega_{\mathcal{C}_x}^\bullet) \\ \downarrow \cong & & \downarrow \cong \\ \ker \left(H^0(\mathcal{F}^0 \otimes k(x)) \xrightarrow{\nabla_{\mathcal{F}^\bullet}} H^0(\mathcal{F}^1 \otimes k(x)) \right)^\vee & \xrightarrow{\iota^\vee} & \ker \left(H^0(\mathcal{O}_{\mathcal{C}_x}) \xrightarrow{d} H^0(\Omega_{\mathcal{C}_x}^1) \right)^\vee. \end{array}$$

We can see by the stability that the endomorphisms of $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\}) \otimes k(x)$ are scalar multiplications. So we have $\ker(H^0(\mathcal{F}^0 \otimes k(x)) \rightarrow H^0(\mathcal{F}^1 \otimes k(x))) = k(x)$ and the canonical inclusion

$$\iota : \ker \left(H^0(\mathcal{O}_{\mathcal{C}_x}) \xrightarrow{d} H^0(\Omega_{\mathcal{C}_x}^1) \right) \longrightarrow \ker \left(H^0(\mathcal{F}^0 \otimes k(x)) \xrightarrow{\nabla_{\mathcal{F}^\bullet}} H^0(\mathcal{F}^1 \otimes k(x)) \right)$$

is an isomorphism. Hence Tr is an isomorphism and $\omega(E, \nabla, \{l_j^{(i)}\}) = 0$ because $\text{Tr}(\omega(E, \nabla, \{l_j^{(i)}\})) = 0$. So we have proved that $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$ is smooth over $T \times \Lambda_r^{(n)}(d)$.

Take an affine open set $M \subset M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$ and an affine open covering $\mathcal{C}_M = \bigcup_\alpha U_\alpha$ such that $\tilde{E}|_{U_\alpha} \cong \mathcal{O}_{U_\alpha}^{\oplus r}$ for any α , $\#\{i \mid t_i|_M \cap U_\alpha \neq \emptyset\} \leq 1$

for any α and $\#\{\alpha | t_i|_M \cap U_\alpha \neq \emptyset\} \leq 1$ for any i . Take a relative tangent vector field $v \in \Theta_{M_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)/T \times \Lambda_r^{(n)}(d)}(M)$. v corresponds to a member $(E_\epsilon, \nabla_\epsilon, \{(l_\epsilon)_j^{(i)}\}) \in M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)(\text{Spec } \mathcal{O}_M[\epsilon])$ such that $(E_\epsilon, \nabla_\epsilon, \{(l_\epsilon)_j^{(i)}\}) \otimes \mathcal{O}_M[\epsilon]/(\epsilon) \cong (\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})|_{\mathcal{C}_M}$, where $\mathcal{O}_M[\epsilon] = \mathcal{O}_M[t]/(t^2)$. There is an isomorphism

$$\varphi_\alpha : E_\epsilon|_{U_\alpha \times \text{Spec } \mathcal{O}_M[\epsilon]} \xrightarrow{\sim} \mathcal{O}_{U_\alpha \times \text{Spec } \mathcal{O}_M[\epsilon]}^{\oplus r} \xrightarrow{\sim} \tilde{E}|_{U_\alpha} \otimes \mathcal{O}_M[\epsilon]$$

such that $\varphi_\alpha \otimes \mathcal{O}_M[\epsilon]/(\epsilon) : E_\epsilon|_{U_\alpha} \otimes \mathcal{O}_M[\epsilon]/(\epsilon) \xrightarrow{\sim} \tilde{E}|_{U_\alpha} \otimes \mathcal{O}_M[\epsilon]/(\epsilon) = \tilde{E}|_{U_\alpha}$ is the given isomorphism and that $\varphi_\alpha|_{t_i \otimes \mathcal{O}_M[\epsilon]}((l_\epsilon)_j^{(i)}) = \tilde{l}_j^{(i)}|_{U_\alpha \times \text{Spec } \mathcal{O}_M[\epsilon]}$ if $t_i|_M \cap U_\alpha \neq \emptyset$. We put

$$u_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1} - \text{id}_{\tilde{E}|_{U_{\alpha\beta} \otimes \mathcal{O}_M[\epsilon]}},$$

$$v_\alpha := (\varphi_\alpha \otimes \text{id}) \circ \nabla_\epsilon|_{U_\alpha \otimes \mathcal{O}_M[\epsilon]} \circ \varphi_\alpha^{-1} - \tilde{\nabla}|_{U_\alpha \otimes \mathcal{O}_M[\epsilon]}.$$

Then $\{u_{\alpha\beta}\} \in C^1((\epsilon) \otimes \mathcal{F}_M^0)$, $\{v_\alpha\} \in C^0((\epsilon) \otimes \mathcal{F}_M^1)$ and

$$d\{u_{\alpha\beta}\} = \{u_{\beta\gamma} - u_{\alpha\gamma} + u_{\alpha\beta}\} = 0, \quad \nabla_{\mathcal{F}^\bullet}\{u_{\alpha\beta}\} = \{v_\beta - v_\alpha\} = d\{v_\alpha\}.$$

So $[(\{u_{\alpha\beta}\}, \{v_\alpha\})]$ determines an element $\sigma_M(v)$ of $\mathbf{H}^1(\mathcal{F}_M^\bullet)$. We can check that $v \mapsto \sigma(v)$ determines an isomorphism

$$\sigma_M : \Theta_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)/T \times \Lambda_r^{(n)}(d)}(M) \xrightarrow{\sim} \mathbf{H}^1(\mathcal{F}_M^\bullet).$$

These isomorphisms σ_M induce a canonical isomorphism

$$\sigma : \Theta_{M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)/T \times \Lambda_r^{(n)}(d)} \xrightarrow{\sim} \mathbf{R}^1(\pi_{T \times \Lambda_r^{(n)}(d)}^*(\mathcal{F}^\bullet)),$$

where $\pi_{T \times \Lambda_r^{(n)}(d)} : \mathcal{C} \rightarrow T \times \Lambda_r^{(n)}(d)$ is the projection. From the hypercohomology spectral sequence $H^q(\mathcal{F}^p \otimes k(x)) \Rightarrow \mathbf{H}^{p+q}(\mathcal{F}^\bullet \otimes k(x))$, we obtain an exact sequence

$$\begin{aligned} 0 \rightarrow k(x) \rightarrow H^0(\mathcal{F}^0(x)) \rightarrow H^0(\mathcal{F}^1(x)) \rightarrow \mathbf{H}^1(\mathcal{F}^\bullet(x)) \rightarrow H^1(\mathcal{F}^0(x)) \\ \rightarrow H^1(\mathcal{F}^1(x)) \rightarrow k(x) \rightarrow 0 \end{aligned}$$

for any $x \in M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$. Here we denote $\mathcal{F}^0 \otimes k(x)$ by $\mathcal{F}^0(x)$ and so on. So we have $\dim \mathbf{H}^1(\mathcal{F}^\bullet(x)) = -2\chi(\mathcal{F}^0(x)) + 2$, because $H^0(\mathcal{F}^1(x)) \cong H^1(\mathcal{F}^0(x))^\vee$ and $H^1(\mathcal{F}^1(x)) \cong H^0(\mathcal{F}^0(x))^\vee$. We define $\mathcal{E}_1$ by the exact sequence

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}nd(\tilde{E}(x)) \longrightarrow \bigoplus_{i=1}^n \text{Hom}\left(\tilde{l}_1^{(i)}(x), \left(\tilde{l}_0^{(i)}/\tilde{l}_1^{(i)}\right)(x)\right) \longrightarrow 0.$$

Inductively we define $\mathcal{E}_k$ by the exact sequence

$$0 \longrightarrow \mathcal{E}_k \longrightarrow \mathcal{E}_{k-1} \longrightarrow \bigoplus_{i=1}^n \text{Hom}\left(\tilde{l}_k^{(i)}(x), \left(\tilde{l}_{k-1}^{(i)}/\tilde{l}_k^{(i)}\right)(x)\right) \longrightarrow 0.$$

Then we have $\mathcal{E}_{r-1} = \mathcal{F}^0(x)$. So we have

$$\chi(\mathcal{F}^0(x)) = \chi(\mathcal{E}nd(\tilde{E}(x))) - nr(r-1)/2 = r^2(1-g) - nr(r-1)/2.$$

Thus we can see that every fiber of $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$ over $T \times \Lambda_r^{(n)}(d)$ is equidimensional and its dimension is $2r^2(g-1) + nr(r-1) + 2$ if it is non-empty. $\square$

Proposition 4.1. *Let $(E, \nabla_E, \{l_j^{(i)}\}) \in \mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)(S)$ be a flat family of parabolic connections over a Noetherian scheme S over $\mathbf{C}$. Then the subset*

$$U^s := \{s \in S \mid (E, \nabla_E, \{l_j^{(i)}\}) \otimes \overline{k(s)} \text{ is } \alpha\text{-stable}\}$$

is a Zariski open subset of S . Take a flat family $(E, \nabla_E, \{l_j^{(i)}\}) \in \mathcal{M}_C(\mathbf{t}, \boldsymbol{\lambda})(S)$ of $(\mathbf{t}, \boldsymbol{\lambda})$ -parabolic connections over a Noetherian scheme S over $\mathbf{C}$. Then

$$U^{irr} := \{s \in S \mid (E, \nabla_E, \{l_j^{(i)}\}) \otimes \overline{k(s)} \text{ is irreducible}\}$$

is a Zariski open subset of S . Note that a parabolic connection $(E, \nabla_E, \{l_j^{(i)}\})$ is reducible if there is a subbundle $0 \neq F \subsetneq E$ such that $\nabla_E(F) \subset F \otimes \Omega_C^1(t_1 + \dots + t_n)$. A parabolic connection $(E, \nabla_E, \{l_j^{(i)}\})$ is irreducible if it is not reducible.

Proof. Take $(E, \nabla_E, \{l_j^{(i)}\}) \in \mathcal{M}(\tilde{\mathbf{t}}, r, d)(S)$. As in the proof of Theorem 2.1, a member of $(E, E, \text{id}, \nabla, F_*(E)) \in \overline{\mathcal{M}_{\mathcal{C}/T \times \Lambda_r^{(n)}(d)}^{D(\tilde{\mathbf{t}})}(r, d, \{d_i\}_{1 \leq i \leq nr})}(S)$, where $\overline{\mathcal{M}_{\mathcal{C}/T \times \Lambda_r^{(n)}(d)}^{D(\tilde{\mathbf{t}})}(r, d, \{d_i\}_{1 \leq i \leq nr})}$ is the moduli functor of parabolic $\Lambda_{D(\tilde{\mathbf{t}})/T}^1$ -triples without stability condition. By [7, Proposition 5.3],

$$S^s = \{s \in S \mid (E, E, \text{id}, \nabla, F_*(E)) \otimes \overline{k(s)} \text{ is } (\alpha', \beta, \gamma)\text{-stable}\}$$

is a Zariski open subset of S . As in the proof of Theorem 2.1, $(E, \nabla, \{l_j^{(i)}\}) \otimes \overline{k(s)}$ is α -stable if and only if the corresponding parabolic $\Lambda_{D(\tilde{\mathbf{t}})_s}^1$ -triple is (α', β, γ) -stable. So we have $S^s = U^s$.

Next take a flat family $(E, \nabla_E, \{l_j^{(i)}\}) \in \mathcal{M}_C(\mathbf{t}, \boldsymbol{\lambda})(S)$ of $(\mathbf{t}, \boldsymbol{\lambda})$ -parabolic connections. Take any subbundle $F \subset E \otimes \overline{k(s)}$ such that $(\nabla_E \otimes \overline{k(s)})(F) \subset F \otimes \Omega_C^1(t_1 + \dots + t_n)$. We have a commutative diagram

$$\begin{array}{ccc} F|_{t_i} & \xrightarrow{\text{res}_{t_i}(\nabla_F)} & F|_{t_i} \\ \downarrow & & \downarrow \\ E|_{t_i} \otimes \overline{k(s)} & \xrightarrow{\text{res}_{t_i}(\nabla_E \otimes \overline{k(s)})} & E|_{t_i} \otimes \overline{k(s)}. \end{array}$$

Then we can see that the eigenvalues $\{\mu_j^{(i)}\}$ of $\text{res}_{t_i}(\nabla_F)$ are eigenvalues of $\text{res}_{t_i}(\nabla_E \otimes \overline{k(s)})$. So the set

$$\mathcal{P} := \left\{ (\text{rank}(F), \deg(F)) \left| \begin{array}{l} F \text{ is a non-zero proper subbundle} \\ \text{of } E \otimes \overline{k(s)} \text{ such that} \\ (\nabla_E \otimes \overline{k(s)})(F) \subset F \otimes \Omega_C^1(t_1 + \cdots + t_n) \end{array} \right. \right\}$$

is finite, because $\deg(F) = -\sum_{i,j} \mu_j^{(i)}$. For $(r', d') \in \mathcal{P}$, put $P_{(r', d')}(m) = r'(m - g + 1) + d'$ which is a polynomial in m . Then consider the Quot-scheme $Q_{(r', d')} := \text{Quot}_{E/C/\mathbf{C}}^{\chi(E(m) \otimes k(s)) - P_{(r', d')}(m)}$ and let $\mathcal{F}^{(r', d')} \subset E_{Q_{(r', d')}}$ be the universal subsheaf. Let $Y_{(r', d')}$ be the maximal closed subscheme of $Q_{(r', d')}$ such that the composite

$$\begin{aligned} \mathcal{F}_{Y_{(r', d')}}^{(r', d')} &\hookrightarrow E_{Y_{(r', d')}} \xrightarrow{(\nabla_E)_{Y_{(r', d')}}} E_{Y_{(r', d')}} \\ &\otimes \Omega_C^1(t_1 + \cdots + t_n) \longrightarrow (E_{Y_{(r', d')}} / \mathcal{F}_{Y_{(r', d')}}^{(r', d')}) \otimes \Omega_C^1(t_1 + \cdots + t_n) \end{aligned}$$

is zero. Since the structure morphism $f_{(r', d')} : Y_{(r', d')} \rightarrow S$ is proper,

$$U^{irr} = S \setminus \bigcup_{(r', d') \in \mathcal{P}} f_{(r', d')}(Y_{(r', d')})$$

is a Zariski open subset of S . □

Proposition 4.2. *The set*

$$RP_r(C, \mathbf{t})^{irr} = \{[\rho] \in RP_r(C, \mathbf{t}) \mid \rho \text{ is an irreducible representation}\}$$

is a Zariski open subset of $RP_r(C, \mathbf{t})$.

Proof. Note that for the quotient morphism

$$p : \text{Hom}(\pi_1(C \setminus \{t_1, \dots, t_n\}, *), GL_r(\mathbf{C})) \longrightarrow RP_r(C, \mathbf{t}),$$

$Z \subset RP_r(C, \mathbf{t})$ is a Zariski closed subset if and only if $p^{-1}(Z)$ is a Zariski closed subset of $\text{Hom}(\pi_1(C \setminus \{t_1, \dots, t_n\}, *), GL_r(\mathbf{C}))$. There is a universal representation $\tilde{\rho}$ given by

$$\begin{aligned} \tilde{\rho}(\alpha_i) &: \mathcal{O}_{\text{Hom}(\pi_1(C \setminus \{t_1, \dots, t_n\}, *), GL_r(\mathbf{C}))}^{\oplus r} \xrightarrow{\sim} \mathcal{O}_{\text{Hom}(\pi_1(C \setminus \{t_1, \dots, t_n\}, *), GL_r(\mathbf{C}))}^{\oplus r} \\ \tilde{\rho}(\beta_i) &: \mathcal{O}_{\text{Hom}(\pi_1(C \setminus \{t_1, \dots, t_n\}, *), GL_r(\mathbf{C}))}^{\oplus r} \xrightarrow{\sim} \mathcal{O}_{\text{Hom}(\pi_1(C \setminus \{t_1, \dots, t_n\}, *), GL_r(\mathbf{C}))}^{\oplus r} \\ \tilde{\rho}(\gamma_j) &: \mathcal{O}_{\text{Hom}(\pi_1(C \setminus \{t_1, \dots, t_n\}, *), GL_r(\mathbf{C}))}^{\oplus r} \xrightarrow{\sim} \mathcal{O}_{\text{Hom}(\pi_1(C \setminus \{t_1, \dots, t_n\}, *), GL_r(\mathbf{C}))}^{\oplus r} \end{aligned}$$

for $i = 1, \dots, g$ and $i = 1, \dots, n$. For $0 < r' < r$, consider the Grassmann variety

$$\text{Gr}' := \text{Grass}_{r'}(\mathcal{O}_{\text{Hom}(\pi_1(C \setminus \{t_1, \dots, t_n\}, *), GL_r(\mathbf{C}))}^{\oplus r})$$

and let $V \subset \mathcal{O}_{\mathrm{Gr}'}^{\oplus r}$ be the universal subbundle. Let $Y_{r'}$ be the maximal closed subscheme of Gr' such that

$$\begin{aligned}\tilde{\rho}(\alpha_i)_{Y_{r'}}(V_{Y_{r'}}) &\subset V_{Y_{r'}} \\ \tilde{\rho}(\beta_i)_{Y_{r'}}(V_{Y_{r'}}) &\subset V_{Y_{r'}} \\ \tilde{\rho}(\gamma_j)_{Y_{r'}}(V_{Y_{r'}}) &\subset V_{Y_{r'}}\end{aligned}$$

for $i = 1, \dots, g$ and $j = 1, \dots, n$. Since the structure morphism

$$f_{r'} : Y_{r'} \rightarrow \mathrm{Hom}(\pi_1(C \setminus \{t_1, \dots, t_n\}, *), GL_r(\mathbf{C}))$$

is a proper morphism, $\bigcup_{r'} f_{r'}(Y_{r'})$ is a Zariski closed subset of $\mathrm{Hom}(\pi_1(C \setminus \{t_1, \dots, t_n\}, *), GL_r(\mathbf{C}))$ which is preserved by the adjoint action of $PGL_r(\mathbf{C})$ on $\mathrm{Hom}(\pi_1(C \setminus \{t_1, \dots, t_n\}, *), GL_r(\mathbf{C}))$. So $p(\bigcup_{r'} f_{r'}(Y_{r'}))$ is a Zariski closed subset of $RP_r(C, \mathbf{t})$. Thus

$$\begin{aligned}\{[\rho] \in RP_r(C, \mathbf{t}) \mid \rho \text{ is an irreducible representation}\} \\ = RP_r(C, \mathbf{t}) \setminus p\left(\bigcup_{r'} f_{r'}(Y_{r'})\right)\end{aligned}$$

is a Zariski open subset of $RP_r(C, \mathbf{t})$. □

Remark 4.1. Assume that $n \geq 3$ if $g = 0$ and $n \geq 1$ if $g \geq 1$. Then $RP_r(C, \mathbf{t})^{irr}$ is non-empty.

Proof. Note that the fundamental group $\pi_1(C \setminus \{t_1, \dots, t_n\})$ is a free group generated by $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \gamma_1, \dots, \gamma_{n-1}$. If $g \geq 1$, consider the $GL_r(\mathbf{C})$ -representation ρ of $\pi_1(C \setminus \{t_1, \dots, t_n\})$ given by

$$\rho(\alpha_1) = \begin{pmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}, \quad \rho(\beta_1) = \begin{pmatrix} \mu & 0 & 0 & 0 & \cdots & 0 \\ 1 & \mu & 0 & 0 & \cdots & 0 \\ 0 & 1 & \mu & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & \mu \end{pmatrix},$$

$\rho(\alpha_2), \rho(\beta_2), \dots, \rho(\alpha_g), \rho(\beta_g), \rho(\gamma_1), \dots, \rho(\gamma_{n-1})$ are arbitrary matrix.

Then we can see that ρ is an irreducible representation. If $g = 0$, then consider the $GL_r(\mathbf{C})$ -representation ρ given by

$$\rho(\gamma_1) = \begin{pmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}, \quad \rho(\gamma_2) = \begin{pmatrix} \mu & 0 & 0 & 0 & \cdots & 0 \\ 1 & \mu & 0 & 0 & \cdots & 0 \\ 0 & 1 & \mu & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & \mu \end{pmatrix},$$

$\rho(\gamma_3), \dots, \rho(\gamma_{n-1})$ are arbitrary matrix.

Then we can see that ρ is an irreducible representation. □

5. Irreducibility of the moduli space

In this section we will prove the irreducibility of $M_{\mathcal{C}_x}^\alpha(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda})$, which is the fiber of $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$ over $(x, \boldsymbol{\lambda}) \in T \times \Lambda_r^{(n)}(d)$. We simply denote $\mathcal{C}_x$ by C , $\tilde{\mathbf{t}}_x$ by $\mathbf{t} = (t_1, \dots, t_n)$ and $t_1 + \dots + t_n$ by $D(\mathbf{t})$. First consider the morphism

$$(10) \quad \det : M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}) \longrightarrow M_C(\mathbf{t}, \det(\boldsymbol{\lambda}))$$

$$(E, \nabla, \{l_j^{(i)}\}) \mapsto (\det E, \det(\nabla)),$$

where $\det(\nabla)$ is the logarithmic connection on $\det E$ induced by ∇ and $\det(\boldsymbol{\lambda}) = (\sum_{j=0}^{r-1} \lambda_j^{(i)})_{1 \leq i \leq n} \in \Lambda_1^{(n)}(d)$. Note that a line bundle with a connection is always stable. We can also construct $M_C(\mathbf{t}, \det(\boldsymbol{\lambda}))$ as an affine space bundle over Pic_C^d whose fiber is of dimension

$$h^0(\mathcal{E}nd(\det E) \otimes \Omega_C^1) = h^0(\Omega_C^1) = g.$$

Thus $M_C(\mathbf{t}, \det(\boldsymbol{\lambda}))$ is a smooth irreducible variety of dimension $2g$.

We can prove the smoothness of the morphism (10). Indeed let A be an Artinian local ring over $M_C(\mathbf{t}, \det(\boldsymbol{\lambda}))$ with maximal ideal m , I an ideal of A satisfying $mI = 0$ and take any A/I -valued point $(E, \nabla, \{l_j^{(i)}\})$ of $M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda})$ over $M_C(\mathbf{t}, \det(\boldsymbol{\lambda}))$. Let (L, ∇_L) be the line bundle with a connection on $(C, \mathbf{t})$ which corresponds to the morphism $\text{Spec } A \rightarrow M_C(\mathbf{t}, \det(\boldsymbol{\lambda}))$. We write $(\bar{E}, \bar{\nabla}, \{\bar{l}_j^{(i)}\}) := (E, \nabla, \{l_j^{(i)}\}) \otimes A/m$ and put

$$\mathcal{F}_0^0 := \left\{ s \in \mathcal{E}nd(\bar{E}) \mid \text{Tr}(s) = 0 \text{ and } s(t_i)(\bar{l}_j^{(i)}) \subset \bar{l}_j^{(i)} \text{ for any } i, j \right\},$$

$$\mathcal{F}_0^1 := \left\{ s \in \mathcal{E}nd(\bar{E}) \otimes \Omega_X^1(D(\mathbf{t})) \mid \text{Tr}(s) = 0 \text{ and } s(t_i)(\bar{l}_j^{(i)}) \subset \bar{l}_{j+1}^{(i)} \text{ for any } i, j \right\}.$$

We define a complex $\mathcal{F}_0^\bullet$ by

$$\nabla_{\mathcal{F}_0^\bullet} : \mathcal{F}_0^0 \ni s \mapsto \bar{\nabla} \circ s - s \circ \bar{\nabla} \in \mathcal{F}_0^1.$$

Then the obstruction class for the lifting of $(E, \nabla, \{l_j^{(i)}\})$ to an A -valued point $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ of $M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda})$ with $(\det(E), \det(\nabla)) \cong (L, \nabla_L)$ lies in $\mathbf{H}^2(\mathcal{F}_0^\bullet) \otimes I$. Using the same method as in the proof of Theorem 2.1, we can see that

$$\mathbf{H}^2(\mathcal{F}_0^\bullet) \cong (\ker(H^0(\mathcal{F}_0^0) \rightarrow H^0(\mathcal{F}_0^1)))^\vee$$

and we can see by the stability of $(\bar{E}, \bar{\nabla}, \{\bar{l}_j^{(i)}\})$ that $\ker(H^0(\mathcal{F}_0^0) \rightarrow H^0(\mathcal{F}_0^1)) = 0$. Hence the morphism $\det$ in (10) is smooth.

From the above argument, it is sufficient to prove the irreducibility of the fibers of the morphism (10) $\det : M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}) \rightarrow M_C(\mathbf{t}, \det(\boldsymbol{\lambda}))$ in order to obtain the irreducibility of $M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda})$. So we fix a line bundle L on C and a connection

$$\nabla_L : L \longrightarrow L \otimes \Omega_C^1(t_1 + \cdots + t_n).$$

We set

$$M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L) := \left\{ (E, \nabla, \{l_j^{(i)}\}) \in M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}) \mid (\det E, \det(\nabla)) \cong (L, \nabla_L) \right\}.$$

Then $M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L)$ is smooth of equidimension $(r-1)(2(r+1)(g-1) + rn)$.

Proposition 5.1. *Assume that $g \geq 2$ and $n \geq 1$. Then $M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L)$ is an irreducible variety of dimension $(r-1)(2(r+1)(g-1) + rn)$.*

Proof. By taking the elementary transform, we can obtain an isomorphism

$$\sigma : \mathcal{M}_C(\mathbf{t}, \boldsymbol{\lambda}, L) \xrightarrow{\sim} \mathcal{M}_C(\mathbf{t}, \boldsymbol{\lambda}', L'),$$

with r and $\det L'$ coprime. Consider the open subscheme

$$N := \left\{ (E, \nabla, \{l_j^{(i)}\}) \in M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L) \mid \sigma(E) \text{ is a stable vector bundle} \right\}$$

of $M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L)$.

Step 1. We will see that N is irreducible.

Let $M_C(r, L')$ be the moduli space of stable vector bundles of rank r with the determinant L' . As is well known, $M_C(r, L')$ is a smooth irreducible variety and there is a universal bundle $\mathcal{E}$ on $C \times M_C(r, L')$. Note that there is a line bundle $\mathcal{L}$ on $M_C(r, L')$ such that $\det(\mathcal{E}) \cong L \otimes \mathcal{L}$. We can parameterize the parabolic structures on $\mathcal{E}_s$ ($s \in M_C(r, L')$) by a product of flag schemes

$$U := \prod_{i=1}^n \text{Flag}(\mathcal{E}|_{t_i \times M_C(r, L')}),$$

which is obviously smooth and irreducible. Let $\{\tilde{l}_j^{(i)}\}$ be the universal family over U , where

$$\mathcal{E}_U|_{t_i \times U} = \tilde{l}_0^{(i)} \supset \tilde{l}_1^{(i)} \supset \cdots \supset \tilde{l}_{r-1}^{(i)} \supset \tilde{l}_r^{(i)} = 0$$

is the filtration by subbundles for $i = 1, \dots, n$ such that $\dim(\tilde{l}_j^{(i)})_s/(\tilde{l}_{j+1}^{(i)})_s = 1$ for any i, j and $s \in U$. Consider the functor $\text{Conn}_U : (\text{Sch}/U) \rightarrow (\text{Sets})$ defined by

$$\text{Conn}_U(S) = \left\{ \nabla : \mathcal{E}_S \rightarrow \mathcal{E}_S \otimes \Omega_C^1(D(\mathbf{t})) \left| \begin{array}{l} \nabla \text{ is a relative connection} \\ \text{satisfying } (\text{res}_{t_i \times S}(\nabla) - \lambda_j^{(i)})(\tilde{l}_j^{(i)})_S \\ \subset (\tilde{l}_{j+1}^{(i)})_S \text{ for any } i, j \text{ and } \det(\nabla) \\ = (\nabla_L)_S \otimes \mathcal{L}_S \end{array} \right. \right\}.$$

We can define a morphism of functors

$$\begin{aligned} \text{Conn}_U &\longrightarrow \text{Quot}_{\Lambda_{D(\mathbf{t})}^1 \otimes \mathcal{E}/C \times U/U} \\ \nabla &\mapsto \left[\Lambda_{D(\mathbf{t})}^1 \otimes \mathcal{E}_U \ni (a, v) \otimes e \mapsto ae + \nabla_v(e) \in \mathcal{E}_U \right], \end{aligned}$$

which is representable by an immersion. So there is a locally closed subscheme Y of $\text{Quot}_{\Lambda_{D(\mathbf{t})}^1 \otimes \mathcal{E}/C \times U/U}$ which represents the functor Conn_U . We will show that Y is smooth over U . So let $\tilde{\nabla} : \mathcal{E}_Y \rightarrow \mathcal{E}_Y \otimes \Omega_C^1(D(\mathbf{t}))$ be the universal connection and put

$$\begin{aligned} \mathcal{F}_0^0 &:= \left\{ s \in \text{End}(\mathcal{E}) \left| \text{Tr}(s) = 0 \text{ and } s|_{t_i \times Y}((\tilde{l}_j^{(i)})_Y) \subset (\tilde{l}_j^{(i)})_Y \text{ for any } i, j \right. \right\}, \\ \mathcal{F}_0^1 &:= \left\{ s \in \text{End}(\mathcal{E}) \otimes \Omega_C^1(D(\mathbf{t})) \left| \text{Tr}(s) = 0 \text{ and } \text{res}_{t_i \times Y}(s)((\tilde{l}_j^{(i)})_Y) \right. \right. \\ &\quad \left. \left. \subset (\tilde{l}_{j+1}^{(i)})_Y \text{ for any } i, j \right. \right\}, \end{aligned}$$

$$\nabla_{\mathcal{F}_0^\bullet} : \mathcal{F}_0^0 \ni s \mapsto \tilde{\nabla} \circ s - s \circ \tilde{\nabla} \in \mathcal{F}_0^1.$$

Let A be an Artinian local ring over U with maximal ideal m and I an ideal of A satisfying $mI = 0$. Take any A/I -valued point x of Y over U . We put $\bar{x} := x \otimes A/m$. Then the obstruction class for the lifting of x to an A -valued point of Y lies in $H^1(\mathcal{F}_0^1 \otimes k(\bar{x})) \otimes I$. We can see that $H^1(\mathcal{F}_0^1 \otimes k(\bar{x})) \cong H^0(\mathcal{F}_0^0 \otimes k(\bar{x}))^\vee = 0$ because $\mathcal{E} \otimes k(\bar{x})$ is a stable bundle. Thus Y is smooth over U . We can see that the fiber Y_s of any point $s \in U$ is isomorphic to the affine space isomorphic to $H^0(\mathcal{F}_0^1 \otimes k(s))$. So Y is irreducible. Consider the open subscheme

$$Y' := \left\{ x \in Y \left| \sigma^{-1}(\mathcal{E}_x, \tilde{\nabla} \otimes k(x), \{(\tilde{l}_j^{(i)})_x\}) \text{ is } \alpha\text{-stable} \right. \right\}$$

of Y . Then we obtain a morphism $Y' \rightarrow M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L)$ whose image is just N . Thus N is irreducible.

