

On well-posedness of some one-dimensional dispersive equations with analytic nonlinearity

By

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Abstract

This note is a résumé of the result obtained in [22]. We consider the Cauchy problem for one-dimensional dispersive equations with a general nonlinearity. As a typical model, our equation includes the generalized Benjamin-Ono equation with the generalized dispersion: $\partial_t u - \partial_x D_x^\alpha u + \partial_x(u^{k+1}) = 0$ for $\alpha \in [1, 2]$ and $1 \leq k \in \mathbb{N}$. We obtain the unconditional local well-posedness of the Cauchy problem in $H^s(\mathbb{T})$ for $s \geq 1 - \frac{\alpha}{2(\alpha+1)}$ in the periodic case. We also discuss the non-periodic problem, which is omitted in [22].

§ 1. Introduction

This note is a résumé of the result obtained in [22]. In Section 1, we present the main results obtained in [22] and explain ideas to show the main theorem. In Section 2, we set some notation and collect useful estimates. In Section 3, improved Strichartz estimates are shown. In this note, we state such estimates in a slightly generalized form compared to Proposition 3.5 in [22]. In Section 4, we present the main idea for the proof of a priori estimate, by focusing on how to recover the derivative loss. We also present the estimate for the difference. In Section 5, we briefly describe how to treat the real line case, which was not contained in [22].

In [22], the authors studied the Cauchy problem associated with the following

Received January 1, 2022. Revised January 10, 2023.

2020 Mathematics Subject Classification(s): 35A01, 35A02, 35Q53

Key Words: Benjamin-Ono equation, dispersion generalized Benjamin-Ono equation, nonlinear dispersive equation, well-posedness, unconditional uniqueness, energy method

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equation:

$$(1.1) \quad \partial_t u + L_{\alpha+1} u + \partial_x(f(u)) = 0, \quad (t, x) \in \mathbb{R} \times \mathcal{M},$$

$$(1.2) \quad u(0, x) = u_0(x), \quad x \in \mathcal{M},$$

where $\mathcal{M} = \mathbb{T} (= \mathbb{R}/2\pi\mathbb{Z})$, under the following two hypotheses on $L_{\alpha+1}$ and f :

Hypothesis 1. $L_{\alpha+1}$ is the Fourier multiplier operator by $-ip_{\alpha+1}$ where $p_{\alpha+1} \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$ is a real-valued odd function satisfying, for some $\xi_0 > 0$, $p'_{\alpha+1}(\xi) \sim \xi^\alpha$ and $p''_{\alpha+1}(\xi) \sim \xi^{\alpha-1}$ for all $\xi \geq \xi_0$.

Hypothesis 2. $f : \mathbb{R} \rightarrow \mathbb{R}$ is an analytic function whose Taylor series around the origin has an infinite radius of convergence.

Famous equations in this class (1.1) are the generalized Korteweg-de Vries equation (gKdV)

$$\partial_t u + \partial_x^3 u + \partial_x(f(u)) = 0$$

that corresponds to $L_{\alpha+1} = \partial_x^3$ with $\alpha = 2$ as well as the generalized Benjamin-Ono equation (gBO)

$$\partial_t u - \mathcal{H}\partial_x^2 u + \partial_x(f(u)) = 0,$$

where $\mathcal{H}$ is the Hilbert transform (Fourier multiplier by $-i \operatorname{sgn}(\xi)$) which corresponds to $L_{\alpha+1} = -\mathcal{H}\partial_x^2$ with $\alpha = 1$.

The Cauchy problem associated with this type of a dispersive equation has been extensively studied by many authors, starting with the work of Kato [7] (see also [1]). See Section 1 in [22] for the development of the well-posedness theory of (1.1).

In what follows, we concentrate on the periodic setting $\mathcal{M} = \mathbb{T}$, except in Section 5. We also always assume that the initial data is real-valued, so we only consider real-valued solutions. Before stating our main result, let us recall our notion of solutions:

Definition 1.1. *Let $s > 1/2$. We will say that $u \in L^\infty(]0, T[; H^s(\mathbb{T}))$ is a solution to (1.1) associated with the initial datum $u_0 \in H^s(\mathbb{T})$ if u satisfies (1.1)-(1.2) in the distributional sense, i.e. for any test function $\phi \in C_c^\infty(]-T, T[\times \mathbb{T})$, it holds*

$$\int_0^\infty \int_{\mathbb{T}} [(\phi_t + L_{\alpha+1}\phi)u + \phi_x f(u)] dx dt + \int_{\mathbb{T}} \phi(0, \cdot) u_0 dx = 0.$$

Remark 1.1. We notice that u satisfies the Duhamel formula associated with (1.1) if u is a solution to (1.1) in the sense of Definition 1.1. See Remark 1.4 in [22] for details.