Step 2.

$$\dim(M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L) \setminus N) < \dim M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L) = (r-1)(2(r+1)(g-1) + rn).$$

For the proof of Step 2, it is sufficient to show that there is an algebraic scheme $\tilde{B}$ with $\dim \tilde{B} < (r-1)(2(r+1)(g-1) + rn)$ and a surjection $\tilde{B} \rightarrow \sigma(M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L) \setminus N)$. Take any member $(E, \nabla_E, \{\iota_j^{(i)}\}) \in \sigma(M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L) \setminus N)$. Let

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_s = E$$

be the Harder-Narasimhan filtration of E . Note that $s > 1$ because r and $\deg E = \deg L'$ are coprime. Put $\bar{E}_k := E_k/E_{k-1}$ for $k = 1, \dots, s$. For each $\bar{E}_k$, there is a Jordan-Hölder filtration

$$0 = E_k^{(0)} \subset E_k^{(1)} \subset E_k^{(2)} \subset \cdots \subset E_k^{(m_k)} = \bar{E}_k.$$

We put $\bar{E}_k^{(i)} := E_k^{(i)}/E_k^{(i-1)}$ for $i = 1, \dots, m_k$. Put $r_k^{(j)} = \text{rank } \bar{E}_k^{(j)}$ and $d_k^{(j)} = \deg \bar{E}_k^{(j)}$. We can parameterize each $\bar{E}_k^{(j)}$ by the moduli space $M(r_k^{(j)}, d_k^{(j)})$ of stable bundles of rank $r_k^{(j)}$ and degree $d_k^{(j)}$ whose dimension is $\dim \text{Ext}^1(\bar{E}_k^{(j)}, \bar{E}_k^{(j)})$. Replacing $M(r_k^{(j)}, d_k^{(j)})$ by an étale covering, we may assume that there is a universal bundle $\bar{\mathcal{E}}_k^{(j)}$ on $C \times M(r_k^{(j)}, d_k^{(j)})$. If we put

$$X := \left\{ (x_k^{(j)}) \in \prod_{1 \leq k \leq s, 1 \leq j \leq m_k} M(r_k^{(j)}, d_k^{(j)}) \left| \bigotimes_{j,k} \det(\bar{\mathcal{E}}_k^{(j)})_{x_k^{(j)}} \cong L \right. \right\},$$

then X is smooth of dimension $-g + \sum_{k=1}^s \sum_{j=1}^{m_k} \dim \text{Ext}^1(\bar{E}_k^{(j)}, \bar{E}_k^{(j)})$. Replacing X by its stratification, we may assume that the relative Ext-sheaf $\text{Ext}_{C \times X/X}^1(\bar{\mathcal{E}}_k^{(2)}, \bar{\mathcal{E}}_k^{(1)})$ becomes locally free. We put $\tilde{W}_k^{(2)} := X$ and $\mathcal{E}_k^{(2)} := \bar{\mathcal{E}}_k^{(1)} \oplus \bar{\mathcal{E}}_k^{(2)}$ if the extension

$$0 \longrightarrow \bar{E}_k^{(1)} \longrightarrow E_k^{(2)} \longrightarrow \bar{E}_k^{(2)} \longrightarrow 0$$

splits. Otherwise we put

$$\tilde{W}_k^{(2)} := \mathbf{P}_* \left(\text{Ext}_{C \times X/X}^1(\bar{\mathcal{E}}_k^{(2)}, \bar{\mathcal{E}}_k^{(1)}) \right)$$

and take a universal extension

$$0 \longrightarrow (\bar{\mathcal{E}}_k^{(1)})_{\tilde{W}_k^{(2)}} \otimes \mathcal{L}_k^{(2)} \longrightarrow \mathcal{E}_k^{(2)} \longrightarrow (\bar{\mathcal{E}}_k^{(2)})_{\tilde{W}_k^{(2)}} \longrightarrow 0$$

for some line bundle $\mathcal{L}_k^{(2)}$ on $\tilde{W}_k^{(2)}$. We define $\tilde{W}_k^{(j)}$ and $\mathcal{E}_k^{(j)}$ inductively on j as follows: Replacing $\tilde{W}_k^{(j-1)}$ by its stratification, we may assume that the relative Ext-sheaf $\text{Ext}_{C \times \tilde{W}_k^{(j-1)}/\tilde{W}_k^{(j-1)}}^1((\bar{\mathcal{E}}_k^{(j)})_{\tilde{W}_k^{(j-1)}}, \mathcal{E}_k^{(j-1)})$ is locally free. Then put $\tilde{W}_k^{(j)} := \tilde{W}_k^{(j-1)}$ and $\mathcal{E}_k^{(j)} := \mathcal{E}_k^{(j-1)} \oplus (\bar{\mathcal{E}}_k^{(j)})_{\tilde{W}_k^{(j)}}$ if the extension

$$0 \longrightarrow E_k^{(j-1)} \longrightarrow E_k^{(j)} \longrightarrow \bar{E}_k^{(j)} \longrightarrow 0$$

splits; otherwise, we put

$$\tilde{W}_k^{(j)} := \mathbf{P}_* \left(\text{Ext}_{C \times \tilde{W}_k^{(j-1)}/\tilde{W}_k^{(j-1)}}^1((\bar{\mathcal{E}}_k^{(j)})_{\tilde{W}_k^{(j-1)}}, \mathcal{E}_k^{(j-1)}) \right)$$

and take a universal extension

$$0 \longrightarrow (\mathcal{E}_k^{(j-1)})_{\tilde{W}_k^{(j)}} \otimes \mathcal{L}_k^{(j)} \longrightarrow \mathcal{E}_k^{(j)} \longrightarrow (\bar{\mathcal{E}}_k^{(j)})_{\tilde{W}_k^{(j)}} \longrightarrow 0$$

for some line bundle $\mathcal{L}_k^{(j)}$ on $\tilde{W}_k^{(j)}$. We can see by the construction that the relative dimension of $\tilde{W}_k^{(m_k)}$ over X at the point corresponding to the extensions

$$0 \longrightarrow E_k^{(j-1)} \longrightarrow E_k^{(j)} \longrightarrow \bar{E}_k^{(j)} \longrightarrow 0 \quad (j = 2, \dots, m_k, E_k^{(1)} = \bar{E}_k^{(1)})$$

is at most $1 - m_k + \sum_{1 \leq i < j \leq m_k} \dim \text{Ext}^1(\bar{E}_k^{(j)}, \bar{E}_k^{(i)})$. We put

$$W := \tilde{W}_1^{(m_1)} \times_X \cdots \times_X \tilde{W}_s^{(m_s)}$$

and $\bar{\mathcal{E}}_k := (\mathcal{E}_k^{(m_k)})_W$ for $k = 1, \dots, s$. Replacing W by its stratification, we assume that the relative Ext-sheaf $\text{Ext}_{C \times W/W}^1(\bar{\mathcal{E}}_2, \bar{\mathcal{E}}_1)$ is locally free. Then we put $W_2 := W$ and $\mathcal{E}_2 := \bar{\mathcal{E}}_1 \oplus \bar{\mathcal{E}}_2$ if the extension

$$0 \longrightarrow \bar{E}_1 \longrightarrow E_2 \longrightarrow \bar{E}_2 \longrightarrow 0$$

splits and otherwise we put

$$W_2 := \mathbf{P}_* \left(\text{Ext}_{C \times W/W}^1(\bar{\mathcal{E}}_2, \bar{\mathcal{E}}_1) \right)$$

and take a universal extension

$$0 \longrightarrow (\bar{\mathcal{E}}_1)_{W_2} \otimes \mathcal{L}_2 \longrightarrow \mathcal{E}_2 \longrightarrow (\bar{\mathcal{E}}_2)_{W_2} \longrightarrow 0$$

for some line bundle $\mathcal{L}_2$. We define W_k and $\mathcal{E}_k$ inductively as follows: Replacing W_{k-1} by its stratification, we assume that the relative Ext-sheaf $\text{Ext}_{C \times W_{k-1}/W_{k-1}}^1((\bar{\mathcal{E}}_k)_{W_{k-1}}, \mathcal{E}_{k-1})$ is locally free. Then we put $W_k := W_{k-1}$ and $\mathcal{E}_k := \bar{\mathcal{E}}_k \oplus \mathcal{E}_{k-1}$ if the extension

$$0 \longrightarrow E_{k-1} \longrightarrow E_k \longrightarrow \bar{E}_k \longrightarrow 0$$

splits and otherwise we put

$$W_k := \mathbf{P}_* \left(\text{Ext}_{C \times W_{k-1}/W_{k-1}}^1((\bar{\mathcal{E}}_k)_{W_{k-1}}, \mathcal{E}_{k-1}) \right)$$

and take a universal extension

$$0 \longrightarrow (\mathcal{E}_{k-1})_{W_k} \otimes \mathcal{L}_k \longrightarrow \mathcal{E}_k \longrightarrow (\bar{\mathcal{E}}_k)_{W_k} \longrightarrow 0$$

for some line bundle $\mathcal{L}_k$ on W_k . We put $r_k := \text{rank } \bar{E}_k$ for $k = 1, \dots, s$.

Claim. $\dim \text{Ext}^1(\bar{E}_k, \bar{E}_{k'}) < r_k r_{k'}(2g - 2)$ for $k' < k$.

Take a general point $t \in C$. We prove this claim in the following three cases.

Case 1. $\deg(\bar{E}_{k'}^\vee \otimes \bar{E}_k(t)) > 0$.

By the Riemann-Roch theorem, we have

$$\begin{aligned} \dim \operatorname{Hom}(\bar{E}_k(t), \bar{E}_{k'}) - \dim H^0((\bar{E}_{k'})^\vee \otimes \bar{E}_k(t) \otimes \omega_C) \\ = r_{k'} r_k (1 - g) + \deg((\bar{E}_k(t))^\vee \otimes \bar{E}_{k'}). \end{aligned}$$

Here we have $\operatorname{Hom}(\bar{E}_k(t), \bar{E}_{k'}) = 0$ because $\bar{E}_k(t)$ and $\bar{E}_{k'}$ are semistable and $\deg((\bar{E}_k(t))^\vee \otimes \bar{E}_{k'}) < 0$. So we have

$$\begin{aligned} \dim \operatorname{Ext}^1(\bar{E}_k, \bar{E}_{k'}) &= \dim H^0(\bar{E}_{k'}^\vee \otimes \bar{E}_k \otimes \omega_C) \\ &\leq \dim H^0(\bar{E}_{k'}^\vee \otimes \bar{E}_k(t) \otimes \omega_C) \\ &= r_k r_{k'} (g - 1) - \deg((\bar{E}_k(t))^\vee \otimes \bar{E}_{k'}) \\ &< r_k r_{k'} (g - 1) + r_k r_{k'} \\ &\leq r_k r_{k'} (2g - 2). \end{aligned}$$

Case 2. $\deg(\bar{E}_{k'}^\vee \otimes \bar{E}_k(t)) < 0$.

Note that there is an exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(\bar{E}_{k'}^\vee \otimes \bar{E}_k \otimes \omega_C(-p_1 - \cdots - p_{2g-3})) \\ \longrightarrow H^0(\bar{E}_{k'}^\vee \otimes \bar{E}_k \otimes \omega_C(-p_1 - \cdots - p_{2g-4})) \\ \longrightarrow H^0(\bar{E}_{k'}^\vee \otimes \bar{E}_k \otimes \omega_C(-p_1 - \cdots - p_{2g-4})|_{p_{2g-3}}) \cong \mathbf{C}^{r_k r_{k'}}. \end{aligned}$$

Since $\deg(\bar{E}_{k'}^\vee \otimes \bar{E}_k \otimes \omega_C(-p_1 - \cdots - p_{2g-3})) < 0$ and $\bar{E}_k$ and $\bar{E}_{k'}$ are semistable, we have

$$H^0(\bar{E}_{k'}^\vee \otimes \bar{E}_k \otimes \omega_C(-p_1 - \cdots - p_{2g-3})) = 0.$$

So we have $\dim H^0(\bar{E}_{k'}^\vee \otimes \bar{E}_k \otimes \omega_C(-p_1 - \cdots - p_{2g-4})) \leq r_k r_{k'}$. Inductively we can see that

$$\dim H^0(\bar{E}_{k'}^\vee \otimes \bar{E}_k \otimes \omega_C(-p_1 - \cdots - p_{2g-2-m})) \leq r_k r_{k'} (m - 1)$$

for $2 \leq m \leq 2g - 2$, because there are exact sequences

$$\begin{aligned} 0 \longrightarrow H^0(\bar{E}_{k'}^\vee \otimes \bar{E}_k \otimes \omega_C(-p_1 - \cdots - p_{2g-2-m+1})) \\ \longrightarrow H^0(\bar{E}_{k'}^\vee \otimes \bar{E}_k \otimes \omega_C(-p_1 - \cdots - p_{2g-2-m})) \\ \longrightarrow H^0(\bar{E}_{k'}^\vee \otimes \bar{E}_k \otimes \omega_C(-p_1 - \cdots - p_{2g-2-m})|_{p_{2g-2-m+1}}) \cong \mathbf{C}^{r_k r_{k'}} \end{aligned}$$

for $2 \leq m \leq 2g - 2$. Thus we have

$$\dim \operatorname{Ext}^1(\bar{E}_k, \bar{E}_{k'}) = \dim H^0(\bar{E}_{k'}^\vee \otimes \bar{E}_k \otimes \omega_C) \leq r_k r_{k'} (2g - 3).$$

Case 3. $\deg(\bar{E}_{k'}^\vee \otimes \bar{E}_k(t)) = 0$.

Put $\mathcal{F} := \bar{E}_{k'}^\vee \otimes \bar{E}_k \otimes \omega_C(-p_1 - \cdots - p_{2g-3})$. Then $\mathcal{F}$ is semistable of degree 0. Let $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_l = \mathcal{F}$ be a Jordan-Hölder filtration of $\mathcal{F}$. If we put $t \in C$ sufficiently general, we may assume that $\mathcal{O}_C(t - p_{2g-3}) \not\cong \mathcal{F}_i/\mathcal{F}_{i-1}$ for any i . Then we have

$$\mathrm{Hom}(\mathcal{O}_C(t - p_{2g-3}), \bar{E}_{k'}^\vee \otimes \bar{E}_k \otimes \omega_C(-p_1 - \cdots - p_{2g-3})) = 0.$$

Since there is an exact sequence

$$\begin{aligned} 0 &= H^0(\bar{E}_{k'}^\vee \otimes \bar{E}_k \otimes \omega_C(-p_1 - \cdots - p_{2g-4} - t)) \\ &\longrightarrow H^0(\bar{E}_{k'}^\vee \otimes \bar{E}_k \otimes \omega_C(-p_1 - \cdots - p_{2g-4})) \\ &\longrightarrow H^0(\bar{E}_{k'}^\vee \otimes \bar{E}_k \otimes \omega_C(-p_1 - \cdots - p_{2g-4})|_t) \cong \mathbf{C}^{r_k r_{k'}}, \end{aligned}$$

we have $\dim H^0(\bar{E}_{k'}^\vee \otimes \bar{E}_k \otimes \omega_C(-p_1 - \cdots - p_{2g-4})) \leq r_k r_{k'}$. Inductively we can see that

$$\dim H^0(\bar{E}_{k'}^\vee \otimes \bar{E}_k \otimes \omega_C(-p_1 - \cdots - p_{2g-2-m})) \leq r_k r_{k'}(m-1)$$

for $2 \leq m \leq 2g-2$. So we have $\dim \mathrm{Ext}^1(\bar{E}_k, \bar{E}_{k'}) = \dim H^0(\bar{E}_{k'}^\vee \otimes \bar{E}_k \otimes \omega_C) \leq r_k r_{k'}(2g-3)$.

Thus the claim has been proved.

We can see that the relative dimension of W_k over W_{k-1} at the point corresponding to the extension

$$0 \longrightarrow E_{k-1} \longrightarrow E_k \longrightarrow \bar{E}_k \longrightarrow 0$$

is at most

$$\begin{aligned} \dim \mathrm{Ext}^1(\bar{E}_k, E_{k-1}) - 1 &\leq -1 + \sum_{k' \leq k-1} \dim \mathrm{Ext}^1(\bar{E}_k, \bar{E}_{k'}) \\ &< -1 + \sum_{1 \leq k' < k} (r_{k'} r_k (2g-2)) \end{aligned}$$

if $0 \rightarrow E_{k-1} \rightarrow E_k \rightarrow \bar{E}_k \rightarrow 0$ does not split. If the extension $0 \rightarrow E_{k-1} \rightarrow E_k \rightarrow \bar{E}_k \rightarrow 0$ splits, the relative dimension of W_k over W_{k-1} is at most $-1 + \sum_{1 \leq k' < k} r_{k'} r_k (2g-2)$. Then we can see that the dimension of W_s at the point corresponding to the extensions

$$\begin{aligned} 0 &= E_0 \subset E_1 \subset \cdots \subset E_s = E, \quad \bar{E}_k = E_k/E_{k-1}, \\ 0 &= E_k^{(0)} \subset E_k^{(1)} \subset \cdots \subset E_k^{(m_k)} = \bar{E}_k, \quad \bar{E}_k^{(j)} = E_k^{(j)}/E_k^{(j-1)}, \end{aligned}$$

is at most

$$\begin{aligned}
& -g + \sum_{k=1}^s \sum_{j=1}^{m_k} \dim \operatorname{Ext}^1(\bar{E}_k^{(j)}, \bar{E}_k^{(j)}) \\
& + \sum_{k=1}^s \left(1 - m_k + \sum_{1 \leq i < j \leq m_k} \dim \operatorname{Ext}^1(\bar{E}_k^{(j)}, \bar{E}_k^{(i)}) \right) \\
& + \sum_{k=2}^s \left(-1 + \sum_{1 \leq k' < k} \dim \operatorname{Ext}^1(\bar{E}_k, \bar{E}_{k'}) \right) \\
& < -g + \sum_{k=1}^s \sum_{j=1}^{m_k} \dim \operatorname{Ext}^1(\bar{E}_k^{(j)}, \bar{E}_k^{(j)}) \\
& + \sum_{k=1}^s \left(1 - m_k + \sum_{1 \leq i < j \leq m_k} \dim \operatorname{Ext}^1(\bar{E}_k^{(j)}, \bar{E}_k^{(i)}) \right) \\
& + 1 - s + \sum_{1 \leq k' < k \leq s} r_{k'} r_k (2g - 2) \\
& \leq \sum_{k=1}^s \sum_{i=1}^{m_k} ((r_k^{(i)})^2 (g - 1) + 1) + \sum_{k=1}^s \left(1 - m_k + \sum_{1 \leq i < j \leq m_k} (r_k^{(i)} r_k^{(j)} (g - 1) + 1) \right) \\
& - g + 1 - s + \sum_{1 \leq k' < k \leq s} \sum_{1 \leq i \leq m_k, 1 \leq j \leq m_{k'}} r_k^{(i)} r_{k'}^{(j)} (2g - 2) \\
& \leq \sum_{k=1}^s \sum_{i=1}^{m_k} (r_k^{(i)})^2 (g - 1) + \sum_{k=1}^s \sum_{1 \leq i < j \leq m_k} 2r_k^{(i)} r_k^{(j)} (g - 1) \\
& - g + 1 + \sum_{k < k'} \sum_{1 \leq i \leq m_k, 1 \leq j \leq m_{k'}} 2r_{k'}^{(i)} r_k^{(j)} (g - 1) \\
& = (r^2 - 1)(g - 1)
\end{aligned}$$

because we have $s > 1$.

Consider the product of flag schemes

$$Z := \prod_{i=1}^n \operatorname{Flag}(\mathcal{E}_s|_{t_i \times W_s})$$

over W_s and put $\mathcal{E} := (\mathcal{E}_s)_Z$. Then there is a universal parabolic structure

$$\mathcal{E}|_{t_i \times Z} = \tilde{l}_0^{(i)} \supset \tilde{l}_1^{(i)} \supset \cdots \supset \tilde{l}_{r-1}^{(i)} \supset \tilde{l}_r^{(i)} = 0$$

for each $i = 1, \dots, n$. The dimension of Z at the point corresponding to the extensions

$$0 = E_0 \subset E_1 \subset \dots \subset E_s = E, \quad \bar{E}_k = E_k/E_{k-1}, \\ 0 = E_k^{(0)} \subset E_k^{(1)} \subset \dots \subset E_k^{(m_k)} = \bar{E}_k, \quad \bar{E}_k^{(j)} = E_k^{(j)}/E_k^{(j-1)},$$

and the parabolic structure

$$E|_{t_i} = l_0^{(i)} \supset l_1^{(i)} \supset \dots \supset l_{r-1}^{(i)} \supset l_r^{(i)} = 0 \quad (i = 1, \dots, n)$$

is less than $(r^2 - 1)(g - 1) + r(r - 1)n/2 = (r - 1)((r + 1)(g - 1) + rn/2)$.

Consider the functor $\text{Conn}_Z : (\text{Sch}/Z) \rightarrow (\text{Sets})$ defined by

$$\text{Conn}_Z(S) = \left\{ \nabla : \mathcal{E}_S \rightarrow \mathcal{E}_S \otimes \Omega_C^1(D(\mathbf{t})) \left| \begin{array}{l} \nabla \text{ is a connection such that} \\ (\text{res}_{t_i \times S}(\nabla) - \lambda_j^{(i)})(\tilde{l}_j^{(i)})_S \\ \subset (\tilde{l}_{j+1}^{(i)})_S \text{ for any } i, j \\ \text{and } \det(\nabla) = \nabla_L \otimes \mathcal{L} \\ \text{via an identification } \mathcal{E} = \\ L \otimes \mathcal{L} \text{ for some line bundle} \\ \mathcal{L} \text{ on } S \end{array} \right. \right\}.$$

We can see that Conn_Z can be represented by a scheme B of finite type over Z . Let

$$\tilde{\nabla} : \mathcal{E}_B \longrightarrow \mathcal{E}_B \otimes \Omega_C^1(D(\mathbf{t}))$$

be the universal connection. If we put

$$\mathcal{F}_0^0 := \left\{ u \in \mathcal{E}nd(\mathcal{E}) \left| \text{Tr}(u) = 0 \text{ and } u|_{t_i \times Y}((\tilde{l}_j^{(i)})_Y) \subset (\tilde{l}_j^{(i)})_Y \text{ for any } i, j \right. \right\},$$

$$\mathcal{F}_0^1 := \left\{ u \in \mathcal{E}nd(\mathcal{E}) \otimes \Omega_C^1(D(\mathbf{t})) \mid \text{Tr}(u) = 0 \text{ and } \right.$$

$$\left. \text{res}_{t_i \times Y}(u)((\tilde{l}_j^{(i)})_Y) \subset (\tilde{l}_{j+1}^{(i)})_Y \text{ for any } i, j \right\},$$

$$\nabla_{\mathcal{F}_0^\bullet} : \mathcal{F}_0^0 \ni u \mapsto \nabla_{\mathcal{F}_0^\bullet} \circ u - u \circ \nabla_{\mathcal{F}_0^\bullet} \in \mathcal{F}_0^1,$$

then the fiber B_x of B over $x \in Z$ is isomorphic to the affine space $H^0(\mathcal{F}_0^1 \otimes k(x))$. Put

$$B' := \left\{ x \in B \left| \sigma^{-1} \left(\mathcal{E}_x, \tilde{\nabla} \otimes k(x), \left\{ \tilde{l}_j^{(i)} \otimes k(x) \right\} \right) \text{ is } \alpha\text{-stable} \right. \right\}.$$

Then there is a canonical morphism

$$\psi : B' \longrightarrow M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L) \setminus N$$

and the homomorphism

$$H^0(\nabla_{\mathcal{F}_0^\bullet} \otimes k(x)) : H^0(\mathcal{F}_0^0 \otimes k(x)) \longrightarrow H^0(\mathcal{F}_0^1 \otimes k(x))$$

is injective for $x \in B'$, because $\text{End}(\mathcal{E} \otimes k(x), \tilde{\nabla} \otimes k(x), \{\tilde{l}_j^{(i)} \otimes k(x)\}) \cong k(x)$. If we put

$$U := \left\{ u : \mathcal{E} \otimes k(x) \xrightarrow{\sim} \mathcal{E} \otimes k(x) \mid u(\tilde{l}_j^{(i)} \otimes k(x)) \subset \tilde{l}_j^{(i)} \otimes k(x) \text{ and } \bigwedge^r u = \text{id} \right\},$$

for $x \in Z$, then we have a morphism

$$\iota : U \rightarrow B'_x; \quad u \mapsto u \circ (\tilde{\nabla} \otimes k(x)) \circ u^{-1}.$$

We can see that the tangent map of ι at id is just the homomorphism $H^0(\nabla_{\mathcal{F}_0^\bullet} \otimes k(x))$ which is injective. We may assume that the morphism $\psi : B' \rightarrow \overline{\psi(B')}$ is flat by replacing B' by a disjoint union of subschemes. So we have for any $x \in B'$,

$$\begin{aligned} \dim_{\psi(x)} \overline{\psi(B')} &= \dim_x(B') - \dim_x \psi^{-1}(\psi(x)) \\ &\leq \dim(Z) + \dim_x B'_x - \dim_x \iota(U) \\ &\leq \dim(Z) + \dim H^0(\mathcal{F}_0^1 \otimes k(x)) - \dim H^0(\mathcal{F}_0^0 \otimes k(x)) \\ &< (r-1)(2(r+1)(g-1) + rn), \end{aligned}$$

because

$$\dim \text{coker}(H^0(\mathcal{F}_0^0 \otimes k(x)) \rightarrow H^0(\mathcal{F}_0^1 \otimes k(x))) = (r-1)((r+1)(g-1) + rn/2).$$

Note that $M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L) \setminus N$ can be covered by a finite union of such $\psi(B')$ s; hence, we have

$$\dim(M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L) \setminus N) < (r-1)(2(r+1)(g-1) + rn).$$

So Step 2 is proved. By Step 1, Step 2 and the fact that the moduli space $M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L)$ is smooth of equidimension $(r-1)(2(r+1)(g-1) + rn)$, the moduli space $M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L)$ is irreducible. $\square$

Proposition 5.2. *Assume that $g = 1$ and $n \geq 2$. Then $M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L)$ is an irreducible variety of dimension $r(r-1)n$.*

Proof. Composing certain elementary transforms, we obtain an isomorphism

$$\sigma : \mathcal{M}_C(\mathbf{t}, \boldsymbol{\lambda}, L) \xrightarrow{\sim} \mathcal{M}_C(\mathbf{t}, \boldsymbol{\lambda}', L'),$$

where r and $\deg L'$ are coprime. Put

$$N := \left\{ (E, \nabla, \{l_j^{(i)}\}) \in M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L) \mid \sigma(E) \text{ is a stable bundle} \right\}.$$

As in the proof of the previous proposition, we can show that N is irreducible. So it suffices to show that $\dim(M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L) \setminus N) < r(r-1)n$ because $M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L)$ is smooth of equidimension $r(r-1)n$. Take any member $(E, \nabla, \{l_j^{(i)}\}) \in \sigma(M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L) \setminus N)$. Let $F_s \subset F_{s-1} \subset \cdots \subset F_1 = E$ be the

Harder-Narasimhan filtration of E . Note that $s > 1$. We can inductively see that the extension

$$0 \longrightarrow F_p/F_{p+1} \longrightarrow E/F_{p+1} \longrightarrow E/F_p \longrightarrow 0$$

must split for $p = 2, \dots, s$, where we put $F_{s+1} = 0$. Then we have a decomposition $E \cong E_1 \oplus \dots \oplus E_s$, where each E_p is semistable and $\mu(E_1) < \mu(E_2) < \dots < \mu(E_s)$. We can write $E_p = \bigoplus_{j=1}^{r_p} F_{p,j}$, where each $F_{p,j}$ is a successive extension of a stable bundle $F_p^{(j)}$ and $F_p^{(j)} \not\cong F_p^{(k)}$ for $j \neq k$. We can see that $F_{p,j} \cong G_1 \oplus \dots \oplus G_m$, where each G_i is a successive non-split extension of $F_p^{(j)}$. We put $r_p^{(j)} := \text{rank } F_p^{(j)}$ and $d_p^{(j)} := \deg F_p^{(j)}$. Let $M_C(r_p^{(j)}, d_p^{(j)})$ be the moduli space of stable bundles on C of rank $r_p^{(j)}$ and degree $d_p^{(j)}$. Replacing $M_C(r_p^{(j)}, d_p^{(j)})$ by an étale covering, we may assume that there is a universal bundle $\mathcal{F}_p^{(j)}$ on $C \times M_C(r_p^{(j)}, d_p^{(j)})$. We can construct non-split extensions $\mathcal{G}_1, \dots, \mathcal{G}_m$ of $\mathcal{F}_p^{(j)}$ such that $F_{p,j} \cong (\mathcal{G}_1 \oplus \dots \oplus \mathcal{G}_m) \otimes k(x_0)$ for the point x_0 of $M_C(r_p^{(j)}, d_p^{(j)})$ corresponding to $F_p^{(j)}$. We put $\mathcal{F}_{p,j} := \mathcal{G}_1 \oplus \dots \oplus \mathcal{G}_m$ and

$$W := \left\{ (x_p^{(j)}) \in \prod_{1 \leq p \leq s, 1 \leq j \leq r_p} M_C(r_p^{(j)}, d_p^{(j)}) \mid \bigotimes_{p,j} \det(\mathcal{F}_{p,j} \otimes k(x_p^{(j)})) \cong L \right\}.$$

Then $\dim W = -1 + \sum_{p=1}^s r_p$. We put $\mathcal{E} := \bigoplus_{j,p} \mathcal{F}_{p,j}$. Then parabolic structure can be parameterized by the product of flag schemes

$$U := \prod_{i=1}^n \text{Flag}(\mathcal{E}|_{t_i \times W})$$

over W . The relative dimension of U over W is $r(r-1)n/2$. Let $\{\tilde{l}_j^{(i)}\}$ be the universal parabolic structure on $\mathcal{E}_U$. Note that

$$\text{End}(\mathcal{E} \otimes k(x)) = \bigoplus_{p=1}^s \bigoplus_{j=1}^{r_p} \text{End}(\mathcal{F}_{p,j} \otimes k(x)) \oplus \bigoplus_{p < q} \bigoplus_{j,k} \text{Hom}(\mathcal{F}_{p,j} \otimes k(x), \mathcal{F}_{q,k} \otimes k(x))$$

for a point $x \in W$ and the group

$$G_x := \left\{ \sum_{p,j} c_{pj} \text{id}_{\mathcal{F}_{p,j} \otimes k(x)} + \sum_{p < q} \sum_{j,k} a_{j,k}^{(p,q)} \right. \\ \left. \in \text{End}(\mathcal{E} \otimes k(x)) \mid \begin{array}{l} c_{pj} \in \mathbf{C}^*, \\ a_{j,k}^{(p,q)} \in \text{Hom}(\mathcal{F}_{p,j} \otimes k(x), \mathcal{F}_{q,k} \otimes k(x)) \end{array} \right\}$$

acts on the fiber U_x of U over x . Note that we can take a section $a \in \text{Hom}(\mathcal{F}_{p,j} \otimes k(x), \mathcal{F}_{q,k} \otimes k(x))$ with $a(t_1) \neq 0$ or $a(t_2) \neq 0$ if $p < q$. So we

may assume that $\alpha(t_2) \neq 0$ for some $\alpha \in \text{Hom}(\mathcal{F}_{p_0,1} \otimes k(x), \mathcal{F}_{p_0+1,1} \otimes k(x))$. Modulo the action of the group

$$\left\{ \sum_{p,j} c_{pj} \text{id}_{\mathcal{F}_{p,j}} \otimes k(x) \mid c_{pj} \in \mathbf{C}^\times \right\},$$

$\tilde{l}_{r-1}^{(1)} \otimes k(y)$ ($y \in U_x$) can be parameterized by an algebraic scheme whose dimension is at most $r - \sum_{p=1}^s r_p$. Moreover, modulo the action of the group

$$\left\{ \text{id}_{\mathcal{E} \otimes k(x)} + \sum_{p < q} \sum_{j,k} a_{j,k}^{(p,q)} \mid a_{j,k}^{(p,q)} \in \text{Hom}(\mathcal{F}_{p,j} \otimes k(x), \mathcal{F}_{q,k} \otimes k(x)) \right\},$$

$\tilde{l}_{r-1}^{(2)} \otimes k(y)$ ($y \in U_x$) can be parameterized by an algebraic scheme whose dimension is at most $r - 2$. Then, modulo the action of G_x , $\tilde{l}_{r-1}^{(1)} \otimes k(y)$ and $\tilde{l}_{r-1}^{(2)} \otimes k(y)$ ($y \in U_x$) can be parameterized by an algebraic scheme whose dimension is at most $2r - \sum_{p=1}^s r_p - 2$. So we can take a finite number of subschemes $Y_1, \dots, Y_l$ of U such that $\dim Y_i \leq -1 + \sum_{p=1}^s r_p + r(r-1)n/2 - \sum_{p=1}^s r_p = r(r-1)n/2 - 1$ and $\bigcup_{i=1}^l G_x(Y_i)_x = U_x$ for any point $x \in W$. We put $Y := \coprod_{i=1}^l Y_i$. Replacing Y by its stratification, we may assume that the dimension of

$$F_y^0 := \left\{ s \in \text{End}(\mathcal{E}_y) \mid \text{Tr}(s) = 0 \text{ and } s(t_i)(\tilde{l}_j^{(i)})_y \subset (\tilde{l}_j^{(i)})_y \text{ for each } i, j \right\}$$

is constant for $y \in Y$. There exists a scheme Z over Y such that for any Y -scheme S , we have

$$Z(S) = \left\{ \nabla : \mathcal{E}_S \rightarrow \mathcal{E}_S \otimes \Omega_C^1(t_1 + \dots + t_n) \mid \begin{array}{l} \nabla \text{ is a relative connection such} \\ \text{that } (\text{res}_{t_i \times S}(\nabla) - \lambda_j^{(i)})(\tilde{l}_j^{(i)})_S \\ \subset (\tilde{l}_{j+1}^{(i)})_S \text{ for any } i, j \text{ and} \\ \det(\nabla) = (\nabla_L) \otimes \mathcal{L} \end{array} \right\}.$$

We can see that each fiber of $Z \rightarrow Y$ over $y \in Y$ is an affine space isomorphic to

$$F_y^1 = \left\{ s \in \text{Hom}(\mathcal{E}_y, \mathcal{E}_y \otimes \Omega_C^1(t_1 + \dots + t_n)) \mid \begin{array}{l} \text{Tr}(s) = 0 \text{ and} \\ \text{res}_{t_i}(s)(\tilde{l}_j^{(i)})_y \subset (\tilde{l}_{j+1}^{(i)})_y \text{ for any } i, j \end{array} \right\}.$$

Let $\tilde{\nabla} : \mathcal{E}_Z \rightarrow \mathcal{E}_Z \otimes \Omega_C^1(t_1 + \dots + t_n)$ be the universal relative connection. If we put

$$Z^s := \left\{ y \in Z \mid \sigma^{-1} \left((\mathcal{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\}) \otimes k(y) \right) \text{ is } \alpha\text{-stable} \right\},$$

then there is a natural morphism $f : Z^s \rightarrow M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L)$. For each $z \in Z$ over $y \in Y$, consider the homomorphism

$$\nabla_z^1 : F_y^0 \otimes k(z) \longrightarrow F_y^1 \otimes k(z); \quad \nabla_z^1(s) := \tilde{\nabla}_z \circ s - s \circ \tilde{\nabla}_z.$$

Then the group Stab_y of automorphisms g of the parabolic bundles $(\mathcal{E}_y, \{(\tilde{l}_j^{(i)})_y\})$ which induce the identity on $\det(\mathcal{E}_y)$ acts on the fiber Z_y and the tangent map of the morphism $\text{Stab}_y \ni g \mapsto gz \in Z^s$ at id is just the homomorphism ∇_z^1 which is injective. Replacing Z^s by a finite disjoint union of subschemes, we may assume that $Z^s \xrightarrow{f} \overline{f(Z^s)}$ is flat. Then we have

$$\begin{aligned} \dim_{f(z)} \overline{f(Z^s)} &= \dim_z Z^s - \dim_z f^{-1}(f(z)) \\ &\leq \dim_z Z^s - \dim_{\text{id}} \text{Stab}_y \\ &= \dim_z Z^s - \dim F_y^0 \\ &\leq \dim_y Y + \dim_z Z_y^s - \dim F_y^0 \\ &\leq \dim_y Y + \dim F_y^1 - \dim F_y^0 \\ &\leq \dim Y + r(r-1)n/2 \leq r(r-1)n-1. \end{aligned}$$

Here $z \in Z^s$ and $Z^s \ni z \mapsto y \in Y$. Since $M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L) \setminus N$ is a finite union of such $f(Z^s)$ s, we have $\dim(M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L) \setminus N) < r(r-1)n$. $\square$

Proposition 5.3. *Assume that $rn - 2(r+1) > 0$ and $r \geq 2$. Then $M_{\mathbf{P}^1}^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L)$ is an irreducible variety of dimension $(r-1)(rn - 2r - 2)$.*

Proof. First we will show that the Zariski open set

$$N := \left\{ (E, \nabla, \{l_j^{(i)}\}) \in M_{\mathbf{P}^1}^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L) \mid \dim \text{End}(E, \{l_j^{(i)}\}) = 1 \right\}$$

is irreducible. There exists an integer n_0 such that $h^1(E(n_0)) = 0$ for any $(E, \nabla, \{l_j^{(i)}\}) \in M_{\mathbf{P}^1}^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L)$. Then we can easily construct a smooth irreducible variety X and a flat family of parabolic bundles $(\tilde{E}, \{\tilde{l}_j^{(i)}\})$ on $\mathbf{P}^1 \times X$ over X whose geometric fibers is just the set of all the simple parabolic bundles $(E, \{l_j^{(i)}\})$ satisfying $h^1(E(n_0)) = 0$ and $\det E = L$. We can take an isomorphism $\det \tilde{E} \cong L \otimes \mathcal{L}$ for some line bundle $\mathcal{L}$ on X . The functor $\text{Conn}_X : (\text{Sch}/X) \longrightarrow (\text{Sets})$ defined by

$$\text{Conn}_X(U) = \left\{ \nabla : \tilde{E}_U \rightarrow \tilde{E}_U \otimes \Omega_{\mathbf{P}^1}^1(D(\mathbf{t})) \mid \begin{array}{l} \nabla \text{ is a relative connection} \\ \text{satisfying } (\text{res}_{t_i}(\nabla) \\ -\lambda_j^{(i)})(\tilde{l}_j^{(i)})_U \subset (\tilde{l}_{j+1}^{(i)})_U \\ \text{for any } i, j \text{ and} \\ \det(\nabla) = (\nabla_L)_U \otimes \mathcal{L} \end{array} \right\}$$

is represented by a scheme Y of finite type over X . We can see by the same proof as Proposition 5.1 that Y is smooth over X whose fiber is an affine

space of dimension $h^0((\mathcal{F}_0^1)_x)$ for $x \in X$, where

$$\mathcal{F}_0^1 = \left\{ s \in \mathcal{E}nd(\tilde{E}) \otimes \Omega_{\mathbf{P}^1}^1(D(\mathbf{t})) \mid \text{Tr}(s) = 0 \text{ and } \text{res}_{t_i \times X}(s)(\tilde{l}_j^{(i)}) \subset \tilde{l}_{j+1}^{(i)} \text{ for any } i, j \right\}.$$

So Y is irreducible. Let

$$\tilde{\nabla} : \tilde{E}_Y \longrightarrow \tilde{E}_Y \otimes \Omega_{\mathbf{P}^1}^1(t_1 + \cdots + t_n)$$

be the universal relative connection and put

$$Y' := \left\{ y \in Y \mid (\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\}) \otimes k(y) \text{ is } \alpha\text{-stable} \right\}.$$

Then Y' is a Zariski open subset of Y and it is irreducible. A morphism, $Y' \rightarrow M_{\mathbf{P}^1}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda}, L)$, whose image is just N is induced. Thus N becomes irreducible.

In order to prove that $M_{\mathbf{P}^1}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda}, L)$ is irreducible, it suffices to show that $\dim(M_{\mathbf{P}^1}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda}, L) \setminus N) < (r-1)(rn-2r-2)$ because $M_{\mathbf{P}^1}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda}, L)$ is smooth of equidimension $(r-1)(rn-2r-2)$. For this, we will construct an algebraic scheme W' of dimension less than $(r-1)(rn-2r-2)$ and a surjection $f : W' \rightarrow M_{\mathbf{P}^1}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda}, L) \setminus N$.

Take any parabolic bundle $(E, \{l_j^{(i)}\})$ such that $\text{rank } E = r$, $\deg E = d$, $\dim \text{End}(E, \{l_j^{(i)}\}) \geq 2$ and $h^1(E(n_0)) = 0$. We will show that such parabolic bundles can be parameterized by an algebraic scheme of dimension less than $(r-1)(rn/2-r-1)$. We can write

$$E \cong \mathcal{O}_{\mathbf{P}^1}(a_1)^{\oplus r_1} \oplus \mathcal{O}_{\mathbf{P}^1}(a_2)^{\oplus r_2} \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(a_s)^{\oplus r_s}$$

with $a_1 < a_2 < \cdots < a_s$. If $l_{r-1}^{(k)} \not\subset \mathcal{O}_{\mathbf{P}^1}(a_2)^{\oplus r_2}|_{t_k} \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(a_s)^{\oplus r_s}|_{t_k}$ for some k , then we replace $(E, \{l_j^{(i)}\})$ by its elementary transform along t_k by $l_{r-1}^{(k)}$, which is isomorphic to the bundle

$$\mathcal{O}_{\mathbf{P}^1}(a_1-1)^{\oplus r_1-1} \oplus \mathcal{O}_{\mathbf{P}^1}(a_1) \oplus \mathcal{O}_{\mathbf{P}^1}(a_2-1)^{\oplus r_2} \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(a_s-1)^{\oplus r_s}$$

with a certain parabolic structure. Repeating this process, we finally obtain two cases:

$$\begin{cases} E \cong \mathcal{O}_{\mathbf{P}^1}(a)^{\oplus r}, \\ l_{r-1}^{(i)} \subset \mathcal{O}_{\mathbf{P}^1}(a_2)^{\oplus r_2}|_{t_i} \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(a_s)^{\oplus r_s}|_{t_i} \quad \text{for all } i \text{ and } r_1 > 0. \end{cases}$$

Case 1. $E \cong \mathcal{O}_{\mathbf{P}^1}(a)^{\oplus r}$.

Tensoring $\mathcal{O}_{\mathbf{P}^1}(-a)$, we may assume $E \cong \mathcal{O}_{\mathbf{P}^1}^{\oplus r}$. For a suitable choice of a basis of $\mathcal{O}_{\mathbf{P}^1}^{\oplus r}$, we may assume that

$$l_{r-i}^{(1)} = k(t_1)e_1 + k(t_1)e_2 + \cdots + k(t_1)e_i$$

for $i = 1, \dots, r$, where e_j is the vector of size r whose j th component is 1 and others are zero. Then the group of automorphisms of E fixing $l_{r-1}^{(1)}, l_{r-2}^{(1)}, \dots, l_1^{(1)}$ is

$$B = \{(a_{ij}) \in GL_r(\mathbf{C}) \mid a_{ij} = 0 \text{ for } i > j\}.$$

We put

$$p(1) := \max \left\{ p \mid v_p \neq 0 \text{ for some } (v_1, \dots, v_r) \in l_{r-1}^{(2)} \right\}.$$

Applying a certain automorphism in B , we may assume that $l_{r-1}^{(2)} = k(t_2)e_{p(1)}$. Inductively we put

$$p(i) := \max \left\{ p \mid v_p \neq 0 \text{ for some } (v_1, \dots, v_r) \in l_{r-i}^{(2)} \text{ and } p \neq p(j) \text{ for any } j < i \right\}$$

for $i = 1, \dots, r$. Applying an automorphism in B which fixes $l_{r-1}^{(2)}, \dots, l_{r-i+1}^{(2)}$, we may assume that

$$l_{r-i}^{(2)} = l_{r-i+1}^{(2)} + k(t_2)e_{p(i)}.$$

Then the group of automorphisms of E fixing both $\{l_j^{(1)}\}$ and $\{l_j^{(2)}\}$ is

$$B' = \{(a_{ij}) \in GL_r(\mathbf{C}) \mid a_{ij} = 0 \text{ for } i > j \text{ and } a_{p(i)p(j)} = 0 \text{ for } i > j\}.$$

Applying a certain automorphism in B' , we may assume that $l_{r-1}^{(3)}$ can be generated by a vector $w = (w_1, \dots, w_r)$, where each w_i is either 1 or 0. Note that we have $n \geq 3$ by the assumption of the proposition. The group of automorphisms of E fixing $\{l_j^{(1)}\}$, $\{l_j^{(2)}\}$ and $l_{r-1}^{(3)}$ is

$$B'' = \left\{ (a_{ij}) \in GL_r(\mathbf{C}) \left| \begin{array}{l} a_{ij} = 0 \text{ for } i > j, \\ a_{p(i)p(j)} = 0 \text{ for } i > j \\ \text{and there is } c \in \mathbf{C}^\times \text{ satisfying} \\ a_{ii}w_i + \sum_{k \neq i} a_{ik}w_k = cw_i \text{ for any } i \end{array} \right. \right\}.$$

Since the parabolic bundle $(E, \{l_j^{(i)}\})$ has a non-trivial endomorphism, there is (i, j) with $i < j$ and $p(i) < p(j)$ or there is some i satisfying $w_i = 0$. Note that $r \geq 3$ or $n \geq 4$ because we assume that $rn - 2r - 2 > 0$. First assume that $r \geq 3$. If $w_{p(i)} = w_{p(j)} = 1$ and $p(i) < p(j)$ for some $i < j$, then we apply an automorphism of the form (a_{kl}) such that $a_{kk} = 1$ if $k \neq p(i)$, $a_{kl} = 0$ if $k \neq l$ and $(k, l) \neq (p(i), p(j))$, $a_{p(i)p(i)} = 1 - c$ and $a_{p(i)p(j)} = c$ for some $c \in \mathbf{C} \setminus \{1\}$. Then we can parameterize $l_{r-2}^{(3)}$ modulo this action by an algebraic scheme whose dimension is less than $r - 2$. If $w_i = 0$ for some i , then we apply an automorphism of the form (a_{kl}) such that $a_{kk} = 1$ for $k \neq i$, $a_{ii} = c$ ($c \in \mathbf{C}^\times$) and $a_{kl} = 0$ for $k \neq l$. Then $l_{r-2}^{(3)}$ can be parameterized modulo this action by an algebraic scheme of dimension less than $r - 2$. Next assume that

$n \geq 4$. If $w_{p(i)} = w_{p(j)} = 1$ and $p(i) < p(j)$ for some $i < j$, then we apply an automorphism of the form (a_{kl}) such that $a_{kk} = 1$ if $k \neq p(i)$, $a_{kl} = 0$ if $k \neq l$ and $(k, l) \neq (p(i), p(j))$, $a_{p(i)p(i)} = 1 - c$ and $a_{p(i)p(j)} = c$ for some $c \in \mathbf{C} \setminus \{1\}$. Then we can parameterize $l_{r-1}^{(4)}$ modulo this action by an algebraic scheme whose dimension is less than $r - 1$. If $w_i = 0$ for some i , then we apply an automorphism of the form (a_{kl}) such that $a_{kk} = 1$ for $k \neq i$, $a_{ii} = c$ ($c \in \mathbf{C}^\times$) and $a_{kl} = 0$ for $k \neq l$. Then $l_{r-1}^{(4)}$ can be parameterized modulo this action by an algebraic scheme of dimension less than $r - 1$. Taking account of all the above arguments, we can parameterize parabolic bundles $(E, \{l_j^{(i)}\})$ satisfying $E \cong \mathcal{O}_{\mathbf{P}^1}^{\oplus r}$ and $\dim \text{End}(E, \{l_j^{(i)}\}) > 1$ by an algebraic scheme of dimension less than $(r - 1)(rn/2 - (r + 1))$.

Case 2. $E \cong \mathcal{O}_{\mathbf{P}^1}(a_1)^{\oplus r_1} \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(a_s)^{\oplus r_s}$ with $r_1 > 0$ and $l_{r-1}^{(i)} \subset \mathcal{O}_{\mathbf{P}^1}(a_2)^{\oplus r_2}|_{t_i} \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(a_s)^{\oplus r_s}|_{t_i}$.

For each p , we consider an identification $\mathcal{O}_{\mathbf{P}^1}(a_p)^{\oplus r_p}|_{t_i} = \mathbf{C}^{r_p}$ and denote by $e_{p,j}^{(i)}$ the element of this vector space whose j th component is 1 and others are zero. We put

$$p(1) := \min \left\{ p \left| v_p \neq 0 \text{ for some } (v_1, \dots, v_s) \in l_{r-1}^{(1)} \text{ where } v_q \in \mathcal{O}_{\mathbf{P}^1}(a_q)^{\oplus r_q}|_{t_1} \text{ for any } q \right. \right\}.$$

Applying an automorphism of E , we may assume that $l_{r-1}^{(1)}$ is generated by $e_{p(1),j(1)}^{(1)}$ with $j(1) = 1$. Inductively we put

$$p(i) := \min \left\{ p \left| v_j^{(p)} \neq 0 \text{ for some } \sum_{q,k} v_k^{(q)} e_{q,k}^{(1)} \in l_{r-i}^{(1)} \text{ where } (p, j) \neq (p(k), j(k)) \text{ for any } k < i \right. \right\}$$

for $i = 1, \dots, r$ and we put

$$j(i) := 1 + \max(\{0\} \cup \{j(i') | i' < i \text{ and } p(i') = p(i)\}).$$

Applying an automorphism of E fixing $l_{r-1}^{(1)}, \dots, l_{r-i+1}^{(1)}$, we may assume that $l_{r-i}^{(1)} = l_{r-i+1}^{(1)} + k(t_1)e_{p(i),j(i)}^{(1)}$. Then each $l_{r-i}^{(1)}$ is generated by $e_{p(1),j(1)}^{(1)}, \dots,$

$e_{p(i),j(i)}^{(1)}$ and the group of automorphisms of E fixing $\{l_j^{(1)}\}$ is

$$B = \left\{ (a_{jk}^{pq})_{\substack{1 \leq p, q \leq s \\ 1 \leq j \leq r_p, 1 \leq k \leq r_q}} \left| \begin{array}{l} (a_{jk}^{pq})_{1 \leq j \leq r_p, 1 \leq k \leq r_q} \in \text{End}(\mathcal{O}(a_q)^{r_q}, \mathcal{O}(a_p)^{r_p}) \\ \text{for each } (p, q) \\ (a_{jk}^{pp}) \in \text{Aut}(\mathcal{O}(a_p)^{r_p}) \text{ and } a_{j(i)j(i')}^{p(i)p(i')}(t_1) = 0 \\ \text{for } i > i' \end{array} \right. \right\}.$$

Note that we have $j(i') < j(i)$ if $i' < i$ and $p(i') = p(i)$. We will also normalize $\{l_j^{(2)}\}$. First we put

$$(p'(1), -j'(1)) := \min \left\{ (p, -j) \left| v_j^{(p)} \neq 0 \text{ for some } \sum_{q,k} v_k^{(q)} e_{q,k}^{(2)} \in l_{r-1}^{(2)} \right. \right\},$$

where we consider the lexicographic order on the pair $(p, -j)$. Applying an automorphism of E fixing $\{l_j^{(1)}\}$, we may assume that $l_{r-1}^{(2)}$ is generated by $e_{p'(1),j'(1)}^{(2)}$. Inductively we put

$$(p'(i), -j'(i)) := \min \left\{ (p, -j) \left| \begin{array}{l} v_j^{(p)} \neq 0 \text{ for some } \sum_{q,k} v_k^{(q)} e_{q,k}^{(2)} \in l_{r-i}^{(2)} \text{ and} \\ (p, j) \neq (p'(k), j'(k)) \text{ for any } k < i \end{array} \right. \right\}$$

for $i = 1, \dots, r$. Applying an automorphism in B fixing $l_{r-1}^{(2)}, \dots, l_{r-i+1}^{(2)}$, we may assume that $l_{r-i}^{(2)} = l_{r-i+1}^{(2)} + k(t_2)e_{p'(i),j'(i)}^{(2)}$. Then the group of automorphisms of E fixing $\{l_j^{(1)}\}$ and $\{l_j^{(2)}\}$ is

$$B' = \left\{ (a_{jk}^{pq})_{\substack{1 \leq p, q \leq s \\ 1 \leq j \leq r_p, 1 \leq k \leq r_q}} \in B \left| a_{j'(i)j'(l)}^{p'(i)p'(l)}(t_2) = 0 \text{ for } i > l \right. \right\}.$$

Applying a certain automorphism in B' , we may assume that $l_{r-1}^{(3)}$ can be generated by a vector $w = \sum w_j^{(p)} e_{p,j}^{(3)}$, where each $w_j^{(p)}$ is either 1 or 0. Then the group of automorphisms of E fixing $\{l_j^{(1)}\}$, $\{l_j^{(2)}\}$ and $l_{r-1}^{(3)}$ is

$$B'' = \left\{ (a_{jk}^{pq})_{\substack{1 \leq p, q \leq s \\ 1 \leq j \leq r_p, 1 \leq k \leq r_q}} \in B' \left| \text{for some } c \in \mathbf{C}^\times, \right. \right. \\ \left. \left. \sum_{q,k} a_{jk}^{pq} w_k^{(q)} = c w_j^{(p)} \text{ for any } p, j \right. \right\}.$$

Note that $p(1) \geq 2$ because $l_{r-1}^{(1)} \subset \mathcal{O}_{\mathbf{P}^1}(a_2)^{\oplus r_2}|_{t_1} \oplus \dots \oplus \mathcal{O}_{\mathbf{P}^1}(a_s)^{\oplus r_s}|_{t_1}$. If $w_1^{(1)} \neq 0$ and $w_{j(1)}^{(p(1))} \neq 0$, then there exist elements (a_{jk}^{pq}) of B'' satisfying $a_{j(1)1}^{p(1)1}(t_3)w_1^{(1)} + a_{j(1)j(1)}^{p(1)p(1)}w_{j(1)}^{(p(1))} = w_{j(1)}^{(p(1))}$, $a_{jj}^{pp} = 1$ for $(p, j) \neq (p(1), j(1))$ and $a_{jk}^{pq} = 0$ for $(p, j) \neq (q, k)$ and $(p, j, q, k) \neq (p(1), j(1), 1, 1)$. Applying such automorphisms, $l_{r-2}^{(3)}$ can be parameterized modulo this action by an algebraic

scheme of dimension less than $r - 2$ when $r \geq 3$ and $l_{r-1}^{(4)}$ can be parameterized modulo this action by an algebraic scheme of dimension less than $r - 1$ when $n \geq 4$. If $w_{j_0}^{(p_0)} = 0$ for some (p_0, j_0) , then we apply an automorphism (a_{jk}^{pq}) in B'' satisfying $a_{jj}^{pp} = 1$ for $(p, j) \neq (p_0, j_0)$, $a_{j_0 j_0}^{p_0 p_0} = c$ ($c \in \mathbf{C}^\times$) and $a_{jk}^{pq} = 0$ for $(p, j) \neq (q, k)$. Then $l_{r-2}^{(3)}$ can be parameterized modulo this action by an algebraic scheme of dimension less than $r - 2$ when $r \geq 3$ and $l_{r-1}^{(4)}$ can be parameterized modulo this action by an algebraic scheme of dimension less than $r - 1$ when $n \geq 4$. Therefore we can parameterize parabolic bundles $(E, \{l_j^{(i)}\})$ satisfying $E \cong \mathcal{O}_{\mathbf{P}^1}(a_1)^{\oplus r_1} \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(a_s)^{\oplus r_s}$ and $l_{r-1}^{(i)} \subset \mathcal{O}_{\mathbf{P}^1}(a_2)^{\oplus r_2}|_{t_i} \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(a_s)^{\oplus r_s}|_{t_i}$ for any i by an algebraic scheme of dimension less than $(r - 1)(rn/2 - r - 1)$.