Let us recall the notion of unconditional local well-posedness that was introduced by Kato [8], which is, roughly speaking, the local well-posedness with uniqueness of solutions in $L^\infty([0, T[; H^s(\mathbb{T}))$.

Definition 1.2. *We will say that the Cauchy problem associated with (1.1) is unconditionally locally well-posed in $H^s(\mathbb{T})$ if for any initial data $u_0 \in H^s(\mathbb{T})$ there exists $T = T(\|u_0\|_{H^s}) > 0$ and a solution $u \in C([0, T]; H^s(\mathbb{T}))$ to (1.1) emanating from u_0 . Moreover, u is the unique solution to (1.1) associated with u_0 that belongs to $L^\infty([0, T[; H^s(\mathbb{T}))$. Finally, for any $R > 0$, the solution-map $u_0 \mapsto u$ is continuous from the ball of $H^s(\mathbb{T})$ with radius R centered at the origin into $C([0, T(R)]; H^s(\mathbb{T}))$.*

Our main result is the following (Theorem 1.1 in [22]):

Theorem 1.3 (Unconditional well-posedness). *Assume that Hypotheses 1-2 are satisfied with $1 \leq \alpha \leq 2$. Then for any $s \geq s(\alpha) = 1 - \frac{\alpha}{2(\alpha+1)}$ the Cauchy problem associated with (1.1) is unconditionally locally well-posed in $H^s(\mathbb{T})$ with a maximal time of existence $T \geq g(\|u_0\|_{H^{s(\alpha)}}) > 0$ where g is a smooth increasing function depending only on $L_{\alpha+1}$ and f .*

The work by Kim-Schippa [11] motivated us to study the Cauchy problem of (1.1). They showed the local well-posedness for the generalized Benjamin-Ono equation in $H^{3/4}(\mathbb{T})$. Our result can be seen as a generalization in terms of class of equations and an improvement in terms of unconditional uniqueness. Remark that they also deduced a priori estimate for the generalized Benjamin-Ono equation in $H^s(\mathbb{T})$ for $s > 1/2$. In this work, they used the short-time $X^{s,b}$ norm method developed by Ionescu-Kenig-Tataru [6], together with modified energy estimates.

Remark 1.2. One may wonder how $s \geq s(\alpha) = 1 - \frac{\alpha}{2(\alpha+1)}$ appears. This is due to the following nonlinear interactions: there are 3 input high frequencies of the same order as the output frequency whereas all other input frequencies are much lower. This is one of resonant interactions. See Remark 1.5 in [22] for details.

Remark 1.3. The proof of Theorem 1.3 also works on the real line $\mathcal{M} = \mathbb{R}$. Moreover, if we restrict ourselves to the purely dispersive case $L_{\alpha+1} = \partial_x D_x^\alpha$, we can exploit the classical Strichartz estimate (Lemma 5.1), instead of Lemma 3.3. Therefore, we can improve Theorem 1.3, i.e., we can show the unconditional local well-posedness for the Cauchy problem associated with (1.1) in $H^s(\mathbb{R})$ for $s \geq 1 - \frac{\alpha}{4}$ with $s > 1/2$. The restriction $s \geq 1 - \frac{\alpha}{4}$ comes from the same nonlinear interaction as discribed in Remark 1.2. In this case, we put two high frequencies in $L_t^\infty L_x^2$ and remaining two high frequencies in $L_t^2 L_x^\infty$ (whereas other ones are put in $L_{t,x}^\infty$). Roughly speaking, Proposition 5.2 allows us to conclude that $D_x^{s-\frac{1}{2}+\frac{\alpha}{4}} u \in L^2([0, T[; L_x^\infty(\mathbb{R}))$ if $u \in L^\infty([0, T[; H^s(\mathbb{R}))$.

Then, one derivative loss can be recovered if $2(s - \frac{1}{2} + \frac{\alpha}{4}) \geq 1 \Leftrightarrow s \geq 1 - \frac{\alpha}{4}$. See also Remark 5.1. However, we frequently use the Sobolev embedding $H^{1/2+}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, so we need to assume $s > 1/2$ at the same time.¹ Notice that, thanks to the Strichartz estimate (Lemma 5.1), the regularity threshold is improved to $s \geq 1 - \frac{\alpha}{4}$ with $s > 1/2$ from $s \geq 1 - \frac{\alpha}{2(\alpha+1)}$. See Section 5.