By Cases 1 and 2, we can construct an algebraic scheme Z of dimension less than $(r - 1)(rn/2 - r - 1)$ and a flat family $(\tilde{E}, \{\tilde{l}_j^{(i)}\})$ of parabolic bundles on $\mathbf{P}_Z^1$ over Z such that the set of the geometric fibers of $(\tilde{E}, \{\tilde{l}_j^{(i)}\})$ is just the set of parabolic bundles $(E, \{l_j^{(i)}\})$ satisfying $\text{rank } E = r$, $\deg E = d$, $\dim \text{End}(E, \{l_j^{(i)}\}) \geq 2$ and $h^1(E(n_0)) = 0$. Note that the functor $\text{Conn}_Z : (\text{Sch}/Z) \rightarrow (\text{Sets})$ defined by

$$\text{Conn}_Z(S) = \left\{ \nabla : \tilde{E}_S \rightarrow \tilde{E}_S \otimes \Omega_{\mathbf{P}^1}^1(t_1 + \cdots + t_n) \left| \begin{array}{l} \nabla \text{ is a relative connection} \\ \text{satisfying } (\text{res}_{t_i \times S}(\nabla) \\ - \lambda_j^{(i)})(\tilde{l}_j^{(i)})_S \subset (\tilde{l}_{j+1}^{(i)})_S \text{ for} \\ \text{any } i, j \text{ and } \det(\nabla) = \\ (\nabla_L)_S \otimes \mathcal{L} \end{array} \right. \right\}$$

can be represented by a scheme W of finite type over Z . Let $\tilde{\nabla} : \tilde{E}_W \rightarrow \tilde{E}_W \otimes \Omega_{\mathbf{P}^1}^1(t_1 + \cdots + t_n)$ be the universal relative connection. If we put

$$\begin{aligned} \mathcal{F}_0^0 &:= \left\{ s \in \mathcal{E}nd(\tilde{E}) \left| \text{Tr}(s) = 0 \text{ and } s(\tilde{l}_j^{(i)}) \subset \tilde{l}_j^{(i)} \text{ for any } i, j \right. \right\}, \\ \mathcal{F}_0^1 &:= \left\{ s \in \mathcal{E}nd(\tilde{E}) \otimes \Omega_{\mathbf{P}^1}^1(t_1 + \cdots + t_n) \left| \text{Tr}(s) = 0 \text{ and} \right. \right. \\ &\quad \left. \left. \text{res}_{t_i \times Z}(s)(\tilde{l}_j^{(i)}) \subset \tilde{l}_{j+1}^{(i)} \text{ for any } i, j \right. \right\}, \\ \nabla^\dagger : \mathcal{F}_0^0 &\longrightarrow \mathcal{F}_0^1; \quad \nabla^\dagger(s) = \tilde{\nabla} \circ s - s \circ \tilde{\nabla}, \end{aligned}$$

then each fiber W_z of $W \rightarrow Z$ over z is an affine space isomorphic to $H^0(\mathcal{F}_0^1 \otimes k(z))$. If we put

$$W' := \left\{ p \in W \left| (\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\}) \otimes k(p) \text{ is } \alpha\text{-stable} \right. \right\},$$

then a morphism $f : W' \rightarrow M_{\mathbf{P}^1}^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L)$ is induced. Replacing W' by a finite disjoint union of subschemes, we may assume that $W' \xrightarrow{f} \overline{f(W')}$ is

flat. There is a canonical action of

$$\begin{aligned} & \text{Aut}_0(\tilde{E} \otimes k(z), \{\tilde{l}_j^{(i)} \otimes k(z)\}) \\ &= \left\{ g : \tilde{E} \otimes k(z) \xrightarrow{\sim} \tilde{E} \otimes k(z) \left| \begin{array}{l} g(\tilde{l}_j^{(i)} \otimes k(z)) \subset \tilde{l}_j^{(i)} \otimes k(z) \text{ for any } i, j \\ \text{and } \bigwedge^r g = \text{id} \end{array} \right. \right\} \end{aligned}$$

on the fiber W'_z of $W' \rightarrow Z$ over $z \in Z$ and we have $f(gp) = p$ for any $p \in W'_z$ and $g \in \text{Aut}_0(E, \{\iota_j^{(i)}\})$. Take $p \in W'$ lying over $z \in Z$. Then the tangent map of $\text{Aut}_0(\tilde{E} \otimes k(p), \{\tilde{l}_j^{(i)} \otimes k(p)\}) \ni g \mapsto gp \in W'_z$ at id is the linear map $H^0(\nabla^\dagger \otimes k(p))$, which is injective because $(\tilde{E} \otimes k(p), \tilde{\nabla} \otimes k(p), \{\tilde{l}_j^{(i)} \otimes k(p)\})$ is α -stable. So we have for any $p \in W'$,

$$\begin{aligned} \dim_{f(p)} \overline{f(W')} &= \dim_p W' - \dim_p f^{-1}(f(p)) \\ &\leq \dim_p W' - \dim_{\text{id}} \text{Aut}_0(\tilde{E} \otimes k(p), \{\tilde{l}_j^{(i)} \otimes k(p)\}) \\ &= \dim_p W' - \dim H^0(\mathcal{F}_0^0 \otimes k(p)) \\ &\leq \dim_z Z + \dim_p W'_z - \dim H^0(\mathcal{F}_0^0 \otimes k(p)) \\ &\leq \dim_z Z + \dim H^0(\mathcal{F}_0^1 \otimes k(p)) - \dim H^0(\mathcal{F}_0^0 \otimes k(p)) \\ &= \dim_z Z + (r-1)(rn/2 - r - 1) \\ &< (r-1)(rn - 2(r+1)). \end{aligned}$$

Here $W' \ni p \mapsto z \in Z$. Since $M_{\mathbf{P}^1}^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L) \setminus N$ can be covered by such $f(W')$ s, we have

$$\dim(M_{\mathbf{P}^1}^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L) \setminus N) < (r-1)(rn - 2(r+1)).$$

Hence $M_{\mathbf{P}^1}^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L)$ is irreducible. $\square$

6. Riemann-Hilbert correspondence

We take a point $x \in T$ and denote $\mathcal{C}_x$ and $\tilde{\mathbf{t}}_x$ simply by C and $\mathbf{t}$. We also denote the fiber of $M_{\mathcal{C}/T}^\alpha(\tilde{\mathbf{t}}, r, d)$ over $x \in T$ by $M_C^\alpha(\mathbf{t}, r, d)$. Sending $(E, \nabla, \{\iota_*^{(i)}\}_{1 \leq i \leq n})$ to $\ker \nabla^{an}|_{C \setminus \{t_1, \dots, t_n\}}$, we obtain a holomorphic mapping

$$\mathbf{RH}_x : M_C^\alpha(\mathbf{t}, r, d) \longrightarrow RP_r(C, \mathbf{t}),$$

which makes the diagram

$$\begin{array}{ccc} M_C^\alpha(\mathbf{t}, r, d) & \xrightarrow{\mathbf{RH}_x} & RP_r(C, \mathbf{t}) \\ \downarrow & & \downarrow \\ \Lambda_r^{(n)}(d) & \xrightarrow{rh} & \mathcal{A}_r^{(n)} \end{array}$$

commute. Here $rh : \Lambda_r^{(n)}(d) \ni \boldsymbol{\lambda} = (\lambda_j^{(i)}) \mapsto \mathbf{a} = (a_j^{(i)}) \in \mathcal{A}_r^{(n)}$ is defined by

$$\prod_{j=0}^{r-1} (X - \exp(-2\pi\sqrt{-1}\lambda_j^{(i)})) = X^r + a_{r-1}^{(i)}X^{r-1} + \cdots + a_1^{(i)}X + a_0^{(i)}$$

for $i = 1, \dots, n$. If we denote the fiber of $M_C^\alpha(\mathbf{t}, r, d)$ over $\boldsymbol{\lambda} \in \Lambda_r^{(n)}(d)$ by $M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda})$, then $\mathbf{RH}_x$ induces a holomorphic mapping

$$\mathbf{RH}_{(x, \boldsymbol{\lambda})} : M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}) \longrightarrow RP_r(C, \mathbf{t})_{\mathbf{a}}$$

for $rh(\boldsymbol{\lambda}) = \mathbf{a}$. In this section we will prove the properness of this morphism $\mathbf{RH}_{(x, \boldsymbol{\lambda})}$ and obtain Theorem 2.2. In order to prove the properness of $\mathbf{RH}_{(x, \boldsymbol{\lambda})}$, it is essential to prove the surjectivity of $\mathbf{RH}_{(x, \boldsymbol{\lambda})}$ and compactness of every fiber of $\mathbf{RH}_{(x, \boldsymbol{\lambda})}$. For the proof of the surjectivity of $\mathbf{RH}_{(x, \boldsymbol{\lambda})}$, it is essential to use the following Langton's type theorem.

Lemma 6.1. *Let R be a discrete valuation ring with residue field $k = R/m$ and quotient field K . Take a flat family $(E, \nabla, \{l_j^{(i)}\})$ of parabolic connections on $C \times \text{Spec } R$ over $\text{Spec } R$ whose generic fiber $(E, \nabla, \{l_j^{(i)}\}) \otimes K$ is α -semistable. Then we can obtain, by repeating elementary transformations of $(E, \nabla, \{l_j^{(i)}\})$ along the special fiber $C \times \text{Spec } k$, a flat family $(E', \nabla', \{l'_j{}^{(i)}\})$ of parabolic connections on $C \times \text{Spec } R$ over $\text{Spec } R$ such that $(E', \nabla', \{l'_j{}^{(i)}\}) \otimes K \cong (E, \nabla, \{l_j^{(i)}\}) \otimes K$ and $(E', \nabla', \{l'_j{}^{(i)}\}) \otimes k$ is α -semistable.*

Proof. This is just a corollary of the proof of [7, Proposition 5.5]. $\square$

Proposition 6.1. *Assume that $n \geq 3$ if $g = 0$ and that $n \geq 1$ if $g \geq 1$. Then the morphism $\mathbf{RH}_{(x, \boldsymbol{\lambda})} : M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}) \rightarrow RP_r(C, \mathbf{t})_{\mathbf{a}}$ is surjective for any $\boldsymbol{\lambda} \in \Lambda_r^{(n)}(d)$ satisfying $rh(\boldsymbol{\lambda}) = \mathbf{a}$.*

Proof. By Proposition 3.1, we obtain an isomorphism

$$(11) \quad \sigma : \mathcal{M}_C(\mathbf{t}, \boldsymbol{\lambda}) \xrightarrow{\sim} \mathcal{M}_C(\mathbf{t}, \boldsymbol{\mu})$$

of functors, where $0 \leq \text{Re}(\mu_j^{(i)}) < 1$ for any i, j . Note that σ induces an isomorphism

$$M_C^{\text{irr}}(\mathbf{t}, \boldsymbol{\lambda}) \xrightarrow[\sim]{\sigma} M_C^{\text{irr}}(\mathbf{t}, \boldsymbol{\mu}),$$

where $M_C^{\text{irr}}(\mathbf{t}, \boldsymbol{\lambda})$ (resp. $M_C^{\text{irr}}(\mathbf{t}, \boldsymbol{\mu})$) is the moduli space of irreducible $(\mathbf{t}, \boldsymbol{\lambda})$ -parabolic connections (resp. $(\mathbf{t}, \boldsymbol{\mu})$ -parabolic connections). By construction, we have $\mathbf{RH}_{(x, \boldsymbol{\mu})} \circ \sigma|_{M_C^{\text{irr}}(\mathbf{t}, \boldsymbol{\lambda})} = \mathbf{RH}_{(x, \boldsymbol{\lambda})}|_{M_C^{\text{irr}}(\mathbf{t}, \boldsymbol{\lambda})}$.

Consider the restriction

$$\mathbf{RH}_{(x, \boldsymbol{\lambda})} : M_C^{\text{irr}}(\mathbf{t}, \boldsymbol{\lambda}) \longrightarrow RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{irr}},$$

of $\mathbf{RH}$, where $M_C^{\text{irr}}(\mathbf{t}, \boldsymbol{\lambda})$ is the moduli space of irreducible parabolic connections and $RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{irr}}$ is the moduli space of irreducible representations. For any point $\rho \in RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{irr}}$, ρ corresponds to a holomorphic vector bundle E°

on $C \setminus \{t_1, \dots, t_n\}$ and a holomorphic connection $\nabla^\circ : E^\circ \rightarrow E^\circ \otimes \Omega_{C \setminus \{t_1, \dots, t_n\}}^1$. By [4, II, Proposition 5.4], there is a unique pair (E, ∇) of a holomorphic vector bundle E on C and a logarithmic connection $\nabla : E \rightarrow E \otimes \Omega_C^1(t_1 + \dots + t_n)$ such that $(E, \nabla)|_{C \setminus \{t_1, \dots, t_n\}} \cong (E^\circ, \nabla^\circ)$ and that all the eigenvalues of $\text{res}_{t_i}(\nabla)$ lie in $\{z \in \mathbf{C} \mid 0 \leq \text{Re}(z) < 1\}$. We can take a parabolic structure $\{l_j^{(i)}\}$ on E which is compatible with the connection ∇ . Then $(E, \nabla, \{l_j^{(i)}\})$ becomes a $(\mathbf{t}, \boldsymbol{\mu})$ -parabolic connection such that $\mathbf{RH}_{(x, \boldsymbol{\mu})}(E, \nabla, \{l_j^{(i)}\}) = \rho$. Thus we have $\sigma^{-1}(E, \nabla, \{l_j^{(i)}\}) \in M_C^{\text{irr}}(\mathbf{t}, \boldsymbol{\lambda}) \subset M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda})$ (note that an irreducible parabolic connection is automatically α -stable) and $\mathbf{RH}_{(x, \boldsymbol{\lambda})}(\sigma^{-1}(E, \nabla, \{l_j^{(i)}\})) = \rho$.

So it is sufficient to show that any member $[\rho] \in RP_r(C, \mathbf{t})_{\mathbf{a}}$ corresponding to a reducible representation is contained in the image of $\mathbf{RH}_{(x, \boldsymbol{\lambda})}$, in order to prove the surjectivity of $\mathbf{RH}_{(x, \boldsymbol{\lambda})}$. The fundamental group $\pi_1(C \setminus \{t_1, \dots, t_n\}, *)$ is generated by cycles $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ and loops γ_i around t_i for $i = 1, \dots, n$ whose relation is given by

$$\prod_{i=1}^g \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1} \prod_{j=1}^n \gamma_j = 1.$$

Since $n > 0$, the fundamental group is isomorphic to the free group generated by the free generators $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \gamma_1, \dots, \gamma_{n-1}$. So the space $\text{Hom}(\pi_1(C \setminus \{t_1, \dots, t_n\}, *), GL_r(\mathbf{C}))$ of representations is isomorphic to $GL_r(\mathbf{C})^{2g+n-1}$ and it is irreducible. By Proposition 4.2 and Remark 4.1, the subset of the points of $GL_r(\mathbf{C})^{2g+n-1}$ corresponding to the irreducible representations is a non-empty Zariski open subset. Thus there exists a smooth affine curve T and a morphism $f : T \rightarrow \text{Hom}(\pi_1(C \setminus \{t_1, \dots, t_n\}, *), GL_r(\mathbf{C}))$ such that the image $f(p_0)$ of a special point $p_0 \in T$ corresponds to the representation ρ and the image $f(\eta)$ of $\eta \in T \setminus \{p_0\}$ corresponds to an irreducible representation. Since the morphism f corresponds to a family of representations of the fundamental group $\pi_1(C \setminus \{t_1, \dots, t_n\}, *)$, we can construct a holomorphic vector bundle $\tilde{E}^\circ$ on $(C \setminus \{t_1, \dots, t_n\}) \times T$ and a holomorphic connection $\tilde{\nabla}^\circ : \tilde{E}^\circ \rightarrow \tilde{E}^\circ \otimes \Omega_{(C \setminus \{t_1, \dots, t_n\}) \times T/T}^1$ such that $(\tilde{E}^\circ, \tilde{\nabla}^\circ)|_{(C \setminus \{t_1, \dots, t_n\}) \times \eta}$ corresponds to the representation $f(\eta)$ which is irreducible for $\eta \in T \setminus \{p_0\}$ and $(\tilde{E}^\circ, \tilde{\nabla}^\circ)|_{(C \setminus \{t_1, \dots, t_n\}) \times p_0}$ corresponds to the representation ρ . By the relative version of [4, II, Proposition 5.4], we can construct, after replacing T by an analytic open neighborhood of p_0 , a holomorphic vector bundle $\tilde{E}$ on $C \times T$ and a relative connection $\tilde{\nabla} : \tilde{E} \rightarrow \tilde{E} \otimes \Omega_C^1(t_1 + \dots + t_n)$ such that $(\tilde{E}, \tilde{\nabla})|_{(C \setminus \{t_1, \dots, t_n\}) \times T} \cong (\tilde{E}^\circ, \tilde{\nabla}^\circ)$ and that all the eigenvalues of $\text{res}_{t_i}(\tilde{\nabla} \otimes k(t))$ lie in $\{z \in \mathbf{C} \mid -\epsilon < \text{Re}(z) < 1 - \epsilon\}$ for any $t \in T$, where $\epsilon > 0$ is a sufficiently small positive real number which depends on $\boldsymbol{\mu}$. We can take

a flat family of parabolic structures $\{\tilde{l}_j^{(i)}\}$ on $\tilde{E}$ which is compatible with $\tilde{\nabla}$ and $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ becomes a flat family of parabolic connections over T .

We denote by (An) the category of analytic spaces. Let $\mathcal{M}_C(\mathbf{t}, r, d)^{an} : (An) \rightarrow (Sets)$ be the functor defined by

$$\mathcal{M}_C(\mathbf{t}, r, d)^{an}(S) := \left\{ (E, \nabla, \{l_j^{(i)}\}) \left| \begin{array}{l} E \text{ is a holomorphic vector bundle of rank } r \\ \text{on } C \times S, \\ \nabla : E \rightarrow E \otimes \Omega_{C \times S/S}^1((t_1 + t_2 + \cdots + t_n)_S) \\ \text{is a relative connection and} \\ E|_{t_i \times S} = l_0^{(i)} \supset l_1^{(i)} \supset \cdots \supset l_{r-1}^{(i)} \supset l_r^{(i)} = 0 \\ \text{is a filtration by subbundles such that} \\ \text{res}_{t_i \times S}(\nabla)(l_j^{(i)}) \subset l_j^{(i)} \text{ for any } i, j \\ \text{and for any } s \in S, \deg(E|_{C \times \{s\}}) = d \\ \text{and } \dim(l_j^{(i)}/l_{j+1}^{(i)})|_s = 1 \text{ for any } i, j \end{array} \right. \right\} / \sim$$

for an analytic space S , where $(E, \nabla, \{l_j^{(i)}\}) \sim (E', \nabla', \{l'_j{}^{(i)}\})$ if there is a line bundle $\mathcal{L}$ on S and an isomorphism $g : E \xrightarrow{\sim} E' \otimes \mathcal{L}$ such that $g|_{t_i}(l_j^{(i)}) = l'_j{}^{(i)}$ for any i, j and the diagram

$$\begin{array}{ccc} E & \xrightarrow{\nabla} & E \otimes \Omega_{C \times S/S}^1((t_1 + \cdots + t_n)_S) \\ g \downarrow \cong & & g \otimes \text{id} \downarrow \cong \\ E' \otimes \mathcal{L} & \xrightarrow{\nabla' \otimes \mathcal{L}} & E' \otimes \mathcal{L} \otimes \Omega_{C \times S/S}^1((t_1 + \cdots + t_n)_S) \end{array}$$

is commutative. We can also define a functor $\mathcal{M}_C(\mathbf{t}, \boldsymbol{\lambda})^{an} : (An) \rightarrow (Sets)$ similarly.

There are canonical morphisms $\pi_{\Lambda_r^{(n)}(d)} : \mathcal{M}_C(\mathbf{t}, r, d)^{an} \rightarrow \Lambda_r^{(n)}(d)$ and $\pi_{\Lambda_r^{(n)}(d')} : \mathcal{M}_C(\mathbf{t}, r, d')^{an} \rightarrow \Lambda_r^{(n)}(d')$. Taking a sufficiently small open neighborhood Δ of $\boldsymbol{\lambda}$ in $\Lambda_r^{(n)}(d)$ and an open neighborhood Δ' of $\boldsymbol{\mu}$ in $\Lambda_r^{(n)}(d')$, σ extends canonically to an isomorphism

$$\tilde{\sigma} : \pi_{\Lambda_r^{(n)}(d)}^{-1}(\Delta) \xrightarrow{\sim} \pi_{\Lambda_r^{(n)}(d')}^{-1}(\Delta')$$

which makes the diagram

$$\begin{array}{ccc} \pi_{\Lambda_r^{(n)}(d)}^{-1}(\Delta) & \longrightarrow & \Delta \\ \tilde{\sigma} \downarrow \cong & & \cong \downarrow \\ \pi_{\Lambda_r^{(n)}(d')}^{-1}(\Delta') & \longrightarrow & \Delta' \end{array}$$

commute. Replacing T by an open neighborhood of p_0 , we have $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\}) \in \pi_{\Lambda_r^{(n)}(d')}^{-1}(\Delta')(T)$. Then we have $\tilde{\sigma}^{-1}(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\}) \in \pi_{\Lambda_r^{(n)}(d)}^{-1}(\Delta)(T) \subset \mathcal{M}_C(\mathbf{t}, r, d)^{an}(T)$. For a sufficiently large integer n_0 , $H^1(\tilde{\sigma}^{-1}(\tilde{E})_t(n_0)) = 0$ and $H^0(\tilde{\sigma}^{-1}(\tilde{E})_t(n_0)) \otimes \mathcal{O}_C(-n_0) \rightarrow \tilde{\sigma}^{-1}(\tilde{E})_t$ is surjective for any $t \in T$. Let $\mathcal{M}'_C(\mathbf{t}, r, d)$ be the subfunctor of $\mathcal{M}_C(\mathbf{t}, r, d)$ defined by

$$\mathcal{M}'_C(\mathbf{t}, r, d)(S) = \left\{ (E, \nabla, \{l^{(i)}\}) \in \mathcal{M}_C(\mathbf{t}, r, d)(S) \left| \begin{array}{l} \text{for any } t \in S, H^1(E_t(n_0)) = 0 \text{ and} \\ H^0(E_t(n_0)) \otimes \mathcal{O}_C(-n_0) \rightarrow E_t \\ \text{is surjective} \end{array} \right. \right\}.$$

Then $\mathcal{M}'_C(\mathbf{t}, r, d) \hookrightarrow \mathcal{M}_C(\mathbf{t}, r, d)$ is representable by an open immersion and we have $\tilde{\sigma}^{-1}(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\}) \in \mathcal{M}'_C(\mathbf{t}, r, d)^{an}(T)$. By the same argument as the proof of Theorem 2.1, there is an algebraic scheme P on which PGL_N acts and there is a flat family $(F, \tilde{\nabla}_F, \{l_{F,j}^{(i)}\})$ of parabolic connections on $C \times P$ over P which induces PGL_N -equivariant morphism $p : P \rightarrow \mathcal{M}'_C(\mathbf{t}, r, d)$ such that the induced morphism $(h_P/h_{PGL_N})^{Zar} \rightarrow \mathcal{M}'_C(\mathbf{t}, r, d)^{Zar}$ is isomorphic, where $(h_P/h_{PGL_N})^{Zar}$ (resp. $\mathcal{M}'_C(\mathbf{t}, r, d)^{Zar}$) is the sheafification of h_P/h_{PGL_N} (resp. $\mathcal{M}'_C(\mathbf{t}, r, d)$) in Zariski topology. We can see that the canonical morphism $(h_{Pan}/h_{PGL_N^{an}})^\sim \rightarrow (\mathcal{M}'_C(\mathbf{t}, r, d)^{an})^\sim$ is also an isomorphism, where $(h_{Pan}/h_{PGL_N^{an}})^\sim$ (resp. $(\mathcal{M}'_C(\mathbf{t}, r, d)^{an})^\sim$) is the analytic sheafification of $h_{Pan}/h_{PGL_N^{an}}$ (resp. $\mathcal{M}'_C(\mathbf{t}, r, d)^{an}$). Note that $\tilde{\sigma}^{-1}(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\}) \in \mathcal{M}'_C(\mathbf{t}, r, d)^{an}(T) \subset (\mathcal{M}'_C(\mathbf{t}, r, d)^{an})^\sim(T)$. Since the morphism $h_{Pan} \rightarrow (\mathcal{M}'_C(\mathbf{t}, r, d)^{an})^\sim$ is surjective as sheaves in analytic topology, there is a morphism $g : T \rightarrow P$ such that $g^*(F, \tilde{\nabla}_F, \{l_{F,j}^{(i)}\})^{an} \cong \tilde{\sigma}^{-1}(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ after replacing T by an open neighborhood of p_0 . There is a ring homomorphism $g^* : \mathcal{O}_{P,g(p_0)}^{an} \rightarrow \mathcal{O}_{T,p_0}^{an} \cong \mathbf{C}\{z\}$, where $\mathbf{C}\{z\}$ is the convergent power series ring. Then we have

$$(F, \tilde{\nabla}_F, \{l_{F,j}^{(i)}\}) \otimes_{\mathcal{O}_{P,g(p_0)}} \mathcal{O}_{T,p_0}^{an} \in \mathcal{M}'_C(\mathbf{t}, r, d)(\mathcal{O}_{T,p_0}^{an}) = \mathcal{M}'_C(\mathbf{t}, r, d)(\mathbf{C}\{z\}).$$

By Lemma 6.1, we can obtain a flat family $(\tilde{E}', \tilde{\nabla}', \{\tilde{l}'_j^{(i)}\})$ of α -semistable parabolic connections over $\mathbf{C}\{z\}$ such that $(\tilde{E}', \tilde{\nabla}', \{\tilde{l}'_j^{(i)}\}) \otimes_{\mathbf{C}\{z\}} \mathbf{C}\{z\}[1/z] \cong (F, \tilde{\nabla}_F, \{l_{F,j}^{(i)}\}) \otimes_{\mathcal{O}_{P,g(p_0)}} \mathbf{C}\{z\}[1/z]$. Since we take α so generic that α -semistable $\Leftrightarrow \alpha$ -stable, we have $(\tilde{E}', \tilde{\nabla}', \{\tilde{l}'_j^{(i)}\}) \in M_C^\alpha(\mathbf{t}, r, d)(\mathbf{C}\{z\})$. Let $h : \text{Spec } \mathbf{C}\{z\} \rightarrow M_C^\alpha(\mathbf{t}, r, d)$ be the induced morphism and $h^* : \mathcal{O}_{M_C^\alpha(\mathbf{t}, r, d), h(q_0)} \rightarrow \mathbf{C}\{z\}$ be the corresponding ring homomorphism, where q_0 is the closed point of $\text{Spec } \mathbf{C}\{z\}$. Then, replacing T by an open neighborhood of p_0 , we have a morphism $h^{an} : T \rightarrow M_C^\alpha(\mathbf{t}, r, d)$ such that $(h^{an})^* = h^* : \mathcal{O}_{M_C^\alpha(\mathbf{t}, r, d), h(q_0)} \rightarrow$

$\mathcal{O}_{T,p_0}^{an} = \mathbf{C}\{z\}$. By construction, we have $h^{an}(p_0) = h(q_0)$ and the diagram

$$\begin{array}{ccc} T \setminus \{p_0\} & \xrightarrow{f} & \mathrm{Hom}(\pi_1(C \setminus \{t_1, \dots, t_n\}, *), GL_r(\mathbf{C})) \\ h^{an} \downarrow & & \downarrow \\ M^\alpha(\mathbf{t}, r, d) & \xrightarrow{\mathbf{RH}} & RP_r(C, \mathbf{t}) \end{array}$$

is commutative after replacing T by an open neighborhood of p_0 . Since $RP_r(C, \mathbf{t})$ is separated over $\mathbf{C}$, we have $\mathbf{RH}(h^{an}(p_0)) = [\rho]$. Hence $\mathbf{RH}_{(x, \lambda)}$ is surjective. $\square$

Remark 6.1. It was not obvious in Propositions 5.1, 5.2 and 5.3 that the moduli space $M_C^\alpha(\mathbf{t}, \lambda, L)$ is non-empty. However, we can see now that it is in fact non-empty in the following way. We first take a simple parabolic bundle $(E, \{l_j^{(i)}\})$ of rank r on C such that $\det E \cong L$. For the existence of such a parabolic bundle, we may assume, by taking certain elementary transform, that r and $\deg L$ are coprime. Then the existence for $g \geq 1$ follows because the moduli space of stable bundles of rank r with the fixed determinant L on a curve of genus $g \geq 1$ is non-empty. For the case of $C = \mathbf{P}^1$, we may assume that $E = \mathcal{O}_{\mathbf{P}^1}^{\oplus r}$ and we can take a parabolic structure $\{l_j^{(i)}\}$ on E such that $\mathrm{End}(E, \{l_j^{(i)}\}) \cong \mathbf{C}$ because $n \geq 3$. We want to construct a connection $\nabla : E \rightarrow E \otimes \Omega_C^1(t_1 + \dots + t_n)$ such that $(E, \nabla, \{l_j^{(i)}\})$ becomes a $(\mathbf{t}, \lambda)$ -parabolic connection with the determinant (L, ∇_L) . The obstruction class for the existence of such a connection lies in $H^1(\mathcal{F}_0^1)$, where

$$\mathcal{F}_0^1 = \left\{ s \in \mathcal{E}nd(E) \otimes \Omega_C^1(t_1 + \dots + t_n) \mid \mathrm{Tr}(s) = 0 \text{ and } \mathrm{res}_{t_i}(a)(l_j^{(i)}) \subset l_{j+1}^{(i)} \text{ for any } i, j \right\}.$$

If we put

$$\mathcal{F}_0^0 := \left\{ s \in \mathcal{E}nd(E) \mid \mathrm{Tr}(s) = 0 \text{ and } s|_{t_i}(l_j^{(i)}) \subset l_j^{(i)} \text{ for any } i, j \right\},$$

then we have an isomorphism $H^1(\mathcal{F}_0^1) \cong H^0(\mathcal{F}_0^0)^\vee$. Since $\mathrm{End}(E, \{l_j^{(i)}\}) \cong \mathbf{C}$, we have $H^0(\mathcal{F}_0^0) = 0$ and so $H^1(\mathcal{F}_0^1) = 0$. Thus there is a connection $\nabla : E \rightarrow E \otimes \Omega_C^1(t_1 + \dots + t_n)$ such that $(E, \nabla, \{l_j^{(i)}\})$ becomes a $(\mathbf{t}, \lambda)$ -parabolic connection. If this connection is irreducible, then $(E, \nabla, \{l_j^{(i)}\})$ becomes α -stable and the moduli space $M_C^\alpha(\mathbf{t}, \lambda, L)$ becomes non-empty. If $(E, \nabla, \{l_j^{(i)}\})$ is reducible, then we take the representation ρ of $\pi_1(C \setminus \{t_1, \dots, t_n\}, *)$ which corresponds to $\ker(\nabla^{an}|_{C \setminus \{t_1, \dots, t_n\}})$. Then $[\rho]$ determines a point of $RP_r(C, \mathbf{t})_{\mathbf{a}}$,

where $rh(\lambda) = \mathbf{a}$. Since $\mathbf{RH}_{(x,\lambda)}$ is surjective, we can take an α -stable $(\mathbf{t}, \lambda)$ -parabolic connection $(E', \nabla', \{l_j^{(i)}\})$ with the determinant (L, ∇_L) such that $\mathbf{RH}_{(x,\lambda)}(E', \nabla', \{l_j^{(i)}\}) = [\rho]$. Hence we have $M_C^\alpha(\mathbf{t}, \lambda, L) \neq \emptyset$.

Proposition 6.2. *Every fiber of $\mathbf{RH}_{(x,\lambda)} : M_C^\alpha(\mathbf{t}, \lambda) \rightarrow RP_r(C, \mathbf{t})_{\mathbf{a}}$ is compact.*

Proof. Take any irreducible representation $\rho \in RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{irr}}$. We want to show that the fiber $\mathbf{RH}_{(x,\lambda)}^{-1}(\rho)$ in $M_C^\alpha(\mathbf{t}, \lambda)$ is compact. By Proposition 3.1, we can obtain an isomorphism

$$\sigma : \mathcal{M}_C(\mathbf{t}, \lambda) \xrightarrow{\sim} \mathcal{M}_C(\mathbf{t}, \mu),$$

of functors, where $0 \leq \text{Re}(\mu_j^{(i)}) < 1$ for any i, j . Then σ induces an isomorphism

$$\sigma : M_C^{\text{irr}}(\mathbf{t}, \lambda) \xrightarrow{\sim} M_C^{\text{irr}}(\mathbf{t}, \mu)$$

between the moduli spaces of irreducible parabolic connections. Note that $M_C^{\text{irr}}(\mathbf{t}, \lambda) \subset M_C^\alpha(\mathbf{t}, \lambda)$. So σ also induces an isomorphism

$$\mathbf{RH}_{(x,\lambda)}^{-1}(\rho) \xrightarrow{\sim} \mathbf{RH}_{(x,\mu)}^{-1}(\rho).$$

By [4, II, Proposition 5.4], there is a unique pair (E, ∇_E) of a bundle E on C and a logarithmic connection $\nabla_E : E \rightarrow E \otimes \Omega_C^1(t_1 + \cdots + t_n)$ such that the local system $\ker(\nabla_E^{an})|_{C \setminus \{t_1, \dots, t_n\}}$ corresponds to the representation ρ and all the eigenvalues of $\text{res}_{t_i}(\nabla_E)$ lie in $\{z \in \mathbf{C} | 0 \leq \text{Re}(z) < 1\}$. Then the fiber $\mathbf{RH}_{(x,\mu)}^{-1}(\rho)$ is just the moduli space of parabolic structures $\{l_j^{(i)}\}$ on E which satisfy $(\text{res}_{t_i}(\nabla_E) - \mu_j^{(i)})(l_j^{(i)}) \subset l_{j+1}^{(i)}$ for any i, j . So $\mathbf{RH}_{(x,\mu)}^{-1}(\rho)$ becomes a Zariski closed subset of a product of flag varieties. Thus $\mathbf{RH}_{(x,\mu)}^{-1}(\rho)$ is compact. Since the fiber $\mathbf{RH}_{(x,\lambda)}^{-1}(\rho)$ of $\mathbf{RH}_{(x,\lambda)}$ over ρ in $M_C^\alpha(\mathbf{t}, \lambda)$ is isomorphic to $\mathbf{RH}_{(x,\mu)}^{-1}(\rho)$ via σ , it is also compact.

Now take a reducible representation $[\rho] \in RP_r(C, \mathbf{t})_{\mathbf{a}}$. We may assume $\rho = \rho_1 \oplus \cdots \oplus \rho_s$ with each ρ_p irreducible. We will show that $\mathbf{RH}_{(x,\lambda)}^{-1}([\rho])$ is an algebraic constructible subset of $M_C^\alpha(\mathbf{t}, \lambda)$. For this, we will construct an algebraic scheme Z over $\mathbf{C}$ and a morphism $f : Z \rightarrow M_C^\alpha(\mathbf{t}, \lambda)$ such that $f(Z) = \mathbf{RH}^{-1}([\rho])$. By Proposition 3.1, we can get an isomorphism

$$\varphi : \mathcal{M}_C(\mathbf{t}, \lambda) \longrightarrow \mathcal{M}_C(\mathbf{t}, \mu)$$

of functors such that $0 \leq \text{Re}(\mu_j^{(i)}) < 1$ for any i, j and $\mathbf{RH}_{(x,\lambda)}(e) = (\mathbf{RH}_{(x,\mu)} \circ \varphi)(e)$ for any $e \in \mathcal{M}_C(\mathbf{t}, \lambda)(\mathbf{C})$. Take any member $e \in M_C^\alpha(\mathbf{t}, \lambda)$ such that $\mathbf{RH}_{(x,\lambda)}(e) = [\rho]$. Put $\varphi(e) = (E, \nabla, \{l_j^{(i)}\})$. Then there is a filtration $0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E$ by subbundles such that $\nabla(F_k) \subset F_k \otimes \Omega_C^1(t_1 + \cdots + t_n)$ and for the induced connections $\nabla_k : F_k/F_{k-1} \rightarrow F_k/F_{k-1} \otimes \Omega_C^1(t_1 + \cdots + t_n)$, there is a permutation σ of $1, \dots, s$ such that

$\ker \nabla_k^{an}|_{C \setminus \{t_1, \dots, t_n\}}$ corresponds to $\rho_{\sigma(k)}$ for any k . Note that $(F_k/F_{k-1}, \nabla_k)$ is uniquely determined by $\rho_{\sigma(k)}$ because $\rho_{\sigma(k)}$ is irreducible and all the eigenvalues of $\text{res}_{t_i}(\nabla_k)$ lie in $\{z \in \mathbf{C} | 0 \leq \text{Re}(z) < 1\}$. Now we will construct Z and f . For each k with $1 \leq k \leq s$, there is a unique pair (E_k, ∇_k) of vector bundle E_k with a connection $\nabla_k : E_k \rightarrow E_k \otimes \Omega_C^1(t_1 + \dots + t_n)$ such that $\ker \nabla_k^{an}|_{C \setminus \{t_1, \dots, t_n\}}$ corresponds to ρ_k and all the eigenvalues of $\text{res}_{t_i}(\nabla)$ lie in $\{z \in \mathbf{C} | 0 \leq \text{Re}(z) < 1\}$ for $i = 1, \dots, n$. Take a permutation σ of $(1, \dots, s)$. Put $Y_1 := \mathbf{V}_*(\text{Ext}_C^1(E_{\sigma(2)}, E_{\sigma(1)}))$ and let

$$0 \longrightarrow (E_{\sigma(1)})_{Y_1} \longrightarrow F_2^{(\sigma)} \longrightarrow (E_{\sigma(2)})_{Y_1} \longrightarrow 0$$

be the universal extension. We may assume that $\text{Ext}_{C \times Y_1/Y_1}^1((E_{\sigma(3)})_{Y_1}, F_2^{(\sigma)})$ is a locally free sheaf on Y_1 by replacing Y_1 by a disjoint union of locally closed subschemes. Inductively we put

$$Y_k := \mathbf{V}_*(\text{Ext}_{C \times Y_{k-1}/Y_{k-1}}^1((E_{\sigma(k+1)})_{Y_{k-1}}, F_k^{(\sigma)}))$$

and let

$$0 \longrightarrow (F_k^{(\sigma)})_{Y_k} \longrightarrow F_{k+1}^{(\sigma)} \longrightarrow (E_{\sigma(k+1)})_{Y_k} \longrightarrow 0$$

be the universal extension. We may assume by replacing Y_k by a disjoint union of locally closed subschemes that $\text{Ext}_{C \times Y_k/Y_k}^1((E_{\sigma(k+2)})_{Y_k}, F_{k+1}^{(\sigma)})$ is a locally free sheaf on Y_k . There is a universal extension

$$0 \longrightarrow (F_{r-1}^{(\sigma)})_{Y_{r-1}} \longrightarrow F_r^{(\sigma)} \longrightarrow (E_{\sigma(r)})_{Y_{r-1}} \longrightarrow 0$$

and we obtain a filtration

$$0 =: F_0^{(\sigma)} \subset F_1^{(\sigma)} := (E_{\sigma(1)})_{Y_{r-1}} \subset (F_2^{(\sigma)})_{Y_{r-1}} \subset \dots \subset (F_{r-1}^{(\sigma)})_{Y_{r-1}} \subset F_r^{(\sigma)}.$$

Consider the functor $\text{Conn}_{C \times Y_{r-1}/Y_{r-1}}$ from the category of algebraic schemes over $\mathbf{C}$ to the category of sets defined by

$$\text{Conn}_{C \times Y_{r-1}/Y_{r-1}}(S) := \left\{ \begin{array}{l} \nabla : F_r^{(\sigma)} \rightarrow F_r^{(\sigma)} \otimes \Omega_C^1(D(\mathbf{t})) \\ \nabla \text{ is a relative connection,} \\ \nabla((F_k^{(\sigma)})_S) \subset (F_k^{(\sigma)})_S \otimes \Omega_C^1(D(\mathbf{t})) \\ \text{for } 1 \leq k \leq r-1 \text{ and for} \\ \text{the connection} \\ \nabla_k^{(\sigma)} \text{ on } (F_k^{(\sigma)})_S / (F_{k-1}^{(\sigma)})_S \\ \text{induced by } \nabla, \\ \nabla_k^{(\sigma)} = (\nabla_{\sigma(k)})_S \text{ for } 1 \leq k \leq s-1 \end{array} \right\}.$$

Then we can see that $\text{Conn}_{Y_{r-1}}$ can be represented by an affine scheme $Y^{(\sigma)}$ over Y_{r-1} . Let

$$\nabla : (F_r^{(\sigma)})_{Y^\sigma} \longrightarrow (F_r^{(\sigma)})_{Y^\sigma} \otimes \Omega_C^1(t_1 + \dots + t_n)$$

be the universal connection. The set of parabolic structures $\{l_j^{(i)}\}$ on $(F_r^{(\sigma)}) \otimes k(y)$ ($y \in Y^{(\sigma)}$) such that $(\text{res}_i(\nabla \otimes k(y)) - \mu_j^{(i)})(l_j^{(i)}) \subset l_{j+1}^{(i)}$ for any i, j can be parametrized by a closed subscheme $Z^{(\sigma)}$ of a product of flag schemes over Y . Let $\{(l^{(\sigma)})_j^{(i)}\}$ be the universal parabolic structure on $(F_r^{(\sigma)})_{Z^{(\sigma)}}$. Put

$$\tilde{Z}^{(\sigma)} := \{z \in Z^{(\sigma)} \mid \varphi^{-1}(F_r^{(\sigma)} \otimes k(z), \nabla \otimes k(z), \{\tilde{l}_j^{(i)} \otimes k(s)\}) \text{ is } \alpha\text{-stable}\}.$$

Then $\tilde{Z}^{(\sigma)}$ is a Zariski open subset of $Z^{(\sigma)}$ and induces a morphism $f_\sigma : \tilde{Z}^{(\sigma)} \rightarrow M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda})$. By construction we have $\mathbf{RH}_{(x, \boldsymbol{\lambda})}^{-1}([\rho]) = \bigcup_\sigma f_\sigma(\tilde{Z}^{(\sigma)})$. Thus we can see that $\mathbf{RH}_{(x, \boldsymbol{\lambda})}^{-1}([\rho])$ is a constructible subset of $M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda})$ by Chevalley's Theorem.

So it is sufficient to show that $\mathbf{RH}_{(x, \boldsymbol{\lambda})}^{-1}([\rho])$ is stable under specialization in the compact moduli space $\overline{M_C^{D(\mathbf{t}), \alpha', \beta, \gamma}}(r, d, \{d_i\}_{1 \leq i \leq nr})$, in order to prove the compactness of $\mathbf{RH}_{(x, \boldsymbol{\lambda})}^{-1}([\rho])$. Here $\overline{M_C^{D(\mathbf{t}), \alpha', \beta, \gamma}}(r, d, \{d_i\}_{1 \leq i \leq nr})$ is the moduli scheme of (α', β, γ) -stable parabolic $\Lambda_{D(\mathbf{t})}^1$ -triples appearing in Theorem 4.1, which contains $M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda})$ as a locally closed subscheme. Take any scheme point $x_1 \in \mathbf{RH}_{(x, \boldsymbol{\lambda})}^{-1}([\rho])$ and a point x_0 of the closure $\overline{\{x_1\}}$ of $\{x_1\}$ in $\overline{M_C^{D(\mathbf{t}), \alpha', \beta, \gamma}}(r, d, \{d_i\}_{1 \leq i \leq nr})$. Let K be the residue field $k(x_1)$ of x_1 . Then there exists a discrete valuation ring R with quotient field K which dominates $\mathcal{O}_{\{x_1\}, x_0}$. The inclusion $\mathcal{O}_{\{x_1\}, x_0} \hookrightarrow R$ induces a flat family $(\tilde{E}_1, \tilde{E}_2, \tilde{\Phi}, F_*(\tilde{E}_1))$ of parabolic $\Lambda_{D(\mathbf{t})}^1$ -triples on $C \times \text{Spec } R$ over R . Here the left $\mathcal{O}_{C \times \text{Spec } R}$ -homomorphism $\tilde{\Phi} : \Lambda_{D(\mathbf{t})}^1 \otimes \tilde{E}_1 \rightarrow \tilde{E}_2$ corresponds to a pair $(\tilde{\phi}, \tilde{\nabla})$ of an $\mathcal{O}_{C \times \text{Spec } R}$ -homomorphism $\tilde{\phi} : \tilde{E}_1 \rightarrow \tilde{E}_2$ and a morphism $\tilde{\nabla} : \tilde{E}_1 \rightarrow \tilde{E}_2 \otimes \Omega_C^1(t_1 + \cdots + t_n)$ satisfying $\tilde{\nabla}(as) = \phi(s) \otimes da + a\tilde{\nabla}(s)$ for $a \in \mathcal{O}_{C \times \text{Spec } R}$ and $s \in \tilde{E}_1$. We also denote $(\tilde{E}_1, \tilde{E}_2, \tilde{\phi}, \tilde{\nabla}, F_*(\tilde{E}_1))$ by $(\tilde{E}_1, \tilde{E}_2, \tilde{\phi}, \tilde{\nabla}, F_*(\tilde{E}_1))$. Let η be the generic point of $\text{Spec } R$ and ξ be the closed point of $\text{Spec } R$ and put

$$\begin{aligned} (E_1, E_2, \phi, \nabla, F_*(E_1)) &:= (\tilde{E}_1, \tilde{E}_2, \tilde{\phi}, \tilde{\nabla}, F_*(\tilde{E}_1)) \otimes K \\ (E'_1, E'_2, \phi', \nabla', F_*(E'_1)) &:= (\tilde{E}_1, \tilde{E}_2, \tilde{\phi}, \tilde{\nabla}, F_*(\tilde{E}_1)) \otimes k(\xi). \end{aligned}$$

Note that ϕ is isomorphic and $\nabla_E = (\phi)^{-1} \circ \nabla$ becomes a connection on $E := E_1$. There exists a filtration

$$(0, 0) = (F^{(0)}, \nabla^{(0)}) \subset (F^{(1)}, \nabla^{(1)}) \subset \cdots \subset (F^{(s)}, \nabla^{(s)}) = (E, \nabla_E) \otimes_K K'$$

such that K' is a finite extension field of K and that each $(F^{(p)}/F^{(p-1)}, \overline{\nabla^{(p)}}) \otimes_{K'} \overline{K'}$ is irreducible, where $\overline{\nabla^{(p)}}$ is the connection on $F^{(p)}/F^{(p-1)}$ induced by $\nabla^{(p)}$. We may assume that $(F^{(p)}/F^{(p-1)}, \overline{\nabla^{(p)}})$ corresponds to the representation ρ_p . Note that there is a discrete valuation ring R' with quotient field

K' which dominates R . Let ξ' be the closed point of $\text{Spec } R'$. The filtration $\{(F^{(p)}, \nabla^{(p)})\}$ induces filtrations

$$\begin{aligned} 0 &= \tilde{F}_1^{(0)} \subset \tilde{F}_1^{(1)} \subset \cdots \subset \tilde{F}_1^{(s)} = \tilde{E}_1 \otimes_R R', \\ 0 &= \tilde{F}_2^{(0)} \subset \tilde{F}_2^{(1)} \subset \cdots \subset \tilde{F}_2^{(s)} = \tilde{E}_2 \otimes_R R', \end{aligned}$$

such that each $\tilde{F}_j^{(p)}/\tilde{F}_j^{(p-1)}$ is flat over R' for $j = 1, 2$ and $\tilde{F}_1^{(p)} \otimes K' = F^{(p)}$, $\tilde{F}_2^{(p)} \otimes K' = (\phi \otimes \text{id})(F^{(p)})$. By construction, we have $(\tilde{\phi} \otimes \text{id})(\tilde{F}_1^{(p)}) \subset \tilde{F}_2^{(p)}$ and $(\tilde{\nabla} \otimes \text{id})(\tilde{F}_1^{(p)}) \subset \tilde{F}_2^{(p)} \otimes \Omega_C^1(t_1 + \cdots + t_n)$. Let

$$\begin{aligned} \overline{\tilde{\phi}^{(p)}} : \tilde{F}_1^{(p)}/\tilde{F}_1^{(p-1)} &\longrightarrow \tilde{F}_2^{(p)}/\tilde{F}_2^{(p-1)} \\ \overline{\tilde{\nabla}^{(p)}} : \tilde{F}_1^{(p)}/\tilde{F}_1^{(p-1)} &\longrightarrow \tilde{F}_2^{(p)}/\tilde{F}_2^{(p-1)} \otimes \Omega_C^1(t_1 + \cdots + t_n) \end{aligned}$$

be the morphisms induced by $\tilde{\phi}$ and $\tilde{\nabla}$. We can construct a parabolic structure on $(F^{(p)}/F^{(p-1)}, \overline{\nabla^{(p)}})$ and obtain $(\mathbf{t}, \boldsymbol{\lambda}^{(p)})$ -parabolic connections of rank r_p and degree d_p . Extending this parabolic structure, we can obtain a flat family of parabolic structures $F_*(\tilde{F}_1^{(p)}/\tilde{F}_1^{(p-1)})$ on $\tilde{F}_1^{(p)}/\tilde{F}_1^{(p-1)}$. Repeating elementary transforms of the flat family

$$(\tilde{F}_1^{(p)}/\tilde{F}_1^{(p-1)}, \tilde{F}_2^{(p)}/\tilde{F}_2^{(p-1)}, \overline{\tilde{\phi}^{(p)}} , \overline{\tilde{\nabla}^{(p)}} , F_*(\tilde{F}_1^{(p)}/\tilde{F}_1^{(p-1)}))$$

along the special fiber $C \times \{\xi'\}$, we obtain a flat family $(\tilde{G}'_1^{(p)}, \tilde{G}'_2^{(p)}, \tilde{\phi}'^{(p)}, \tilde{\nabla}'^{(p)}, F_*(\tilde{G}'_1^{(p)}))$ of $(\boldsymbol{\alpha}'_p, \boldsymbol{\beta}_p, \gamma_p)$ -semistable $\Lambda_{D(\mathbf{t})}^1$ -triples for some weight $(\boldsymbol{\alpha}'_p, \boldsymbol{\beta}_p, \gamma_p)$. We may assume that $(\boldsymbol{\alpha}'_p, \boldsymbol{\beta}_p, \gamma_p)$ -semistable $\Leftrightarrow (\boldsymbol{\alpha}'_p, \boldsymbol{\beta}_p, \gamma_p)$ -stable. This flat family defines a morphism

$$f : \text{Spec } R' \longrightarrow \overline{M_C^{D(\mathbf{t}), \boldsymbol{\alpha}'_p, \boldsymbol{\beta}_p, \gamma_p}}(r_p, d_p, \{d_i^{(p)}\}_{1 \leq i \leq nr_p}).$$

Note that the moduli space $M_C^{\boldsymbol{\alpha}_p}(\mathbf{t}, \boldsymbol{\lambda}^{(p)})$ of $\boldsymbol{\alpha}_p$ -stable $(\mathbf{t}, \boldsymbol{\lambda}^{(p)})$ -parabolic connections for a certain weight $\boldsymbol{\alpha}_p$ is a locally closed subscheme of $\overline{M_C^{D(\mathbf{t}), \boldsymbol{\alpha}'_p, \boldsymbol{\beta}_p, \gamma_p}}(r_p, d_p, \{d_i^{(p)}\}_{1 \leq i \leq nr_p})$ and the image $f(\text{Spec } K')$ is contained in the moduli space $M_C^{\boldsymbol{\alpha}_p}(\mathbf{t}, \boldsymbol{\lambda}^{(p)})$. Since $(\tilde{G}'_1^{(p)}, \tilde{G}'_2^{(p)}, \tilde{\phi}'^{(p)}, \tilde{\nabla}'^{(p)}) \otimes K'$ is equivalent to the irreducible connection $(F^{(p)}/F^{(p-1)}, \overline{\nabla^{(p)}})$ which corresponds to a constant family of the irreducible representation ρ_p , we have $f(\text{Spec } K') \subset \mathbf{RH}_{(x, \boldsymbol{\lambda}^{(p)})}^{-1}(\rho_p)$, where $\mathbf{RH}_{(x, \boldsymbol{\lambda}^{(p)})}$ is the morphism

$$\mathbf{RH}_{(x, \boldsymbol{\lambda}^{(p)})} : M_C^{\boldsymbol{\alpha}_p}(\mathbf{t}, \boldsymbol{\lambda}^{(p)}) \longrightarrow RP_{r_p}(C, \mathbf{t})_{\mathbf{a}^{(p)}}$$

determined by the Riemann-Hilbert correspondence and $rh(\boldsymbol{\lambda}^{(p)}) = \mathbf{a}^{(p)}$. Since $\mathbf{RH}_{(x, \boldsymbol{\lambda}^{(p)})}^{-1}(\rho_p)$ is compact by the previous proof and $f(\text{Spec } K')$ is contained in $\mathbf{RH}_{(x, \boldsymbol{\lambda}^{(p)})}^{-1}(\rho_p)$, the closure of $f(\text{Spec } K')$ must be contained in $\mathbf{RH}_{(x, \boldsymbol{\lambda}^{(p)})}^{-1}(\rho_p)$. In particular, $f(\xi')$ is contained in $\mathbf{RH}_{(x, \boldsymbol{\lambda}^{(p)})}^{-1}(\rho_p) \subset$

$M_C^{\alpha_p}(\mathbf{t}, \boldsymbol{\lambda}^{(p)})$. So the special fiber $(\tilde{G}'_1, \tilde{G}'_2, \tilde{\phi}'^{(p)}, \tilde{\nabla}'^{(p)}, F_*(\tilde{G}'_1)) \otimes k(\xi')$ becomes a $(\mathbf{t}, \boldsymbol{\lambda}^{(p)})$ -parabolic connection which implies that $\tilde{\phi}'^{(p)} \otimes k(\xi')$ must be isomorphic. Then $\tilde{\phi}^{(p)} \otimes k(\xi')$ also becomes isomorphic and $\phi' = \tilde{\phi} \otimes k(\xi)$ must be an isomorphism. Thus $\tilde{\phi}$ is an isomorphism. Let $\{\tilde{l}_j^{(i)}\}$ be the flat family of parabolic structures corresponding to $F_*(\tilde{E}_1)$. Then we have $(\text{res}_{t_i}(\tilde{\nabla}) - \lambda_j^{(i)} \tilde{\phi}|_{t_i \otimes R})(\tilde{l}_j^{(i)}) \subset \tilde{\phi}(\tilde{l}_{j+1}^{(i)})$. Thus $(E'_1, E'_2, \phi', \nabla', F_*(E'_1)) \in M_C^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})(R)$ which is equivalent to $x_0 \in M_C^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})$. Then we have $x_0 \in \overline{\mathbf{RH}_{(x, \boldsymbol{\lambda})}^{-1}([\rho])}$. Therefore $\mathbf{RH}_{(x, \boldsymbol{\lambda})}^{-1}([\rho])$ is stable under specialization in $\overline{M_C^{D(\mathbf{t}), \alpha', \beta, \gamma}(r, d, \{d_i\}_{1 \leq i \leq nr})}$. Hence $\mathbf{RH}_{(x, \boldsymbol{\lambda})}^{-1}([\rho])$ becomes a compact subset of $M_C^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})$. $\square$

Proof of Theorem 2.2. We will prove that

$$\mathbf{RH}_{(x, \boldsymbol{\lambda})} : M_C^{\alpha}(\mathbf{t}, \boldsymbol{\lambda}) \longrightarrow RP_r(C, \mathbf{t})_{\mathbf{a}}$$

is an analytic isomorphism for generic $\boldsymbol{\lambda}$ and gives an analytic resolution of singularities of $RP_r(C, \mathbf{t})_{\mathbf{a}}$ for special $\boldsymbol{\lambda}$. First note that $RP_r(C, \mathbf{t})_{\mathbf{a}}$ is an irreducible variety since by Proposition 6.1 it is the image of the irreducible variety $M_C^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})$ by $\mathbf{RH}_{(x, \boldsymbol{\lambda})}$. We set

$$RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{sing}} := \left\{ [\rho] \in RP_r(C, \mathbf{t})_{\mathbf{a}} \left| \begin{array}{l} \rho \text{ is reducible or} \\ \dim \left(\ker \left(\rho(\gamma_i) - \exp(-2\pi\sqrt{-1}\lambda_j^{(i)}) I_r \right) \right) \right. \\ \left. \geq 2 \text{ for some } i, j \right. \end{array} \right\},$$

where γ_i is a loop around t_i for $1 \leq i \leq n$. Then we can see that $RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{sing}}$ is a Zariski closed subset of $RP_r(C, \mathbf{t})_{\mathbf{a}}$. We will see that it is a proper closed subset. Since $\dim M_C^{\alpha}(\mathbf{t}, \boldsymbol{\lambda}) = 2r^2(g-1) + nr(r-1) + 2$ and $\mathbf{RH}_{(x, \boldsymbol{\lambda})}$ is surjective, we have $\dim RP_r(C, \mathbf{t})_{\mathbf{a}} \leq 2r^2(g-1) + nr(r-1) + 2$. On the other hand, $RP_r(C, \mathbf{t})_{\mathbf{a}}$ is a Zariski closed subset of the $r^2(2g+n-2)+1$ dimensional irreducible variety $RP_r(C, \mathbf{t})$ defined by $nr-1$ equations given by

$$(12) \quad \det(XI_r - \rho(\gamma_i)) = X^r + a_{r-1}^{(i)} X^{r-1} + \cdots + a_1^{(i)} X + a_0^{(i)} \quad (i = 1, \dots, n).$$

Note that the equation $\det(\rho(\gamma_1)) \cdots \det(\rho(\gamma_n)) = 1 = (-1)^{nr} a_0^{(1)} \cdots a_0^{(n)}$ is automatically satisfied and so the condition (12) is in fact equivalent to $nr-1$ equations. So we have

$$\dim RP_r(C, \mathbf{t})_{\mathbf{a}} \geq r^2(2g+n-2) + 1 - (nr-1) = 2r^2(g-1) + nr(r-1) + 2$$

and then we have $\dim RP_r(C, \mathbf{t})_{\mathbf{a}} = 2r^2(g-1) + nr(r-1) + 2$.

First consider the locus $RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{irr}} \cap RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{sing}}$ in $RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{sing}}$ corresponding to the irreducible representations. By Proposition 3.1, we can obtain

an isomorphism

$$\sigma : \mathcal{M}_C(\mathbf{t}, \boldsymbol{\lambda}) \xrightarrow{\sim} \mathcal{M}_C(\mathbf{t}, \boldsymbol{\mu}),$$

where $0 \leq \operatorname{Re}(\mu_j^{(i)}) < 1$ for any i, j . We note that σ also induces an isomorphism

$$\sigma : M_C^{\text{irr}}(\mathbf{t}, \boldsymbol{\lambda}) \xrightarrow{\sim} M_C^{\text{irr}}(\mathbf{t}, \boldsymbol{\mu}).$$

Take any point $[\rho] \in RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{irr}} \cap RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{sing}}$. By [4, II, Proposition 5.4], we can take a unique pair (E, ∇) of a vector bundle E on C and a logarithmic connection $\nabla : E \rightarrow E \otimes \Omega_C^1(t_1 + \cdots + t_n)$ such that $\ker \nabla^{an}|_{C \setminus \{t_1, \dots, t_n\}}$ corresponds to the representation ρ and all the eigenvalues of $\operatorname{res}_{t_i}(\nabla)$ that lie in $\{z \in \mathbf{C} | 0 \leq \operatorname{Re}(z) < 1\}$. Then the fiber $\mathbf{RH}_{(x, \boldsymbol{\mu})}^{-1}([\rho])$ is isomorphic to the moduli space of parabolic structures $\{l_j^{(i)}\}$ on E whose dimension should be positive because $\mu_j^{(i)} = \mu_{j'}^{(i)}$ for some i and $j \neq j'$ and the automorphisms of (E, ∇) are only scalar multiplications. So the fiber $\mathbf{RH}_{(x, \boldsymbol{\lambda})}^{-1}([\rho])$ also has positive dimension. For a general point ξ of $RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{irr}} \cap RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{sing}}$, we have

$$\begin{aligned} \dim \mathbf{RH}_{(x, \boldsymbol{\lambda})}^{-1}(\xi) + \dim_{\xi} (RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{irr}} \cap RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{sing}}) \\ \leq \dim M_C^{\text{irr}}(\mathbf{t}, \boldsymbol{\lambda}) = 2r^2(g-1) + nr(r-1) + 2. \end{aligned}$$

Thus we have $\dim (RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{irr}} \cap RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{sing}}) < 2r^2(g-1) + nr(r-1) + 2 = \dim RP_r(C, \mathbf{t})_{\mathbf{a}}$.