Our aim in this note is to explain ideas to show the unconditional local well-posedness for (1.1). The proof of Theorem 1.3 is based on the energy method. In the energy method, it is important to obtain a priori estimate (which corresponds to Proposition 4.7) and the difference estimate (which corresponds to Proposition 4.8) at the desired regularity. As studied in [1], the standard energy method allows us to get the unconditional local well-posedness in $H^s(\mathbb{T})$ for $s > 3/2$. By the standard energy method, we mean a series of techniques that consists of the parabolic regularization, the Kato-Ponce commutator estimate [9], and the Bona-Smith argument [4]. See Subsection 9.4 in [15] for instance. In this argument, however, the dispersive part $L_{\alpha+1}$ does not play any role. To go below $s \leq \frac{3}{2}$, we must utilize the dispersive nature of the equation. For that purpose, it is reasonable to decompose functions with respect to frequencies (such as the Littlewood-Paley decomposition), and then we look into nonlinear interactions closely. We also use the symmetrization argument, which roughly corresponds to the integration by parts (in space) in the standard energy method. In [22], the authors followed the argument developed by the first author and Vento [17] in which they utilized the dispersion for nonresonant nonlinear interactions. The key observation in this argument originates from Zhou [25]: it holds that $u \in X_T^{s-1,1}$ if $u \in L_T^\infty H_x^s$ is a solution.² See Lemma 4.6 and Case 2 in Subsection 4.3. This argument has the same underlying idea as the normal form reduction conducted by Babin-Ilyin-Titi [2]. They showed the unconditional uniqueness for the KdV equation in $L^2(\mathbb{T})$. The normal form reduction is based on the integration by parts in time and has been widely applied to dispersive equations. For the recent development of the normal form reduction, see [5, 12, 14] and references therein. This result [2] is recovered by the first author and Vento [17]. On the other hand, if we have resonant interactions, we need additional work. In [17], they studied (1.1) with $f(u) = u^2$, and showed the unconditional local well-posedness in $H^s(\mathcal{M})$ for $s \geq 1 - \frac{\alpha}{2}$ with $(s, \alpha) \neq (1/2, 1)$ and $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$. Note that when $f(u) = u^2$, the oscillation is always strong enough to close estimates, i.e., they did not face resonant interactions in [17]. However, in our case, (1.1) includes high degrees of polynomials, so we need to treat resonant interactions. For that purpose, we exploit the improved Strichartz estimate (Proposition 3.4) introduced by Koch-Tzvetkov [13]. This

¹Of course, even for nonresonant cases, our analysis depends on this embedding, so $s > 1/2$ is used in many steps (also in our definition of solution (Definition 1.1)).

²However, there is a subtle problem in applying this observation to our argument, i.e., the boundedness of the multiplication of $\mathbf{1}_{[0,T]}$ in $X^{s-1,1}$. See (4.1) and (4.3).

estimate is favorable if the nonlinearity has a derivative. In the proof of Proposition 4.7, we chop the time interval $[0, T]$ into small pieces, depending on the space frequency. This enables us to transfer a part of derivatives from the Duhamel part to the linear part, which results in giving us a better estimate than the Sobolev embedding. See Case 1 in Subsection 4.3. In this note, we state this estimate in a slightly more generalized manner than that in [22].

Finally, we also remark that in [22], a priori estimate (Proposition 4.8 in [22]) is stated with the frequency envelope. The frequency envelope is a weight in the frequency space which is increasing like $\log N$ when $|\xi| \sim N$, and the authors used this technique to show the continuous dependence. Depending on the sequence of initial data $\{u_{0,n}\}_{n \geq 1}$ converging to u_0 in $H^s(\mathbb{T})$ as $n \rightarrow \infty$, we can choose envelopes $\{\omega_N\}_{N \in 2^{\mathbb{N}} \cup \{0\}}$ of positive numbers such that

- $\omega_N \leq \omega_{2N} \leq \delta \omega_N$ for $1 < \delta \leq 2$ and $\omega_N \nearrow +\infty$,
- $\|u_0\|_{H_\omega^s} := \|\omega_N(1 \vee N)^s \|P_N u_0\|_{L_x^2} \| \ell_N^2 \| < \infty$,
- $\sup_{n \geq 1} \|u_{0,n}\|_{H_\omega^s} < \infty$.