Next we consider the locus $RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{red}} \subset RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{sing}}$ corresponding to the reducible representations. For a point $[\rho]$ of $RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{red}}$ we may write $\rho = \rho_1 \oplus \rho_2$. If ρ_i is a representation of dimension r_i and its local monodromy data is $\mathbf{a}_i$ for $i = 1, 2$, then the set of such $[\rho]$ can be parameterized by $RP_{r_1}(C, \mathbf{t})_{\mathbf{a}_1} \times RP_{r_2}(C, \mathbf{t})_{\mathbf{a}_2}$. Note that the assumption $rn - 2(r+1) > 0$ for $g = 0$, $n \geq 2$ for $g = 1$ and $n \geq 1$ for $g \geq 2$ implies

$$\begin{aligned} \dim (RP_{r_1}(C, \mathbf{t})_{\mathbf{a}_1} \times RP_{r_2}(C, \mathbf{t})_{\mathbf{a}_2}) \\ = 2r_1^2(g-1) + nr_1(r_1-1) + 2 + 2r_2^2(g-1) + nr_2(r_2-1) + 2 \\ \leq 2(r_1 + r_2)^2(g-1) + n(r_1 + r_2)(r_1 + r_2 - 1) \\ = \dim RP_r(C, \mathbf{t})_{\mathbf{a}} - 2. \end{aligned}$$

Since $RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{red}}$ can be covered by a finite number of the images of such spaces, we have

$$\dim RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{red}} < 2r^2(g-1) + nr(r-1) + 2 = \dim RP_r(C, \mathbf{t})_{\mathbf{a}}.$$

Thus $RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{sing}}$ becomes a proper closed subset of $RP_r(C, \mathbf{t})_{\mathbf{a}}$. If we set $RP_r(C, \mathbf{t})_{\mathbf{a}}^{\sharp} := RP_r(C, \mathbf{t})_{\mathbf{a}} \setminus RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{sing}}$, then it becomes a non-empty Zariski

open subset of $RP_r(C, \mathbf{t})_{\mathbf{a}}$. We put $M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda})^\sharp := \mathbf{RH}_{(x, \boldsymbol{\lambda})}^{-1}(RP_r(C, \mathbf{t})_{\mathbf{a}}^\sharp)$ and consider the restriction

$$\mathbf{RH}_{(x, \boldsymbol{\lambda})}|_{M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda})^\sharp} : M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda})^\sharp \longrightarrow RP_r(C, \mathbf{t})_{\mathbf{a}}^\sharp.$$

Recall that σ induces an isomorphism $\sigma : M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda})^\sharp \xrightarrow{\sim} M_C^\alpha(\mathbf{t}, \boldsymbol{\mu})^\sharp$ which is compatible with $\mathbf{RH}$. For any point $[\rho]$ of $RP_r(C, \mathbf{t})_{\mathbf{a}}^\sharp$, we can see by [4, II, Proposition 5.4] that there is a unique $(\mathbf{t}, \boldsymbol{\mu})$ -parabolic connection $(E, \nabla, \{l_j^{(i)}\})$ such that $\mathbf{RH}_{(x, \boldsymbol{\mu})}(E, \nabla, \{l_j^{(i)}\}) = [\rho]$ because the parabolic structure $\{l_j^{(i)}\}$ is uniquely determined by (E, ∇) . So $\mathbf{RH}_{(x, \boldsymbol{\mu})}$ gives a one-to-one correspondence between the points of $M_C^\alpha(\mathbf{t}, \boldsymbol{\mu})^\sharp$ and the points of $RP_r(C, \mathbf{t})_{\mathbf{a}}^\sharp$. We can extend this correspondence to a correspondence between flat families. Thus the morphism $M_C^\alpha(\mathbf{t}, \boldsymbol{\mu})^\sharp \xrightarrow{\mathbf{RH}_{(x, \boldsymbol{\mu})}} RP_r(C, \mathbf{t})_{\mathbf{a}}^\sharp$ becomes an isomorphism and so $M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda})^\sharp \xrightarrow{\mathbf{RH}_{(x, \boldsymbol{\lambda})}} RP_r(C, \mathbf{t})_{\mathbf{a}}^\sharp$ is also an isomorphism. Hence the morphism $\mathbf{RH}_{(x, \boldsymbol{\lambda})} : M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}) \longrightarrow RP_r(C, \mathbf{t})_{\mathbf{a}}$ becomes a bimeromorphic morphism. If $\boldsymbol{\lambda}$ is generic, we have $M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda})^\sharp = M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda})$, $RP_r(C, \mathbf{t})_{\mathbf{a}}^\sharp = RP_r(C, \mathbf{t})_{\mathbf{a}}$, and so

$$\mathbf{RH}_{(x, \boldsymbol{\lambda})} : M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}) \longrightarrow RP_r(C, \mathbf{t})_{\mathbf{a}}$$

is an analytic isomorphism.

In order to prove the properness of $\mathbf{RH}_{(x, \boldsymbol{\lambda})}$, we will use the following lemma due to A. Fujiki:

Lemma 6.2 ([7], Lemma 10.3). *Let $f : X \rightarrow Y$ be a surjective holomorphic mapping of irreducible analytic varieties. Assume that an analytic closed subset S of Y exists such that $\text{codim}_Y S \geq 2$, $X^\sharp := f^{-1}(Y^\sharp)$ is dense in X , where $Y^\sharp = Y \setminus S$ and that the restriction $f|_{X^\sharp} : X^\sharp \rightarrow Y^\sharp$ is an analytic isomorphism. Moreover assume that the fibers $f^{-1}(y)$ are compact for all $y \in Y$. Then f is a proper mapping.*

Applying the above lemma and using Propositions 6.1 and 6.2, it suffices to prove that $\text{codim}_{RP_r(C, \mathbf{t})_{\mathbf{a}}}(RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{sing}}) \geq 2$ in order to obtain the properness of $\mathbf{RH}_{(x, \boldsymbol{\lambda})}$. Recall that we have $\dim RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{red}} \leq \dim RP_r(C, \mathbf{t})_{\mathbf{a}} - 2$. On the other hand, $M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda})^\sharp$ is a non-empty Zariski open subset of $M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda})$ and so we have $\dim(M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}) \setminus M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda})^\sharp) \leq \dim M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}) - 1$. Since the dimension of every fiber of the surjective morphism

$$M_C^{\text{irr}}(\mathbf{t}, \boldsymbol{\lambda}) \setminus M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda})^\sharp \xrightarrow{\mathbf{RH}_{(x, \boldsymbol{\lambda})}} RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{sing}} \cap RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{irr}}$$

has positive dimension, we have

$$\begin{aligned} \dim(RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{sing}} \cap RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{irr}}) \\ \leq \dim(M_C^{\text{irr}}(\mathbf{t}, \boldsymbol{\lambda}) \setminus M_C^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})^{\sharp}) - 1 \\ \leq \dim M_C^{\alpha}(\mathbf{t}, \boldsymbol{\lambda}) - 1 - 1 = \dim RP_r(C, \mathbf{t})_{\mathbf{a}} - 2. \end{aligned}$$

Thus we have $\text{codim}_{RP_r(C, \mathbf{t})_{\mathbf{a}}}(RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{sing}}) \geq 2$ and obtain the properness of $\mathbf{RH}_{(x, \boldsymbol{\lambda})}$ by Lemma 6.2. In particular, $\mathbf{RH}_{(x, \boldsymbol{\lambda})} : M_C^{\alpha}(\mathbf{t}, \boldsymbol{\lambda}) \rightarrow RP_r(C, \mathbf{t})_{\mathbf{a}}$ gives an analytic resolution of singularities of $RP_r(C, \mathbf{t})_{\mathbf{a}}$ for special $\boldsymbol{\lambda}$. By the same argument, we can see that the morphism

$$\mathbf{RH} : M_{C/T}^{\alpha}(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T} \longrightarrow (RP_r(C, \tilde{\mathbf{t}}) \times_T \tilde{T}) \times_{\mathcal{A}_r^{(n)}} \Lambda_r^{(n)}(d)$$

is also a proper mapping. The existence of a symplectic structure is given in Proposition 7.1. $\square$

Remark 6.2. Under the assumption of Theorem 2.2, $RP_r(C, \mathbf{t})_{\mathbf{a}}$ is a normal variety and the locus

$$RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{sing}} = \left\{ [\rho] \in RP_r(C, \mathbf{t})_{\mathbf{a}} \left| \begin{array}{l} \rho \text{ is reducible or} \\ \dim \left(\ker \left(\rho(\gamma_i) - \exp(-2\pi\sqrt{-1}\lambda_j^{(i)})I_r \right) \right) \right. \\ \left. \geq 2 \text{ for some } i, j \right. \end{array} \right\}$$

is just the singular locus of $RP_r(C, \mathbf{t})_{\mathbf{a}}$.

Proof. Note that $RP_r(C, \mathbf{t})$ is a Cohen-Macaulay irreducible variety by [5, Theorem 4(2)]. In the proof of Theorem 2.2, we showed that $RP_r(C, \mathbf{t})_{\mathbf{a}}$ is a Zariski closed subset of $RP_r(C, \mathbf{t})$ defined by $nr - 1$ equations and it is of codimension $nr - 1$ in $RP_r(C, \mathbf{t})$. Thus $RP_r(C, \mathbf{t})_{\mathbf{a}}$ is Cohen-Macaulay. Since $RP_r(C, \mathbf{t})_{\mathbf{a}}^{\sharp}$ is isomorphic to $M_C^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})^{\sharp}$, $RP_r(C, \mathbf{t})_{\mathbf{a}}^{\sharp}$ is smooth. So $RP_r(C, \mathbf{t})_{\mathbf{a}}$ is regular in codimension one because $\text{codim}_{RP_r(C, \mathbf{t})_{\mathbf{a}}}(RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{sing}}) \geq 2$. Therefore $RP_r(C, \mathbf{t})_{\mathbf{a}}$ is a normal variety by Serre's criterion for normality.

We will prove that every fiber of $\mathbf{RH}_{(x, \boldsymbol{\lambda})}$ over a point of $RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{sing}}$ has positive dimension. For a point $[\rho] \in RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{irr}} \cap RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{sing}}$, we have already showed in the proof of Theorem 2.2 that $\dim \mathbf{RH}_{(x, \boldsymbol{\lambda})}^{-1}([\rho]) \geq 1$. So take a point $[\rho] \in RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{red}}$. By Proposition 3.1, we can take an isomorphism

$$\sigma : \mathcal{M}_C(\mathbf{t}, \boldsymbol{\lambda}) \xrightarrow{\sim} \mathcal{M}_C(\mathbf{t}, \boldsymbol{\mu}),$$

where $0 \leq \text{Re}(\mu_j^{(i)}) < 1$ for any i, j .

First we assume that ρ is an extension of two irreducible representations ρ_1, ρ_2 and that $\mu_j^{(i)} \neq \mu_{j'}^{(i)}$ for any $j \neq j'$ and any i . We denote the dimension of ρ_i by r_i . Since $\mathbf{RH}_{(x, \boldsymbol{\lambda})}$ is surjective by Proposition 6.1, there is an α -stable $(\mathbf{t}, \boldsymbol{\lambda})$ -parabolic connection $(E, \nabla_E, \{l_j^{(i)}\})$ such that $\mathbf{RH}_{(x, \boldsymbol{\lambda})}(E, \nabla_E, \{l_j^{(i)}\}) = [\rho]$. We put $\sigma(E, \nabla_E, \{l_j^{(i)}\}) = (E', \nabla_{E'}, \{l_{j'}^{(i)}\})$. Since $\mu_j^{(i)} \neq \mu_{j'}^{(i)}$ for any

$j \neq j'$ and any i , the parabolic structure $\{l_j^{(i)}\}$ is uniquely determined by $(E', \nabla_{E'})$. So we have

$$\mathrm{End}(E', \nabla_{E'}) = \mathrm{End}(E', \nabla_{E'}, \{l_j^{(i)}\}) \cong \mathrm{End}(E, \nabla_E, \{l_j^{(i)}\})$$

which is isomorphic to $\mathbf{C}$ because $(E, \nabla_E, \{l_j^{(i)}\})$ is α -stable. Then $(E', \nabla_{E'})$ cannot split. Thus the representation $\rho_{E'}$ corresponding to $\ker \nabla_{E'}^{an}|_{C \setminus \{t_1, \dots, t_n\}}$ cannot split and so we may assume that $\rho_{E'}$ is given by matrices

$$\begin{aligned} \rho_{E'}(\alpha_k) &= \begin{pmatrix} \rho_1(\alpha_k) & A_k^{(0)} \\ 0 & \rho_2(\alpha_k) \end{pmatrix}, & \rho_{E'}(\beta_k) &= \begin{pmatrix} \rho_1(\beta_k) & B_k^{(0)} \\ 0 & \rho_2(\beta_k) \end{pmatrix}, \\ \rho_{E'}(\gamma_i) &= \begin{pmatrix} \rho_1(\gamma_i) & C_i^{(0)} \\ 0 & \rho_2(\gamma_i) \end{pmatrix} \quad (1 \leq k \leq g, 1 \leq i \leq n-1). \end{aligned}$$

Note that $\rho_{E'}(\gamma_n)$ is uniquely determined by the above data. Consider the family of representations over $M(r_1, r_2, \mathbf{C})^{2g+n-1}$ given by matrices

$$\begin{aligned} \tilde{\rho}(\alpha_k) &= \begin{pmatrix} \rho_1(\alpha_k) & A_k^{(0)} + A_k \\ 0 & \rho_2(\alpha_k) \end{pmatrix}, & \tilde{\rho}(\beta_k) &= \begin{pmatrix} \rho_1(\beta_k) & B_k^{(0)} + B_k \\ 0 & \rho_2(\beta_k) \end{pmatrix}, \\ \tilde{\rho}(\gamma_i) &= \begin{pmatrix} \rho_1(\gamma_i) & C_i^{(0)} + C_i \\ 0 & \rho_2(\gamma_i) \end{pmatrix} \quad (1 \leq k \leq g, 1 \leq i \leq n-1), \end{aligned}$$

where matrices $A_k, B_k (1 \leq k \leq g), C_i (1 \leq i \leq n-1)$ move around in $M(r_1, r_2, \mathbf{C})^{2g+n-1}$. Applying the relative version of [4, II, Proposition 5.4] and the transform σ^{-1} , we can obtain an analytic flat family $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ of $(\mathbf{t}, \boldsymbol{\lambda})$ -parabolic connections on $C \times M(r_1, r_2, \mathbf{C})^{2g+n-1}$ over $M(r_1, r_2, \mathbf{C})^{2g+n-1}$ which corresponds to $\tilde{\rho}$. We denote the point of $M(r_1, r_2, \mathbf{C})^{2g+n-1}$ corresponding to $A_k = B_k = C_i = 0$ ($1 \leq k \leq g, 1 \leq i \leq n-1$) by p_0 . Then the fiber of $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ over p_0 is just $(E, \nabla_E, \{l_j^{(i)}\})$ which is α -stable. So there is an analytic open neighborhood U of p_0 such that every fiber of $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ over a point of U is α -stable. Then the family $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})_U$ induces a morphism $f : U \rightarrow M_{\mathbf{C}}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})$ such that $f(U) \subset \mathbf{RH}_{(x, \boldsymbol{\lambda})}^{-1}([\rho])$. The group

$$G = \left\{ \begin{pmatrix} uI_{r_1} & V \\ 0 & I_{r_2} \end{pmatrix} \middle| \begin{array}{l} u \in \mathbf{C}^{\times} \\ V \in M(r_1, r_2, \mathbf{C}) \end{array} \right\}$$

acts on $M(r_1, r_2, \mathbf{C})^{2g+n-1}$ by the adjoint action. If $f(x) = f(y)$, then there is an element $g \in G$ such that $x = gy$. So we have $\dim f(U) \geq (2g + n - 1)r_1r_2 - (r_1r_2 + 1) \geq 1$, since we assume $rn - 2r - 2 > 0$ if $g = 0, n \geq 2$ if $g = 1$ and $n \geq 1$ if $g \geq 2$. In particular, we have $\dim \mathbf{RH}_{(x, \boldsymbol{\lambda})}^{-1}([\rho]) \geq 1$.

Next we consider the case $r \geq 3$ or $n \geq 2$ and no assumption on the reducible representation ρ . We may assume that ρ is an extension of two representations ρ_1, ρ_2 and denote the dimension of ρ_i by r_i . For each $p = 1, 2$, there is an α_p -stable $(\mathbf{t}, \mu_p)$ -parabolic connection $(E_p, \nabla_{E_p}, \{(l_p)_j^{(i)}\})$ for some weight α_p such that $\mathbf{RH}_{(x, \mu_p)}(E_p, \nabla_{E_p}, \{(l_p)_j^{(i)}\}) = [\rho_p]$ and $0 \leq \operatorname{Re}((\mu_p)_j^{(i)}) < 1$ for any i, j . We put $d_p := \deg E_p$. If we put $\Delta := \{z \in \mathbf{C} \mid |z| < \epsilon\}$ ($\epsilon > 0$), then we can take morphisms

$$m_p : \Delta \longrightarrow \Lambda_{r_p}^{(n)}(d_p) \quad (p = 1, 2)$$

such that $m_p(0) = \mu_p$ and for $m_p(t) = \{(\mu'_p)_j^{(i)}\}$ ($t \in \Delta \setminus \{0\}$), we have $(\mu'_p)_j^{(i)} \neq (\mu'_{p'})_{j'}^{(i')}$ for $(p, i, j) \neq (p', i', j')$. Replacing Δ by a neighborhood of 0, we can take a morphism $\varphi_p : \Delta \longrightarrow M_C^{\alpha_p}(\mathbf{t}, r_p, d_p)$ which is a lift of m_p and satisfies $\varphi_p(0) = (E_p, \nabla_{E_p}, \{(l_p)_j^{(i)}\})$. The composite $\mathbf{RH} \circ \varphi_p$ determines a family $\tilde{\rho}_p$ of representations over Δ and $\tilde{\rho}_1 \oplus \tilde{\rho}_2$ determines a morphism $\varphi : \Delta \rightarrow RP_r(C, \mathbf{t})$ such that $\varphi(0) = [\rho]$ and for $t \in \Delta \setminus \{0\}$, the representation corresponding to $\varphi(t)$ satisfies the condition considered in the former case. Then we have $\dim \mathbf{RH}^{-1}(\varphi(t)) \geq 1$ for any $t \in \Delta \setminus \{0\}$ by the proof of the former case. Since the base change $\mathbf{RH}_\Delta : M_C^\alpha(\mathbf{t}, r, d)_\Delta \rightarrow \Delta$ is a proper morphism, we can see by the upper semi-continuity of the fiber dimension of a holomorphic mapping that $\dim(\mathbf{RH}_\Delta)^{-1}(0) \geq 1$, which means $\dim \mathbf{RH}_{(x, \lambda)}^{-1}([\rho]) \geq 1$.

Finally assume that $r = 2$ and $n = 1$. Let ρ be a reducible representation such that $[\rho] = [\rho_1 \oplus \rho_2]$ in $RP_2(C, \mathbf{t})$. In this case we have $g \geq 2$ by the assumption of Theorem 2.2. By Proposition 6.1, we can take an α -stable $(\mathbf{t}, \lambda)$ -parabolic connection $(E, \nabla, \{l_j^{(i)}\})$ such that $\mathbf{RH}(E, \nabla, \{l_j^{(i)}\}) = [\rho]$. We put $\sigma(E, \nabla, \{l_j^{(i)}\}) = (E', \nabla', \{(l')_j^{(i)}\})$. Note that the local monodromy $\rho_i(\gamma_1)$ is trivial for $i = 1, 2$, because ρ_i is a one dimensional representation and $n = 1$. So the representation $\rho_{E'}$ corresponding to $\ker(\nabla')^{an}|_{C \setminus \{t_1, \dots, t_n\}}$ can be given by the following data:

$$\rho_{E'}(\alpha_k) = \begin{pmatrix} \rho_1(\alpha_k) & a_k^{(0)} \\ 0 & \rho_2(\alpha_k) \end{pmatrix}, \quad \rho_{E'}(\beta_k) = \begin{pmatrix} \rho_1(\beta_k) & b_k^{(0)} \\ 0 & \rho_2(\beta_k) \end{pmatrix} \quad (1 \leq k \leq g).$$

Note that the monodromy matrix

$$\rho_{E'}(\gamma_1) = \begin{pmatrix} 1 & c^{(0)} \\ 0 & 1 \end{pmatrix}$$

is given by

$$\rho_{E'}(\beta_g) \rho_{E'}(\alpha_g) \rho_{E'}(\beta_g)^{-1} \rho_{E'}(\alpha_g)^{-1} \cdots \rho_{E'}(\beta_1) \rho_{E'}(\alpha_1) \rho_{E'}(\beta_1)^{-1} \rho_{E'}(\alpha_1)^{-1}.$$

By the upper semi-continuity of fiber dimension, we may assume that $\rho_1 \not\cong \rho_2$. If $c^{(0)} \neq 0$, then we can prove in the same manner as the first case that $\dim_{(E, \nabla, \{\tilde{l}_j^{(i)}\})} \mathbf{RH}_{(x, \lambda)}^{-1}([\rho]) \geq 2g - 2 > 0$. If $c^{(0)} = 0$, then we may assume by the upper semi-continuity of fiber dimension that $l_1^{(1)} \neq \{(*)\}$. Consider the family of representations given by matrices

$$\tilde{\rho}(\alpha_k) = \begin{pmatrix} \rho_1(\alpha_k) & a_k^{(0)} + a_k \\ 0 & \rho_2(\alpha_k) \end{pmatrix}, \quad \tilde{\rho}(\beta_k) = \begin{pmatrix} \rho_1(\beta_k) & b_k^{(0)} + b_k \\ 0 & \rho_2(\beta_k) \end{pmatrix} \quad (1 \leq k \leq g),$$

where a_k, b_k ($1 \leq k \leq g$) move around in $\{c = 0\} \subset \mathbf{C}^{2g}$. Here c_k is given by

$$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = \tilde{\rho}(\beta_g) \tilde{\rho}(\alpha_g) \tilde{\rho}(\beta_g)^{-1} \tilde{\rho}(\alpha_g)^{-1} \cdots \tilde{\rho}(\beta_1) \tilde{\rho}(\alpha_1) \tilde{\rho}(\beta_1)^{-1} \tilde{\rho}(\alpha_1)^{-1}.$$

Adding parabolic structure, we can obtain a family $(\tilde{E}', \tilde{\nabla}', \{(\tilde{l}')_j^{(i)}\})$ of $(\mathbf{t}, \boldsymbol{\mu})$ -parabolic connections over a variety Y of dimension at least $2g$. If we put

$$Y' := \left\{ y \in Y \mid \sigma^{-1}(\tilde{E}', \tilde{\nabla}', \{(\tilde{l}')_j^{(i)}\}) \otimes k(y) \text{ is } \boldsymbol{\alpha}\text{-stable} \right\},$$

we obtain a morphism $f' : Y' \rightarrow M_C^\alpha(\mathbf{t}, \lambda)$. We can see that $f'(y_1) = f'(y_2)$ if and only if there is an isomorphism $\varphi : (\tilde{E}', \tilde{\nabla}') \otimes k(y_1) \xrightarrow{\sim} (\tilde{E}', \tilde{\nabla}') \otimes k(y_2)$ such that $\varphi((\tilde{l}')_j^{(i)} \otimes k(y_1)) = (\tilde{l}')_j^{(i)} \otimes k(y_2)$. The isomorphism φ corresponds to the isomorphisms between the representations $\tilde{\rho}_{y_1}, \tilde{\rho}_{y_2}$ which is given by an element of the group

$$\left\{ \begin{pmatrix} c & a \\ 0 & 1 \end{pmatrix} \mid c \in \mathbf{C}^\times, a \in \mathbf{C} \right\}.$$

So we have $\dim f'(Y') = \dim Y' - 2 \geq 2g - 2 > 0$. Since $f'(Y') \subset \mathbf{RH}_{(x, \lambda)}^{-1}([\rho])$, we have $\dim \mathbf{RH}_{(x, \lambda)}^{-1}([\rho]) > 0$.

From all the above arguments, we have in any case that $\dim \mathbf{RH}_{(x, \lambda)}^{-1}([\rho]) \geq 1$ for any $[\rho] \in RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{sing}}$. Now take any point $p \in RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{sing}}$. We want to show that p is a singular point of $RP_r(C, \mathbf{t})_{\mathbf{a}}$. Assume that p is a non-singular point of $RP_r(C, \mathbf{t})_{\mathbf{a}}$. By Proposition 7.2, there is a non-degenerate 2-form ω on $M_C^\alpha(\mathbf{t}, \lambda)$. Via an isomorphism $M_C^\alpha(\mathbf{t}, \lambda)^\# \xrightarrow{\mathbf{RH}} RP_r(C, \mathbf{t})_{\mathbf{a}}^\#$, we can obtain a non-degenerate 2-form ω' on $RP_r(C, \mathbf{t})_{\mathbf{a}}^\#$ such that $\mathbf{RH}_{(x, \lambda)}^*(\omega') = \omega|_{M_C^\alpha(\mathbf{t}, \lambda)^\#}$. Then ω' can be extended to a form also defined in a neighborhood of p because $\text{codim}_{RP_r(C, \mathbf{t})_{\mathbf{a}}}(RP_r(C, \mathbf{t})_{\mathbf{a}}^{\text{sing}}) \geq 2$ and we are assuming that p is a non-singular point. Since $\dim_x \mathbf{RH}_{(x, \lambda)}^{-1}(p) \geq 1$ for a point $x \in \mathbf{RH}_{(x, \lambda)}^{-1}(p)$, there is a tangent vector $0 \neq v \in \Theta_{M_C^\alpha(\mathbf{t}, \lambda), x}$ such that $(\mathbf{RH}_{(x, \lambda)})_*(v) = 0$. For any tangent vector $w \in \Theta_{M_C^\alpha(\mathbf{t}, \lambda), x}$, we have

$$\omega(v, w) = (\mathbf{RH}_{(x, \lambda)}^*(\omega'))(v, w) = \omega'((\mathbf{RH}_{(x, \lambda)})_*(v), (\mathbf{RH}_{(x, \lambda)})_*(w)) = 0.$$

Since ω determines a non-degenerate pairing on the tangent space $\Theta_{M_{\mathcal{C}}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda}), x}$, we have $v = 0$ which is a contradiction. Thus p is a singular point of $RP_r(C, \mathbf{t})_{\mathbf{a}}$. $\square$

7. Symplectic structure on the moduli space

Proposition 7.1. *Put $(C, \mathbf{t}) = (\mathcal{C}_x, \tilde{\mathbf{t}}_x)$ for a point $x \in T$ and take a point $\boldsymbol{\lambda} \in \Lambda_r^{(n)}(d)$. Then the moduli space $M_{\mathcal{C}}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})$ has a holomorphic symplectic structure (more precisely an algebraic symplectic structure).*

We can obtain the above proposition by the following two propositions.

Proposition 7.2. *There is a non-degenerate relative 2-form $\omega \in H^0(M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d), \Omega_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)/T \times \Lambda_r^{(n)}(d)}^2)$.*

Proof. Take a universal family $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ on $\mathcal{C} \times_T M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)$. We define a complex $\mathcal{F}^{\bullet}$ by

$$\begin{aligned} \mathcal{F}^0 &= \left\{ s \in \mathcal{E}nd(\tilde{E}) \mid s|_{\tilde{\mathbf{t}}_i \times M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)}(\tilde{l}_j^{(i)}) \subset \tilde{l}_j^{(i)} \text{ for any } i, j \right\}, \\ \mathcal{F}^1 &= \left\{ s \in \mathcal{E}nd(\tilde{E}) \otimes \Omega_{\mathcal{C}/T}^1(D(\tilde{\mathbf{t}})) \mid \text{res}_{\tilde{\mathbf{t}}_i \times M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)}(s)(\tilde{l}_j^{(i)}) \subset \tilde{l}_{j+1}^{(i)} \text{ for any } i, j \right\}, \end{aligned}$$

and

$$\nabla_{\mathcal{F}^{\bullet}} : \mathcal{F}^0 \ni s \mapsto \tilde{\nabla} \circ s - s \circ \tilde{\nabla} \in \mathcal{F}^1.$$

Notice that there is an isomorphism of sheaves

$$\mathbf{R}^1(\pi_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)})_*(\mathcal{F}^{\bullet}) \xrightarrow{\sim} \Theta_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)/T \times \Lambda_r^{(n)}(d)},$$

where $\pi_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)} : \mathcal{C} \times_T M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d) \rightarrow M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)$ is the projection and $\Theta_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)/T \times \Lambda_r^{(n)}(d)}$ is the relative algebraic tangent bundle on the moduli space. For each affine open subset $U \subset M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)$, we define a pairing

$$\begin{aligned} \mathbf{H}^1(\mathcal{C} \times_T U, \mathcal{F}_U^{\bullet}) \otimes \mathbf{H}^1(\mathcal{C} \times_T U, \mathcal{F}_U^{\bullet}) &\longrightarrow \mathbf{H}^2(\mathcal{C} \times_T U, \Omega_{\mathcal{C} \times_T U/U}^{\bullet}) \cong H^0(\mathcal{O}_U) \\ &[\{u_{\alpha\beta}\}, \{v_{\alpha}\}] \otimes [\{u'_{\alpha\beta}\}, \{v'_{\alpha}\}] \\ &\mapsto \{[\text{Tr}(u_{\alpha\beta} \circ u'_{\beta\gamma})] - [\text{Tr}(u_{\alpha\beta} \circ v'_{\beta}) - \text{Tr}(v_{\alpha} \circ u'_{\alpha\beta})]\}, \end{aligned}$$

where we consider in Čech cohomology with respect to an affine open covering $\{U_{\alpha}\}$ of $\mathcal{C} \times_T U$, $\{u_{\alpha\beta}\} \in C^1(\mathcal{F}^0)$, $\{v_{\alpha}\} \in C^0(\mathcal{F}^1)$ and so on. This pairing determines a pairing

$$(13) \quad \omega : \mathbf{R}^1(\pi_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)})_*(\mathcal{F}^{\bullet}) \otimes \mathbf{R}^1(\pi_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)})_*(\mathcal{F}^{\bullet}) \longrightarrow \mathcal{O}_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)}.$$

Take any point $x \in M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)$ which corresponds to a $(\mathbf{t}, \boldsymbol{\lambda})$ -parabolic connection $(E, \nabla, \{l_j^{(i)}\})$. A tangent vector $v \in \Theta_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)/T \times \Lambda_r^{(n)}(d)}|_x$ corresponds to a flat family of $(\mathbf{t}, \boldsymbol{\lambda})$ -parabolic connections $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ on $\mathcal{C}_x \times \text{Spec } \mathbf{C}[\epsilon]/(\epsilon^2)$ over $\mathbf{C}[\epsilon]/(\epsilon^2)$ which is a lift of $(E, \nabla, \{l_j^{(i)}\})$. We can see that $\omega|_x(v \otimes v) \in \mathbf{H}^0(\Omega_{\mathcal{C}_x}^{\bullet}) \xrightarrow[\sim]{\text{Tr}^{-1}} \mathbf{H}^2(\mathcal{F}_x^{\bullet})$ is just the obstruction class for the lifting of $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ to a flat family over $\mathbf{C}[\epsilon]/(\epsilon^3)$. Since the moduli space $M_{\mathcal{C}_x}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})$ is smooth, this obstruction class vanishes: $\omega|_x(v \otimes v) = 0$. Thus ω is skew symmetric and determines a 2-form. For each point $x \in M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)$, let

$$\mathbf{H}^1(\mathcal{F}_x^{\bullet}) \xrightarrow{\xi} \mathbf{H}^1(\mathcal{F}_x^{\bullet})^{\vee}$$

be the homomorphism induced by ω . Then we have the following exact commutative diagram

$$\begin{array}{ccccccccc} H^0(\mathcal{F}_x^0) & \longrightarrow & H^0(\mathcal{F}_x^1) & \longrightarrow & \mathbf{H}^1(\mathcal{F}_x^{\bullet}) & \longrightarrow & H^1(\mathcal{F}_x^0) & \longrightarrow & H^1(\mathcal{F}_x^1) \\ b_1 \downarrow & & b_2 \downarrow & & \xi \downarrow & & b_3 \downarrow & & b_4 \downarrow \\ H^1(\mathcal{F}_x^1)^{\vee} & \longrightarrow & H^1(\mathcal{F}_x^0)^{\vee} & \longrightarrow & \mathbf{H}^1(\mathcal{F}_x^{\bullet})^{\vee} & \longrightarrow & H^0(\mathcal{F}_x^1)^{\vee} & \longrightarrow & H^0(\mathcal{F}_x^0)^{\vee}, \end{array}$$

where $b_1, \dots, b_4$ are isomorphisms induced by the isomorphisms $\mathcal{F}^0 \cong (\mathcal{F}^1)^{\vee} \otimes \Omega_{\mathcal{C}/T}^1$, $\mathcal{F}^1 \cong (\mathcal{F}^0)^{\vee} \otimes \Omega_{\mathcal{C}/T}^1$ and Serre duality. So ξ becomes an isomorphism by the five lemma. Hence ω is non-degenerate. $\square$

Proposition 7.3. *For the 2-form constructed in Proposition 7.2, we have $d\omega = 0$.*

Proof. It suffices to show that the restriction of ω to any general fiber $M_{\mathcal{C}_x}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})$ over $T \times \Lambda_r^{(n)}(d)$ is d -closed. We may assume $\boldsymbol{\lambda}$ generic, and so $\lambda_j^{(i)} - \lambda_{j'}^{(i)} \notin \mathbf{Z}$ for any i and $j \neq j'$. We also denote the restriction by ω . Let $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ be a universal family on $\mathcal{C}_x \times M_{\mathcal{C}_x}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})$ over $M_{\mathcal{C}_x}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})$. We set

$$\begin{aligned} \mathcal{F}^0 &:= \left\{ s \in \text{End}(\tilde{E}) \middle| s|_{t_i \times M_{\mathcal{C}_x}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})}(\tilde{l}_j^{(i)}) \subset \tilde{l}_j^{(i)} \text{ for any } i, j \right\} \\ \mathcal{F}^1 &:= \left\{ s \in \text{End}(\tilde{E}) \otimes \Omega_{\mathcal{C}_x}^1(t_1 + \dots + t_n) \middle| \text{res}_{t_i \times M_{\mathcal{C}_x}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})}(s)(\tilde{l}_j^{(i)}) \subset \tilde{l}_{j+1}^{(i)} \right. \\ &\quad \left. \text{for any } i, j \right\} \end{aligned}$$

$$\nabla_{\mathcal{F}^{\bullet}} : \mathcal{F}^0 \longrightarrow \mathcal{F}^1; \quad \nabla_{\mathcal{F}^{\bullet}}(s) = \tilde{\nabla} \circ s - s \circ \tilde{\nabla}.$$

We also put

$$\mathbf{V} := \ker \tilde{\nabla}^{an}|_{(\mathcal{C}_x \setminus \{t_1, \dots, t_n\}) \times M_{\mathcal{C}_x}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})}.$$

Since there is a canonical isomorphism

$$\ker \nabla_{\mathcal{F}^\bullet}^{an}|_{(\mathcal{C}_x \setminus \{t_1, \dots, t_n\}) \times M_{\mathcal{C}_x}^\alpha(\mathbf{t}, \boldsymbol{\lambda})} \xrightarrow{\sim} \mathcal{E}nd(\mathbf{V}),$$

we obtain a canonical homomorphism

$$(14) \quad \ker \nabla_{\mathcal{F}^\bullet}^{an}|_{\mathcal{C}_x \times M_{\mathcal{C}_x}^\alpha(\mathbf{t}, \boldsymbol{\lambda})} \longrightarrow j_*(\mathcal{E}nd(\mathbf{V})),$$

where $j : (\mathcal{C}_x \setminus \{t_1, \dots, t_n\}) \times M_{\mathcal{C}_x}^\alpha(\mathbf{t}, \boldsymbol{\lambda}) \hookrightarrow \mathcal{C}_x \times M_{\mathcal{C}_x}^\alpha(\mathbf{t}, \boldsymbol{\lambda})$ is the canonical inclusion. We will show by local calculations that the homomorphism (14) is in fact an isomorphism. We can see by [4, II, Proposition 5.4] that there is an isomorphism

$$(\tilde{E}_{(t_i, y)}^{an}, \tilde{\nabla}_{(t_i, y)}^{an}) \cong \left((\mathcal{O}_{\mathcal{C}_x \times M_{\mathcal{C}_x}^\alpha(\mathbf{t}, \boldsymbol{\lambda}), (t_i, y)}^{an})^{\oplus r}, \nabla^\lambda \right),$$

on the stalks at $(t_i, y) \in \mathcal{C}_x \times M_{\mathcal{C}_x}^\alpha(\mathbf{t}, \boldsymbol{\lambda})$, where $\nabla^\lambda : \left(\mathcal{O}_{\mathcal{C}_x \times M_{\mathcal{C}_x}^\alpha(\mathbf{t}, \boldsymbol{\lambda}), (t_i, y)}^{an} \right)^{\oplus r} \rightarrow \left(\mathcal{O}_{\mathcal{C}_x \times M_{\mathcal{C}_x}^\alpha(\mathbf{t}, \boldsymbol{\lambda}), (t_i, y)}^{an} \right)^{\oplus r} \otimes \Omega_{\mathcal{C}_x \times M_{\mathcal{C}_x}^\alpha(\mathbf{t}, \boldsymbol{\lambda}) / M_{\mathcal{C}_x}^\alpha(\mathbf{t}, \boldsymbol{\lambda})}^1(t_i \times M_{\mathcal{C}_x}^\alpha(\mathbf{t}, \boldsymbol{\lambda}))$ is given by

$$\nabla^\lambda \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix} = \begin{pmatrix} df_1 \\ \vdots \\ df_r \end{pmatrix} + \begin{pmatrix} \lambda_{r-1}^{(i)} z_i^{-1} dz_i & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_0^{(i)} z_i^{-1} dz_i \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix},$$

where z_i is a generator of the maximal ideal of $\mathcal{O}_{\mathcal{C}_x, t_i}$. With respect to this isomorphism, we have

$$(\mathcal{F}^0)_{(t_i, y)}^{an} = \left\{ \begin{pmatrix} f_{1,1} & \cdots & f_{1,r} \\ \vdots & \ddots & \vdots \\ f_{r,1} & \cdots & f_{r,r} \end{pmatrix} \middle| \begin{array}{l} f_{p,q} \in z_i \mathcal{O}_{\mathcal{C}_x \times M_{\mathcal{C}_x}^\alpha(\mathbf{t}, \boldsymbol{\lambda}), (t_i, y)}^{an} \text{ for } p > q \text{ and} \\ f_{p,q} \in \mathcal{O}_{\mathcal{C}_x \times M_{\mathcal{C}_x}^\alpha(\mathbf{t}, \boldsymbol{\lambda}), (t_i, y)}^{an} \text{ for } p \leq q \end{array} \right\},$$

$$(\mathcal{F}^1)_{(t_i, y)}^{an} = \left\{ \begin{pmatrix} g_{1,1} & \cdots & g_{1,r} \\ \vdots & \ddots & \vdots \\ g_{r,1} & \cdots & g_{r,r} \end{pmatrix} \middle| \begin{array}{l} g_{p,q} \in \mathcal{O}_{\mathcal{C}_x \times M_{\mathcal{C}_x}^\alpha(\mathbf{t}, \boldsymbol{\lambda}), (t_i, y)}^{an} dz_i \text{ for } p \geq q \text{ and} \\ g_{p,q} \in \mathcal{O}_{\mathcal{C}_x \times M_{\mathcal{C}_x}^\alpha(\mathbf{t}, \boldsymbol{\lambda}), (t_i, y)}^{an} dz_i / z_i \text{ for } p < q \end{array} \right\},$$

and $(\nabla_{\mathcal{F}^\bullet})_{(t_i, y)}^{an} : (\mathcal{F}^0)_{(t_i, y)}^{an} \rightarrow (\mathcal{F}^1)_{(t_i, y)}^{an}$ is given by

$$\begin{pmatrix} f_{1,1} & f_{1,2} & \cdots & f_{1,r} \\ f_{2,1} & f_{2,2} & \cdots & f_{2,r} \\ \vdots & \vdots & \ddots & \vdots \\ f_{r,1} & f_{r,2} & \cdots & f_{r,r} \end{pmatrix} \mapsto \begin{pmatrix} df_{1,1} & df_{1,2} & \cdots & df_{1,r} \\ df_{2,1} & df_{2,2} & \cdots & df_{2,r} \\ \vdots & \vdots & \ddots & \vdots \\ df_{r,1} & df_{r,2} & \cdots & df_{r,r} \end{pmatrix} + \begin{pmatrix} 0 & (\lambda_{r-1}^{(i)} - \lambda_{r-2}^{(i)}) f_{1,2} dz_i / z_i & \cdots & (\lambda_{r-1}^{(i)} - \lambda_0^{(i)}) f_{1,r} dz_i / z_i \\ (\lambda_{r-2}^{(i)} - \lambda_{r-1}^{(i)}) f_{2,1} dz_i / z_i & 0 & \cdots & (\lambda_{r-2}^{(i)} - \lambda_0^{(i)}) f_{2,r} dz_i / z_i \\ \vdots & \vdots & \ddots & \vdots \\ (\lambda_0^{(i)} - \lambda_{r-1}^{(i)}) f_{r,1} dz_i / z_i & (\lambda_0^{(i)} - \lambda_{r-2}^{(i)}) f_{r,2} dz_i / z_i & \cdots & 0 \end{pmatrix}.$$

Take any element

$$\begin{pmatrix} f_{1,1} & f_{1,2} & \cdots & f_{1,r} \\ f_{2,1} & f_{2,2} & \cdots & f_{2,r} \\ \vdots & \vdots & \ddots & \vdots \\ f_{r,1} & f_{r,2} & \cdots & f_{r,r} \end{pmatrix} \in j_*(\mathcal{E}nd(\mathbf{V}))_{(t_i,y)} \\ = j_* \left(\ker \nabla_{\mathcal{F}^\bullet}^{an} |_{\mathcal{C}_x \setminus \{t_1, \dots, t_n\}} \times M_{\mathcal{C}_x}^\alpha(\mathbf{t}, \boldsymbol{\lambda}) \right)_{(t_i,y)}.$$

For $j \neq k$, we have $f_{j,k} = c_{j,k} z_i^{\lambda_{r-k}^{(i)} - \lambda_{r-j}^{(i)}}$ for some function $c_{j,k} \in \mathcal{O}_{M_{\mathcal{C}_x}^\alpha(\mathbf{t}, \boldsymbol{\lambda}), y}^{an}$. Since $f_{j,k}$ is a single valued function on $(U \setminus \{t_i\}) \times M$ for some open neighborhood U of t_i in $\mathcal{C}_x$ and some open neighborhood M of y in $M_{\mathcal{C}_x}^\alpha(\mathbf{t}, \boldsymbol{\lambda})$ and $\lambda_{r-k}^{(i)} - \lambda_{r-j}^{(i)} \notin \mathbf{Z}$, we have $c_{j,k} = 0$, namely $f_{j,k} = 0$. On the other hand, we have $df_{j,j} = 0$ for $j = 1, \dots, r$. So we have $f_{j,j} \in \mathcal{O}_{M_{\mathcal{C}_x}^\alpha(\mathbf{t}, \boldsymbol{\lambda}), y}^{an}$. Thus we have

$$\begin{pmatrix} f_{1,1} & f_{1,2} & \cdots & f_{1,r} \\ f_{2,1} & f_{2,2} & \cdots & f_{2,r} \\ \vdots & \vdots & \ddots & \vdots \\ f_{r,1} & f_{r,2} & \cdots & f_{r,r} \end{pmatrix} \in (\mathcal{F}^0)_{(t_i,y)}^{an} \cap j_* (\ker \nabla_{\mathcal{F}^\bullet}^{an} |_{\mathcal{C}_x \setminus \{t_1, \dots, t_n\}})_{(t_i,y)} \\ = \left(\ker \nabla_{\mathcal{F}^\bullet}^{an} |_{\mathcal{C}_x \times M_{\mathcal{C}_x}^\alpha(\mathbf{t}, \boldsymbol{\lambda})} \right)_{(t_i,y)}.$$

So the homomorphism (14) is surjective. The injectivity of (14) is obvious. Thus (14) is isomorphic.

Next we will show that $\nabla_{\mathcal{F}^\bullet}^{an} : (\mathcal{F}^0)^{an} \rightarrow (\mathcal{F}^1)^{an}$ is surjective. For this it is essential to show the surjectivity of the morphism

$$(\nabla_{\mathcal{F}^\bullet})_{(t_i,y)}^{an} : (\mathcal{F}^0)_{(t_i,y)}^{an} \longrightarrow (\mathcal{F}^1)_{(t_i,y)}^{an} \quad ((t_i, y) \in \mathcal{C}_x \times M_{\mathcal{C}_x}^\alpha(\mathbf{t}, \boldsymbol{\lambda}))$$

at the stalks. Take any member

$$\begin{pmatrix} g_{1,1} & \cdots & g_{1,r} \\ \vdots & \ddots & \vdots \\ g_{r,1} & \cdots & g_{r,r} \end{pmatrix} \in (\mathcal{F}^1)_{(t_i,y)}^{an}.$$

For $p > q$ we can write $g_{p,q} = \sum_{k=0}^{\infty} b_{p,q}^{(k)} z_i^k dz_i$, where $b_{p,q}^{(k)} \in \mathcal{O}_{M_{\mathcal{C}_x}^\alpha(\mathbf{t}, \boldsymbol{\lambda}), y}^{an}$ for $k \geq 0$. We define a power series $f_{p,q} = \sum_{k=0}^{\infty} a_{p,q}^{(k)} z_i^k$ by $a_{p,q}^{(0)} := 0$ and $a_{p,q}^{(k)} := (k + \lambda_{r-p}^{(i)} - \lambda_{r-q}^{(i)})^{-1} b_{p,q}^{(k-1)}$ for $k \geq 1$. Since $\sum_{k=0}^{\infty} b_{p,q}^{(k)} z_i^k$ is a convergent power series, we can see that $\sum_{k=0}^{\infty} a_{p,q}^{(k)} z_i^k$ is also convergent. So we have $f_{p,q} := \sum_{k=0}^{\infty} a_{p,q}^{(k)} z_i^k \in \mathcal{O}_{\mathcal{C}_x \times M_{\mathcal{C}_x}^\alpha(\mathbf{t}, \boldsymbol{\lambda}), (t_i,y)}^{an}$ and $df_{p,q} + (\lambda_{r-p}^{(i)} - \lambda_{r-q}^{(i)}) f_{p,q} dz_i / z_i = g_{p,q}$. For $g_{p,p} \in \mathcal{O}_{\mathcal{C}_x \times M_{\mathcal{C}_x}^\alpha(\mathbf{t}, \boldsymbol{\lambda}), (t_i,y)}^{an} dz_i$, we can find $f_{p,p} \in \mathcal{O}_{\mathcal{C}_x \times M_{\mathcal{C}_x}^\alpha(\mathbf{t}, \boldsymbol{\lambda}), (t_i,y)}^{an}$

such that $df_{p,p} = g_{p,p}$. For $p < q$, we can write $g_{p,q} = \sum_{k=0}^{\infty} b_{p,q}^{(k)} z_i^{k-1} dz_i$ with $b_{p,q}^{(k)} \in \mathcal{O}_{M_{\mathcal{C}_x}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda}), y}^{an}$ for $k \geq 0$. For each $k \geq 0$, we put $a_{p,q}^{(k)} := (k + \lambda_{r-p}^{(i)} - \lambda_{r-q}^{(i)})^{-1} b_{p,q}^{(k)}$. Since $\sum_{k=0}^{\infty} b_{p,q}^{(k)} z_i^k$ is convergent, we can see that $f_{p,q} := \sum_{k=0}^{\infty} a_{p,q}^{(k)} z_i^k$ is also convergent. Hence, we have $f_{p,q} \in \mathcal{O}_{\mathcal{C}_x \times M_{\mathcal{C}_x}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda}), (t_i, y)}^{an}$ and $df_{p,q} + (\lambda_{r-p}^{(i)} - \lambda_{r-q}^{(i)}) f_{p,q} dz_i / z_i = g_{p,q}$. By construction, we have

$$\begin{pmatrix} f_{1,1} & \cdots & f_{1,r} \\ \vdots & \ddots & \vdots \\ f_{r,1} & \cdots & f_{r,r} \end{pmatrix} \in (\mathcal{F}^0)_{(t_i, y)}^{an}$$

and

$$(\nabla_{\mathcal{F}^\bullet})_{(t_i, y)}^{an} \begin{pmatrix} f_{1,1} & \cdots & f_{1,r} \\ \vdots & \ddots & \vdots \\ f_{r,1} & \cdots & f_{r,r} \end{pmatrix} = \begin{pmatrix} g_{1,1} & \cdots & g_{1,r} \\ \vdots & \ddots & \vdots \\ g_{r,1} & \cdots & g_{r,r} \end{pmatrix}.$$

Hence we obtain the surjectivity of $\nabla_{\mathcal{F}^\bullet}^{an} : (\mathcal{F}^0)^{an} \rightarrow (\mathcal{F}^1)^{an}$.

So we obtain an isomorphism

$$R^1(\pi_{M_{\mathcal{C}_x}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})})_*((\mathcal{F}^\bullet)^{an}) \xrightarrow{\sim} R^1(\pi_{M_{\mathcal{C}_x}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})})_*(j_*(\mathcal{E}nd(\mathbf{V})))$$

of analytic sheaves. Then ω corresponds to a canonical pairing

$$\begin{aligned} R^1(\pi_{M_{\mathcal{C}_x}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})})_*(j_*(\mathcal{E}nd(\mathbf{V}))) \times R^1(\pi_{M_{\mathcal{C}_x}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})})_*(j_*(\mathcal{E}nd(\mathbf{V}))) \\ \longrightarrow R^2(\pi_{M_{\mathcal{C}_x}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})})_*(\pi_{M_{\mathcal{C}_x}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})}^{-1}(\mathcal{O}_{M_{\mathcal{C}_x}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})})) \\ ([\{c_{\alpha\beta}\}], [\{c'_{\alpha\beta}\}]) \mapsto [\{\text{Tr}(c_{\alpha\beta} \circ c'_{\beta\gamma})\}]. \end{aligned}$$

Note that $R^2(\pi_{M_{\mathcal{C}_x}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})})_*(\pi_{M_{\mathcal{C}_x}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})}^{-1}(\mathcal{O}_{M_{\mathcal{C}_x}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})})) \cong \mathcal{O}_{M_{\mathcal{C}_x}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})}$.

Take any small open subset $M \subset M_{\mathcal{C}_x}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda})$ and an open covering $\{U_\alpha\}$ of $\mathcal{C}_x$ such that $\sharp\{\alpha | t_i \in U_\alpha\} = 1$ for any i and that $\sharp(U_\alpha \cap \{t_1, \dots, t_n\}) \leq 1$ for any α . If we replace U_α and M sufficiently smaller, there exists a sheaf E_α on U_α such that $E_\alpha|_{U_\alpha \cap U_\beta} \cong \mathbf{C}_{U_\alpha \cap U_\beta}^{\oplus r^2}$ for any $\beta \neq \alpha$ and an isomorphism $\phi_\alpha : j_*(\mathbf{V})|_{U_\alpha \times M} \xrightarrow{\sim} E_\alpha \otimes \mathcal{O}_M$. For each α, β , we put

$$\begin{aligned} \varphi_{\alpha\beta} &:= \phi_\alpha \circ \phi_\beta^{-1} : E_\beta \otimes \mathcal{O}_M|_{(U_\alpha \cap U_\beta) \times M} \\ &\xrightarrow{\phi_\beta^{-1}} j_*(\mathbf{V})|_{(U_\alpha \cap U_\beta) \times M} \xrightarrow{\phi_\alpha} E_\alpha \otimes \mathcal{O}_M|_{(U_\alpha \cap U_\beta) \times M}. \end{aligned}$$

Hence, $j_*(\mathbf{V})|_{\mathcal{C}_x \times M}$ is given by the transition functions $\{\varphi_{\alpha\beta}\}$. Next we consider a vector field $v \in H^0(M, \Theta_M)$. Then v corresponds to a derivation $D_v : \mathcal{O}_M \rightarrow \mathcal{O}_M$ which naturally induces a morphism

$$D_v : \mathcal{H}om(E_\beta|_{U_\alpha \cap U_\beta}, E_\alpha|_{U_\alpha \cap U_\beta}) \otimes \mathcal{O}_M \longrightarrow \mathcal{H}om(E_\beta|_{U_\alpha \cap U_\beta}, E_\alpha|_{U_\alpha \cap U_\beta}) \otimes \mathcal{O}_M.$$

We note that v also corresponds to a morphism $f_v : \text{Spec } \mathcal{O}_M[\epsilon] \rightarrow M$, where $\epsilon^2 = 0$. Then $(1_{\mathcal{C}_x} \times f_v)^*(j_*(\mathbf{V}))$ corresponds to the gluing data $\{(1_{\mathcal{C}_x} \times f_v)^*(\varphi_{\alpha\beta})\}$. We can see the following equality:

$$\begin{aligned} (1_{\mathcal{C}_x} \times f_v)^*(\varphi_{\alpha\beta}) &= \varphi_{\alpha\beta} + \epsilon D_v(\varphi_{\alpha\beta}) : \\ E_\beta|_{U_\alpha \cap U_\beta} \otimes \mathcal{O}_M[\epsilon] &\xrightarrow{\sim} E_\alpha|_{U_\alpha \cap U_\beta} \otimes \mathcal{O}_M[\epsilon] \\ a + \epsilon b &\mapsto \varphi_{\alpha\beta}(a) + \epsilon(\varphi_{\alpha\beta}(b) + D_v(\varphi_{\alpha\beta})(a)). \end{aligned}$$

So the isomorphism $\Theta_M \cong R^1(\pi_M)_*(j_*(\mathcal{E}nd(\mathbf{V}))_M)$ is given by

$$\Theta_M \ni v \mapsto [\{\phi_\alpha^{-1} \circ D_v(\varphi_{\alpha\beta}) \circ \phi_\beta\}] \in R^1(\pi_M)_*(j_*(\mathcal{E}nd(\mathbf{V}_M)))$$

and

$$\omega(u, v) = [\{\text{Tr}(D_u(\varphi_{\alpha\beta}) \circ D_v(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha})\}] \in R^2(\pi_M)_*(\pi_M^{-1}(\mathcal{O}_M)) \cong \mathcal{O}_M.$$

Thus we have

$$\begin{aligned} d\omega(u, v, w) &= D_u(\omega(v, w)) + D_v(\omega(w, u)) + D_w(\omega(u, v)) \\ &\quad + \omega(w, [u, v]) + \omega([u, w], v) + \omega(u, [v, w]) \\ &= \left\{ \text{Tr} \left(D_u(D_v(\varphi_{\alpha\beta}) \circ D_w(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha}) \right. \right. \\ &\quad + D_v(D_w(\varphi_{\alpha\beta}) \circ D_u(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha}) \\ &\quad + D_w(D_u(\varphi_{\alpha\beta}) \circ D_v(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha}) \\ &\quad + D_w(\varphi_{\alpha\beta}) \circ (D_u D_v - D_v D_u)(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} \\ &\quad + (D_u D_w - D_w D_u)(\varphi_{\alpha\beta}) \circ D_v(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} \\ &\quad \left. \left. + D_u(\varphi_{\alpha\beta}) \circ (D_v D_w - D_w D_v)(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} \right) \right\} \\ &= \left\{ \text{Tr} \left(D_u D_v(\varphi_{\alpha\beta}) \circ D_w(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} \right. \right. \\ &\quad + D_w(\varphi_{\alpha\beta}) \circ D_u D_v(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} \\ &\quad + D_u D_w(\varphi_{\alpha\beta}) \circ D_v(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} \\ &\quad + D_v(\varphi_{\alpha\beta}) \circ D_u D_w(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} \\ &\quad + D_v D_w(\varphi_{\alpha\beta}) \circ D_u(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} \\ &\quad + D_u(\varphi_{\alpha\beta}) \circ D_v D_w(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} \\ &\quad + D_u(\varphi_{\alpha\beta}) \circ D_v(\varphi_{\beta\gamma}) \circ D_w(\varphi_{\gamma\alpha}) \\ &\quad + D_v(\varphi_{\alpha\beta}) \circ D_w(\varphi_{\beta\gamma}) \circ D_u(\varphi_{\gamma\alpha}) \\ &\quad \left. \left. + D_w(\varphi_{\alpha\beta}) \circ D_u(\varphi_{\beta\gamma}) \circ D_v(\varphi_{\gamma\alpha}) \right) \right\}. \end{aligned}$$

On the other hand, applying $D_u D_v D_w$ to $\varphi_{\alpha\beta} \circ \varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$, we obtain the equality

$$\begin{aligned} & -D_u D_v D_w(\varphi_{\alpha\beta})\varphi_{\beta\gamma} + D_u D_v D_w(\varphi_{\alpha\gamma}) - \varphi_{\alpha\beta} D_u D_v D_w(\varphi_{\beta\gamma}) \\ & = D_u D_v(\varphi_{\alpha\beta}) \circ D_w(\varphi_{\beta\gamma}) + D_w(\varphi_{\alpha\beta}) \circ D_u D_v(\varphi_{\beta\gamma}) \\ & \quad + D_u D_w(\varphi_{\alpha\beta}) \circ D_v(\varphi_{\beta\gamma}) + D_v(\varphi_{\alpha\beta}) \circ D_u D_w(\varphi_{\beta\gamma}) \\ & \quad + D_v D_w(\varphi_{\alpha\beta}) \circ D_u(\varphi_{\beta\gamma}) + D_u(\varphi_{\alpha\beta}) \circ D_v D_w(\varphi_{\beta\gamma}) \end{aligned}$$

which implies

$$\begin{aligned} & -d\{\mathrm{Tr}(D_u D_v D_w(\varphi_{\alpha\beta})\varphi_{\beta\alpha})\} \\ & = \left\{ \mathrm{Tr} \left(D_u D_v(\varphi_{\alpha\beta}) \circ D_w(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} + D_w(\varphi_{\alpha\beta}) \circ D_u D_v(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} \right. \right. \\ & \quad + D_u D_w(\varphi_{\alpha\beta}) \circ D_v(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} + D_v(\varphi_{\alpha\beta}) \circ D_u D_w(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} \\ & \quad + D_v D_w(\varphi_{\alpha\beta}) \circ D_u(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} \\ & \quad \left. \left. + D_u(\varphi_{\alpha\beta}) \circ D_v D_w(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} \right) \right\}. \end{aligned}$$

We can also check the equality

$$\begin{aligned} & \mathrm{Tr} \left(D_u(\varphi_{\alpha\beta}) \circ D_v(\varphi_{\beta\gamma}) \circ D_w(\varphi_{\gamma\alpha}) + D_v(\varphi_{\alpha\beta}) \circ D_w(\varphi_{\beta\gamma}) \circ D_u(\varphi_{\gamma\alpha}) \right. \\ & \quad \left. + D_w(\varphi_{\alpha\beta}) \circ D_u(\varphi_{\beta\gamma}) \circ D_v(\varphi_{\gamma\alpha}) \right) \\ & = \mathrm{Tr} \left(-D_u(\varphi_{\alpha\beta}) D_v(\varphi_{\beta\alpha}) D_w(\varphi_{\alpha\beta}) \varphi_{\beta\alpha} + D_u(\varphi_{\alpha\gamma}) D_v(\varphi_{\gamma\alpha}) D_w(\varphi_{\alpha\gamma}) \varphi_{\gamma\alpha} \right. \\ & \quad \left. - \varphi_{\alpha\beta} D_u(\varphi_{\beta\gamma}) D_v(\varphi_{\gamma\beta}) D_w(\varphi_{\beta\gamma}) \varphi_{\gamma\alpha} \right) \end{aligned}$$

which means

$$\begin{aligned} & -d\{\mathrm{Tr}(D_u(\varphi_{\alpha\beta}) D_v(\varphi_{\beta\alpha}) D_w(\varphi_{\alpha\beta}) \varphi_{\beta\alpha})\} \\ & = \left\{ \mathrm{Tr} \left(D_u(\varphi_{\alpha\beta}) \circ D_v(\varphi_{\beta\gamma}) \circ D_w(\varphi_{\gamma\alpha}) + D_v(\varphi_{\alpha\beta}) \circ D_w(\varphi_{\beta\gamma}) \circ D_u(\varphi_{\gamma\alpha}) \right. \right. \\ & \quad \left. \left. + D_w(\varphi_{\alpha\beta}) \circ D_u(\varphi_{\beta\gamma}) \circ D_v(\varphi_{\gamma\alpha}) \right) \right\}. \end{aligned}$$

Thus we can see that $d\omega(u, v, w) = 0$. $\square$

Remark 7.1. (1) Put $(C, \mathbf{t}) = (C_x, \tilde{\mathbf{t}}_x)$ for a point $x \in T$ and take $\boldsymbol{\lambda} \in \Lambda_r^{(n)}(d)$. Let (L, ∇_L) be a pair of a line bundle L on C and a connection $\nabla_L : L \rightarrow L \otimes \Omega_C^1(D(\mathbf{t}))$ such that $\mathrm{res}_{t_i}(\nabla_L) = \sum_{j=0}^{r-1} \lambda_j^{(i)}$. Consider the moduli space $M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L)$ of α -stable parabolic connections with the determinant (L, ∇_L) . Then we can see by the same proof as Proposition 7.2 that the restriction $\omega|_{M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L)}$ of ω to $M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L)$ is also non-degenerate. So $M_C^\alpha(\mathbf{t}, \boldsymbol{\lambda}, L)$ also has a symplectic structure.

(2) By Proposition 8.1, we obtain a splitting $\Theta_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)/\Lambda_r^{(n)}(d)} \cong \pi^* \Theta_T \oplus \Theta_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)/T \times \Lambda_r^{(n)}(d)}$. With respect to this splitting, we can lift ω to $\tilde{\omega} \in H^0(\Omega^2_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)/\Lambda_r^{(n)}(d)})$ and we have $d\tilde{\omega} = 0$. This 2-form $\tilde{\omega}$ is nothing but the 2-form considered in [10] and [11]. Note that we have $\tilde{\omega} \cdot v = 0$ for $v \in D(\pi^* \Theta_T)$, where D is the homomorphism given in Proposition 8.1.

8. Isomonodromic deformation

Let $\tilde{T} \rightarrow T$ be a universal covering. Then $RP_r(\mathcal{C}, \tilde{\mathbf{t}}) \times_T \tilde{T} \rightarrow \tilde{T}$ becomes a trivial fibration and we can consider the set of constant sections

$$\mathcal{F}_R^{\sharp} = \left\{ \sigma : \tilde{T} \rightarrow RP_r(\mathcal{C}, \tilde{\mathbf{t}})^{\sharp} \times_T \tilde{T} \right\},$$

where

$$RP_r(\mathcal{C}, \tilde{\mathbf{t}})^{\sharp} = \coprod_{(x, \mathbf{a}) \in T \times \mathcal{A}_r^{(n)}} RP_r(\mathcal{C}_x, \tilde{\mathbf{t}}_x)^{\sharp}.$$

The pull back $\tilde{\mathcal{F}}_M^{\sharp} = \{\mathbf{RH}^{-1}(\sigma)\}$ of these constant sections determines a foliation on $M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)^{\sharp} \times_T \tilde{T}$ where $M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)^{\sharp} = \mathbf{RH}^{-1}(RP_r(\mathcal{C}, \tilde{\mathbf{t}})^{\sharp})$. This foliation corresponds to a subbundle determined by a splitting

$$D^{\sharp} : \tilde{\pi}^*(\Theta_{\tilde{T}}^{an}) \rightarrow \Theta_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)^{\sharp} \times_T \tilde{T}}^{an}$$

of the analytic tangent map

$$\Theta_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)^{\sharp} \times_T \tilde{T}}^{an} \longrightarrow \tilde{\pi}^* \Theta_{\tilde{T}}^{an} \rightarrow 0,$$

where $\tilde{\pi} : M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)^{\sharp} \times_T \tilde{T} \rightarrow \tilde{T}$ is the projection. We will show that this splitting $D^{\sharp}$ is in fact induced by a splitting

$$D : \pi^*(\Theta_T) \rightarrow \Theta_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)}$$

of the algebraic tangent bundle, where $\pi : M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d) \rightarrow T$ is the projection. This splitting is nothing but the differential equation determined by the isomonodromic deformation.

Usually, the equation of the isomonodromic deformation is given by the following way: Assume $C = \mathbf{P}^1$ and take a Zariski open set $U \subset M_{\mathbf{P}^1 \times T/T}^{\alpha}(\tilde{\mathbf{t}}, r, 0)$ such that a universal family on $\mathbf{P}^1 \times U$ is given by $(\mathcal{O}_{\mathbf{P}^1 \times U}^{\oplus r}, \nabla, \{\tilde{l}_j^{(i)}\})$, where ∇ is given by a connection matrix

$$\sum_{i=1}^n \frac{A_i dz}{z - t_i} \quad (A_i = (a_{jk}^{(i)})_{1 \leq j, k \leq r} \in M_r(\mathcal{O}_U)).$$

Then the differential equation of the isomonodromic deformation is given by

$$(15) \quad \frac{\partial A_i}{\partial t_j} = -\frac{[A_i, A_j]}{t_i - t_j} \quad (t_i \neq t_j), \quad \frac{\partial A_i}{\partial t_i} = \sum_{k \neq i} \frac{[A_i, A_k]}{t_i - t_k}.$$

The equation (15) is called the Schlesinger equation and, needless to say, this equation plays a great role in the explicit description of many differential equations arising from the isomonodromic deformation. However, U and the equation (15) do not have the geometric Painlevé property. For example, take a point x of U over $\mathbf{t} \in T$ which corresponds to $(\mathcal{O}_{\mathbf{P}_V^{\oplus r}}, \nabla, \{l_j^{(i)}\})$. Take a path γ in $M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)$ starting at x and coming back to a point y over $\mathbf{t}$ such that γ is a solution of the differential equation determined by the isomonodromic deformation. Let $(F, \nabla_F, \{l_F^{(i)}\})$ be the parabolic connection corresponding to y . Then F may not be trivial and so y may not lie in U . Thus U and (15) do not have the geometric Painlevé property. Moreover, we want to describe the isomonodromic deformation even on the locus contracted by **RH**. So the equation (15) itself is not enough for the explicit geometric description of the isomonodromic deformation, because (15) is defined only on a Zariski open set and is not even defined in the higher genus case. In order to obtain the splitting D , we will go back to how the equation (15) arises as the isomonodromic deformation. Take a local constant section $\sigma : T' \rightarrow RP_r(\mathcal{C}, \tilde{\mathbf{t}})^{\sharp} \times_T T'$, where T' is an analytic open subset of T . Then σ corresponds to a local system $\mathbf{V}_{\sigma}$ on $\mathcal{C}_{T'} \setminus ((\tilde{t}_1)_{T'} + \cdots + (\tilde{t}_n)_{T'})$ after shrinking T' . The canonical flat connection on $\mathbf{V}_{\sigma} \otimes \mathcal{O}_{\mathcal{C}_{T'} \setminus ((\tilde{t}_1)_{T'} + \cdots + (\tilde{t}_n)_{T'})}$ extends to an integrable logarithmic connection

$$(16) \quad \nabla^{\sigma} : \tilde{E}_{\mathbf{RH}^{-1}(\sigma(T'))} \longrightarrow \tilde{E}_{\mathbf{RH}^{-1}(\sigma(T'))} \otimes \Omega_{\mathcal{C}_{T'}}^1(\log((\tilde{t}_1)_{T'} + \cdots + (\tilde{t}_n)_{T'}))$$

whose induced relative connection is just $\tilde{\nabla}|_{\mathcal{C} \times_T \mathbf{RH}^{-1}(\sigma(T'))}$. The equation (15) comes from the integrability condition for ∇^{σ} . Taking the tangent direction of $\sigma(T')$, we obtain the splitting D . Precisely, we have the following proposition:

Proposition 8.1. *There exists an algebraic splitting*

$$D : \pi^*(\Theta_T) \rightarrow \Theta_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)}$$

of the tangent map $\Theta_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)} \rightarrow \pi^*(\Theta_T)$ whose pull-back to $M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)^{\sharp} \times_T \tilde{T}$ coincides with $D^{\sharp}$.

Proof. We will define the corresponding homomorphism $\Theta_T \rightarrow \pi_* \Theta_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)}$. Take any affine open set $U \subset T$ and a vector field $v \in H^0(U, \Theta_T)$. Then v corresponds to a morphism $\iota^v : \text{Spec } \mathcal{O}_U[\epsilon] \rightarrow T$ with $\epsilon^2 = 0$ such that the composite $U \hookrightarrow \text{Spec } \mathcal{O}_U[\epsilon] \rightarrow T$ is just the inclusion

$U \hookrightarrow T$. We denote the restriction of the universal family to $\mathcal{C} \times_T \pi^{-1}(U)$ simply by $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$. Consider the fiber product $\mathcal{C} \times_T \text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon]$ with respect to the canonical projection $\mathcal{C} \rightarrow T$ and the composite $\text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon] \rightarrow \text{Spec } \mathcal{O}_U[\epsilon] \xrightarrow{\iota^v} T$. We denote the pull-back of $D(\tilde{\mathbf{t}})$ by the morphism $\mathcal{C} \times_T \text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon] \rightarrow \mathcal{C}$ simply by $D(\tilde{\mathbf{t}})_{\mathcal{O}_{\pi^{-1}(U)}[\epsilon]}$. We call $(\mathcal{E}, \nabla^{\mathcal{E}}, \{(l_{\mathcal{E}})_j^{(i)}\})$ a horizontal lift of $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ if

- (1) $\mathcal{E}$ is a vector bundle on $\mathcal{C} \times_T \text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon]$,
- (2) $\mathcal{E}|_{\tilde{t}_i \times \text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon]} = (l_{\mathcal{E}})_0^{(i)} \supset \cdots \supset (l_{\mathcal{E}})_r^{(i)} = 0$ is a filtration by subbundles for $i = 1, \dots, n$,
- (3) $\nabla^{\mathcal{E}} : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_{\mathcal{C} \times_T \text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon]/\pi^{-1}(U)}^1 \left(\log \left(D(\tilde{\mathbf{t}})_{\mathcal{O}_{\pi^{-1}(U)}[\epsilon]} \right) \right)$ is a connection satisfying

(a)

$$\nabla^{\mathcal{E}}(F_j^{(i)}(\mathcal{E})) \subset F_j^{(i)}(\mathcal{E}) \otimes \Omega_{\mathcal{C} \times_T \text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon]/\pi^{-1}(U)}^1 \left(\log D(\tilde{\mathbf{t}})_{\mathcal{O}_{\pi^{-1}(U)}[\epsilon]} \right),$$

where $F_j^{(i)}(\mathcal{E})$ is given by

$$F_j^{(i)}(\mathcal{E}) := \ker \left(\mathcal{E} \rightarrow \mathcal{E}|_{\tilde{t}_i \times \text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon]} / (l_{\mathcal{E}})_j^{(i)} \right).$$

- (b) The curvature

$$\nabla^{\mathcal{E}} \circ \nabla^{\mathcal{E}} : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_{\mathcal{C} \times_T \text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon]/\pi^{-1}(U)}^2 \left(\log \left(D(\tilde{\mathbf{t}})_{\mathcal{O}_{\pi^{-1}(U)}[\epsilon]} \right) \right)$$

is zero.

- (c) $(\text{res}_{\tilde{t}_i \times \text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon]}(\tilde{\nabla}^{\mathcal{E}}) - \tilde{\lambda}_j^{(i)})(l_{\mathcal{E}})_j^{(i)} \subset (l_{\mathcal{E}})_{j+1}^{(i)}$ for any i, j , where $\tilde{\nabla}^{\mathcal{E}}$ is the relative connection over $\text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon]$ induced by $\nabla^{\mathcal{E}}$.
- (d) $(\mathcal{E}, \tilde{\nabla}^{\mathcal{E}}, \{(l_{\mathcal{E}})_j^{(i)}\}) \otimes \mathcal{O}_{\pi^{-1}(U)}[\epsilon]/(\epsilon) \cong (\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$.

Here we define the sheaf $\Omega_{\mathcal{C} \times_T \text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon]/\pi^{-1}(U)}^1 \left(\log \left(D(\tilde{\mathbf{t}})_{\mathcal{O}_{\pi^{-1}(U)}[\epsilon]} \right) \right)$ as the coherent subsheaf of $\Omega_{\mathcal{C} \times_T \text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon]/\pi^{-1}(U)}^1 \left(D(\tilde{\mathbf{t}})_{\mathcal{O}_{\pi^{-1}(U)}[\epsilon]} \right)$ locally generated by $\tilde{g}^{-1}d\tilde{g}$ and $d\epsilon$ for a local defining equation $\tilde{g}$ of $D(\tilde{\mathbf{t}})_{\mathcal{O}_{\pi^{-1}(U)}[\epsilon]}$ and the sheaf $\Omega_{\mathcal{C} \times_T \text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon]/\pi^{-1}(U)}^2 \left(\log \left(D(\tilde{\mathbf{t}})_{\mathcal{O}_{\pi^{-1}(U)}[\epsilon]} \right) \right)$ as the coherent subsheaf of $\Omega_{\mathcal{C} \times_T \text{Spec } \mathcal{O}_{\pi^{-1}(U)}[\epsilon]/\pi^{-1}(U)}^2 \left(D(\tilde{\mathbf{t}})_{\mathcal{O}_{\pi^{-1}(U)}[\epsilon]} \right)$ locally generated by $\tilde{g}^{-1}d\tilde{g} \wedge d\epsilon$. Assume that the parabolic connection $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})$ is locally given in a small affine open subset W of $\mathcal{C} \times_T \pi^{-1}(U)$ by a connection matrix $Ag^{-1}dg$, where g is a local defining equation of $(\tilde{t}_i \times \pi^{-1}(U)) \cap W$, $A \in M_r(\mathcal{O}_W)$, $A((\tilde{t}_i \times \pi^{-1}(U)) \cap W)$ is an upper triangular matrix and the parabolic structure $\{\tilde{l}_j^{(i)}\}_W$ is given by $(\tilde{l}_j^{(i)})_W = (\overbrace{*, \dots, *}^{r-j}, 0, \dots, 0)$. Let

$\tilde{W}$ be the affine open subscheme of $\mathcal{C} \times \operatorname{Spec} \mathcal{O}_{\pi^{-1}(U)}[\epsilon]$ whose underlying space is W and let $\tilde{g}$ be a lift of g which is a local defining equation of $(\tilde{t}_i \times \operatorname{Spec} \mathcal{O}_{\pi^{-1}(U)}[\epsilon]) \cap \tilde{W}$. If we denote the composite $\mathcal{O}_{\tilde{W}} \xrightarrow{d} \Omega_{\tilde{W}/\pi^{-1}(U)}^1 = \mathcal{O}_{\tilde{W}} d\tilde{g} \oplus \mathcal{O}_{\tilde{W}} d\epsilon \rightarrow \mathcal{O}_{\tilde{W}} d\epsilon$ by d_ϵ , we can take a lift $\tilde{A} \in M_r(\mathcal{O}_{\tilde{W}})$ of A such that $d_\epsilon(\tilde{A}) = 0$ and $\tilde{A}((\tilde{t}_i \times \operatorname{Spec} \mathcal{O}_{\pi^{-1}(U)}[\epsilon]) \cap \tilde{W})$ is an upper-triangular matrix. Then the connection matrix $\tilde{A}\tilde{g}^{-1}d\tilde{g}$ gives a local horizontal lift of $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_j^{(i)}\})|_W$. We put

$$\begin{aligned} \mathcal{F}^0 &:= \left\{ u \in \mathcal{E}nd(\tilde{E}) \mid u|_{\tilde{t}_i \times \pi^{-1}(U)}(\tilde{l}_j^{(i)}) \subset \tilde{l}_j^{(i)} \quad (1 \leq i \leq n, 0 \leq j \leq r) \right\}, \\ \mathcal{F}^1 &:= \left\{ u \in \mathcal{E}nd(\tilde{E}) \otimes \tilde{\Omega}^1 \mid \begin{array}{l} u(F_j^{(i)}(\tilde{E})) \subset F_j^{(i)}(\tilde{E}) \otimes \tilde{\Omega}^1 \\ \text{for any } i, j \text{ and the image of} \\ F_j^{(i)}(\tilde{E}) \xrightarrow{u} \tilde{E} \otimes \tilde{\Omega}^1 \xrightarrow{\operatorname{res}_{\tilde{t}_i}} \tilde{E}|_{\tilde{t}_i \times \pi^{-1}(U)} \\ \text{is contained in } \tilde{l}_{j+1}^{(i)} \text{ for any } i, j \end{array} \right\}, \\ \mathcal{F}^2 &:= \left\{ u \in \mathcal{E}nd(\tilde{E}) \otimes \tilde{\Omega}^2 \mid u(F_j^{(i)}(\tilde{E})) \subset F_{j+1}^{(i)}(\tilde{E}) \otimes \tilde{\Omega}^2 \text{ for any } i, j \right\}, \end{aligned}$$

where we put

$$\begin{aligned} \tilde{\Omega}^1 &:= \Omega_{\mathcal{C} \times_{T\pi^{-1}(U)}/\pi^{-1}(U)}^1(D(\tilde{\mathbf{t}})) \oplus \mathcal{O}_{\mathcal{C} \times_{T\pi^{-1}(U)}} d\epsilon, \\ \tilde{\Omega}^2 &:= \Omega_{\mathcal{C} \times_{T\pi^{-1}(U)}/\pi^{-1}(U)}^1(D(\tilde{\mathbf{t}})) \wedge d\epsilon, \\ F_j^{(i)}(\tilde{E}) &:= \ker \left(\tilde{E} \rightarrow \tilde{E}|_{\tilde{t}_i \times \pi^{-1}(U)} / \tilde{l}_j^{(i)} \right). \end{aligned}$$

Consider the complex

$$\mathcal{F}^0 \xrightarrow{d^0} \mathcal{F}^1 \xrightarrow{d^1} \mathcal{F}^2$$

defined by $d^0(u) = \tilde{\nabla} \circ u - u \circ \tilde{\nabla} + u d\epsilon$ and $d^1(\omega + a d\epsilon) = d\epsilon \wedge \omega + (\tilde{\nabla} \circ a - a \circ \tilde{\nabla}) \wedge d\epsilon$ for $u \in \mathcal{F}^0$, $\omega \in \mathcal{E}nd(\tilde{E}) \otimes g^{-1}dg$ and $a \in \mathcal{F}^0$ such that $\omega + a d\epsilon \in \mathcal{F}^1$.

We can see that an obstruction class for the existence of a horizontal lift of $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_i\})$ is in $\mathbf{H}^2(\mathcal{F}^\bullet)$ and the set of horizontal lifts is isomorphic to $\mathbf{H}^1(\mathcal{F}^\bullet)$ if it is not empty. It is obvious that d^0 is injective and d^1 is surjective. Take any member $\omega + a d\epsilon \in \ker(d^1)$. Then we have $d^1(\omega + a d\epsilon) = d\epsilon \wedge \omega + (\tilde{\nabla} \circ a - a \circ \tilde{\nabla}) \wedge d\epsilon = 0$. So we have $\omega = \tilde{\nabla} \circ a - a \circ \tilde{\nabla}$ and $d^0(a) = \tilde{\nabla} \circ a - a \circ \tilde{\nabla} + a d\epsilon = \omega + a d\epsilon$. Thus $\mathcal{F}^\bullet$ is exact at $\mathcal{F}^1$. Hence $\mathcal{F}^\bullet$ is an exact complex. Thus we have $\mathbf{H}^2(\mathcal{F}^\bullet) = 0$, $\mathbf{H}^1(\mathcal{F}^\bullet) = 0$ and there is a unique horizontal lift $(\mathcal{E}, \nabla^\mathcal{E}, \{(l^\mathcal{E})_j^{(i)}\})$ of $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_i\})$. We can check that $v \mapsto (\mathcal{E}, \tilde{\nabla}^\mathcal{E}, \{(l^\mathcal{E})_j^{(i)}\})$ gives an $\mathcal{O}_T$ -homomorphism $\Theta_T \rightarrow \pi_*(\Theta_{M_{\mathcal{C}/T}^\mathcal{E}(\tilde{\mathbf{t}}, r, d)})$. Thus we obtain an $\mathcal{O}_{M_{\mathcal{C}/T}^\mathcal{E}(\tilde{\mathbf{t}}, r, d)}$ -homomorphism

$$D : \pi^*(\Theta_T) \rightarrow \Theta_{M_{\mathcal{C}/T}^\mathcal{E}(\tilde{\mathbf{t}}, r, d)}$$

which is a splitting of the surjection $\Theta_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)} \rightarrow \pi^*(\Theta_T)$. By construction we can see that the pull-back of D to $M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)^{\sharp} \times \tilde{T}$ coincides with $D^{\sharp}$. $\square$

Remark 8.1. (1) We can see by construction that the image $D(\pi^*\Theta_T)$ is in fact contained in the relative tangent bundle $\Theta_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)/\Lambda_r^{(n)}(d)}$.

(2) As is explained in section 2, the subbundle $D(\pi^*\Theta_T) \subset \Theta_{M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)}$ satisfies an integrability condition and determines a foliation $\mathcal{F}_M$ on $M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)$. By the proof of Proposition 8.1, we can see that $\{\mathbf{RH}(\mathcal{L}) | \mathcal{L} \in \mathcal{F}_M\}$ coincides with $\mathcal{F}_R$.

Proof of Theorem 2.3. Take any path $\gamma : [0, 1] \rightarrow T$ and a point $x \in M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)$ such that $\pi(x) = \gamma(0)$. We take a lift $\tilde{x} \in M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T}$ of x . We can easily lift γ to a path $\tilde{\gamma} : [0, 1] \rightarrow (RP_r(\mathcal{C}, \tilde{\mathbf{t}}) \times_T \tilde{T}) \times_{\mathcal{A}_r^{(n)}} \Lambda_r^{(n)}(d)$ such that $\tilde{\gamma}[0, 1]$ is contained in a leaf of $\tilde{\mathcal{F}}_R \times_{\mathcal{A}_r^{(n)}} \Lambda_r^{(n)}(d)$ and $\mathbf{RH}(\tilde{x}) = \tilde{\gamma}(0)$. Since a leaf of $\tilde{\mathcal{F}}_M$ passing through $\tilde{x}$ is locally isomorphic to $\tilde{T}$, we can take a unique local lift $\delta_{\epsilon} : [0, \epsilon] \rightarrow M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T}$ of $\tilde{\gamma}$ for small $\epsilon > 0$ such that $\delta_{\epsilon}(0) = \tilde{x}$ and $\delta_{\epsilon}([0, \epsilon])$ is contained in a leaf of $\tilde{\mathcal{F}}_M$. Consider the set

$$A := \left\{ t \in (0, 1] \left| \begin{array}{l} \text{there exists a lift } \delta_t : [0, t] \rightarrow M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T} \\ \text{of } \tilde{\gamma}|_{[0, t)} \text{ such that } \delta(0) = \tilde{x} \text{ and } \delta([0, t)) \text{ is contained} \\ \text{in a leaf of } \tilde{\mathcal{F}}_M \end{array} \right. \right\}$$

and put $a := \sup A$. Assume that $a < 1$. Then there is a sequence $\{t_n\}_{n \geq 0}$ in A such that $\lim_{n \rightarrow \infty} t_n = a$. Since $\mathbf{RH} : M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T} \rightarrow (RP_r(\mathcal{C}, \tilde{\mathbf{t}}) \times_T \tilde{T}) \times_{\mathcal{A}_r^{(n)}} \Lambda_r^{(n)}(d)$ is a proper morphism, $\mathbf{RH}^{-1}(\tilde{\gamma}[0, a])$ is compact. Thus the sequence $\{\delta_{t_n}(t_n)\}$ in $\mathbf{RH}^{-1}(\tilde{\gamma}[0, a])$ has a subsequence $\{\delta_{t_{n_k}}(t_{n_k})\}$ which is convergent in $\mathbf{RH}^{-1}(\tilde{\gamma}[0, a])$. We put $\tilde{x}_a := \lim_{k \rightarrow \infty} \delta_{t_{n_k}}(t_{n_k}) \in \mathbf{RH}^{-1}(\tilde{\gamma}[0, a])$. Then $\mathbf{RH}(\tilde{x}_a) = \tilde{\gamma}(a)$ and we can take a local lift $\tilde{\delta}_a : (a - \epsilon, a + \epsilon) \rightarrow M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T}$ of $\tilde{\gamma}|_{(a - \epsilon, a + \epsilon)}$ for small $\epsilon > 0$ such that $\tilde{\delta}_a(a) = \tilde{x}_a$ and $\tilde{\delta}_a((a - \epsilon, a + \epsilon))$ is contained in a leaf of $\tilde{\mathcal{F}}_M$. Gluing $\tilde{\delta}_a$ and δ_{t_n} for some $t_n \in A$ with $t_n > a - \epsilon$, we can obtain a lift $\delta_a : [0, a + \epsilon) \rightarrow M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T}$ of $\tilde{\gamma}|_{[0, a + \epsilon)}$ whose image is contained in a leaf of $\tilde{\mathcal{F}}_M$. Thus $a + \epsilon \in A$ which contradicts the choice of a . So we have $a = 1$ and we can obtain a lift $\delta_1 : [0, 1] \rightarrow M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T}$ of $\tilde{\gamma}$ whose image is contained in a leaf of $\tilde{\mathcal{F}}_M$. Taking the image by the projection $M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T} \rightarrow M_{\mathcal{C}/T}^{\alpha}(\tilde{\mathbf{t}}, r, d)$, we obtain an $\mathcal{F}_M$ -horizontal lift δ of γ such that $\delta(0) = x$. $\square$

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References

- [1] D. Arinkin and S. Lysenko, Isomorphisms between moduli spaces of $\mathrm{SL}(2)$ -bundles with connections on $\mathbf{P}^1 \setminus \{x_1, \dots, x_4\}$, *Math. Res. Lett.* 4, 1997 no. 2-3, 181–190. MR1453052 (98h:14012)
- [2] D. Arinkin and S. Lysenko, On the moduli of $\mathrm{SL}(2)$ -bundles with connections on $\mathbf{P}^1 \setminus \{x_1, \dots, x_4\}$, *Internat. Math. Res. Notices* 1997, no. 19, 983–999. MR1488348 (99c:14013)
- [3] D. Arinkin, Orthogonality of natural sheaves on moduli stacks of $\mathrm{SL}(2)$ -bundles with connections on $\mathbb{P}^1$ minus 4 points, *Selecta Math.* 7, 2001, no. 2, 213–239. MR1860014 (2003a:14010)
- [4] P. Deligne, *Équations différentielles à points singuliers réguliers*, Springer-Verlag, Berlin, 1970, *Lecture Notes in Mathematics*, Vol. 163. MR0417174 (54:5232)
- [5] E. Formanek, The invariants of $n \times n$ matrices, *Invariant Theory* (S.S. Koh, eds.), *Lecture Notes in Math.* vol. 1278, Springer Verlag, 1987, pp. 18–43. MR924163 (88m:16018)
- [6] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, *Aspects of Mathematics*, E31. Friedr. Vieweg & Sohn, Braunschweig, 1997. MR1450870 (98g:14012)
- [7] M. Inaba, K. Iwasaki and M.-H. Saito, Moduli of Stable Parabolic Connections, Riemann-Hilbert correspondence and Geometry of Painlevé equation of type VI , Part I, *Publ. Res. Inst. Math. Sci.* 42 (2006), no. 4, 987–1089 (math.AG/0309342) MR2289083 (2008k:32040)
- [8] M. Inaba, K. Iwasaki and M.-H. Saito, Moduli of Stable Parabolic Connections, Riemann-Hilbert correspondence and Geometry of Painlevé equation of type VI , Part II, *Moduli spaces and arithmetic geometry*, 387–432, *Adv. Stud. Pure Math.*, 45, Math. Soc. Japan, Tokyo, 2006.
- [9] M. Inaba, K. Iwasaki and M.-H. Saito, Dynamics of the sixth Painlevé Equations, *Théories asymptotiques et équations de Painlevé*, 103–167, *Sémin. Congr.*, 14, Soc. Math. France, Paris, 2006. (math.AG/0501007)
- [10] K. Iwasaki, Moduli and deformation for Fuchsian projective connections on a Riemann surface, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **38** (1991), no. 3, 431–531. MR1146359 (92m:32039)
- [11] K. Iwasaki, Fuchsian moduli on a Riemann surface—its Poisson structure and Poincaré-Lefschetz duality, *Pacific J. Math.* **155** (1992), no. 2, 319–340. MR1178029 (93j:32028)

- [12] M. Jimbo, T. Miwa and K. Ueno, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients I. General theory and τ -function, *Physica 2D* (1981), 306–352. MR630674 (83k:34010a)
- [13] M. Jimbo and T. Miwa, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients II., *Physica 2D* (1981), 407–448. MR625446 (83k:34010b)
- [14] H. Nakajima, Hyper-Kähler structures on moduli spaces of parabolic Higgs bundles on Riemann surfaces, *Moduli of vector bundles* (Sanda, 1994; Kyoto, 1994), 199–208, *Lecture Notes in Pure and Appl. Math.*, 179, Dekker, New York, 1996. MR1397988 (97k:32030)
- [15] N. Nitsure, Moduli of semistable logarithmic connections, *J. Amer. Math. Soc.* 6 (1993), no. 3, 597–609. MR1182671 (93i:32025)
- [16] K. Okamoto, Sur les feuilletages associés aux équations du second ordre à points critiques fixes de P. Painlevé, *Japan. J. Math.* 5, 1979, no. 1, 1–79. MR614694 (83m:58005)
- [17] C. T. Simpson, Moduli of representations of the fundamental group of a smooth projective variety. I, *Inst. Hautes Études Sci. Publ. Math.* (1994), no. 79, 47–129. MR1307297 (96e:14012)
- [18] C. T. Simpson, Moduli of representations of the fundamental group of a smooth projective variety. II, *Inst. Hautes Études Sci. Publ. Math.* (1994), no. 80, 5–79 (1995). MR1320603 (96e:14013)

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