See Lemma 4.1 in [13] for the construction of envelopes. The key observation is the following: we would like to show the convergence $u^{(n)} \rightarrow u$ in $L_T^\infty H_x^s$ whereas we have a priori bound for $u^{(n)}, u$ in $L_T^\infty H_\omega^s$, which is slightly stronger than $L_T^\infty H_x^s$. Here, $u^{(n)}$ (resp. u) is the solution to (1.1) emanating from $u_{0,n}$ (resp. u_0). This method enables us to bypass the Bona-Smith approximation argument. We believe that the frequency envelope makes the proof much simpler than the Bona-Smith argument. However, in this note, we do not treat this method for the sake of simplicity. Modification from a priori estimate (4.5) in the Sobolev setting to estimates with frequency envelopes ((4.8) in [22]) (and other related estimates such as (2.1)) is straightforward.

For the proof of Theorem 1.3, see Section 6 in [22]. We also point out that we can show that the solution constructed in Theorem 1.3 is global when we can control the potential energy $\int_{\mathbb{T}} F(u) dx$, where $F(x) = \int_0^x f(y) dy$. See Theorems 1.2 and 1.3 in [22].

§ 2. Notation, function spaces and basic estimates

§ 2.1. Notation

Throughout this note, $\mathbb{N}$ denotes the set of nonnegative integers. For any positive numbers a and b , we write $a \lesssim b$ when there exists a positive constant C such that $a \leq Cb$. We also write $a \sim b$ when $a \lesssim b$ and $b \lesssim a$ hold. Moreover, we denote $a \ll b$ if the estimate $b \lesssim a$ does not hold. For two nonnegative numbers a, b , we denote $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$. We also write $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$.

For $u = u(t, x)$, $\mathcal{F}u = \tilde{u}$ denotes its space-time Fourier transform, whereas $\mathcal{F}_x u = \hat{u}$ (resp. $\mathcal{F}_t u$) denotes its Fourier transform in space (resp. time). We define the Riesz potentials by $D_x^s g := \mathcal{F}_x^{-1}(|\xi|^s \mathcal{F}_x g)$. We also denote the unitary group associated to the linear part of (1.1) by $U_\alpha(t) = e^{-tL_{\alpha+1}}$, i.e.,

$$U_\alpha(t)u = \mathcal{F}_x^{-1}(e^{itp_{\alpha+1}(\xi)} \mathcal{F}_x u).$$

Let $\chi \in C_0^\infty(\mathbb{R})$ satisfy

$$0 \leq \chi \leq 1, \quad \chi|_{[-1,1]} = 1 \quad \text{and} \quad \text{supp } \chi \subset [-2, 2].$$

We set $\phi(\xi) := \chi(\xi) - \chi(2\xi)$. For any $l \in \mathbb{N}$, we define

$$\phi_{2^l}(\xi) := \phi(2^{-l}\xi), \quad \psi_{2^l}(\tau, \xi) := \phi_{2^l}(\tau - p_{\alpha+1}(\xi)),$$

where $ip_{\alpha+1}(\xi)$ is the Fourier symbol of $L_{\alpha+1}$. By convention, we also denote

$$\phi_0(\xi) = \chi(2\xi) \quad \text{and} \quad \psi_0(\tau, \xi) = \chi(2(\tau - p_{\alpha+1}(\xi))).$$

Any summations over capitalized variables such as K, L, M or N are presumed to be dyadic. We work with non-homogeneous dyadic decompositions, i.e., these variables ranges over numbers of the form $\{2^k; k \in \mathbb{N}\} \cup \{0\}$. We call those numbers *non-homogeneous dyadic numbers*. It is worth pointing out that $\sum_N \phi_N(\xi) = 1$ for any $\xi \in \mathbb{Z}$,

$$\text{supp}(\phi_N) \subset \{N/2 \leq |\xi| \leq 2N\}, \quad N \geq 1, \quad \text{and} \quad \text{supp}(\phi_0) \subset \{|\xi| \leq 1\}.$$

Finally, we define the Littlewood-Paley multipliers P_N and Q_L by

$$P_N u = \mathcal{F}_x^{-1}(\phi_N \mathcal{F}_x u) \quad \text{and} \quad Q_L u = \mathcal{F}^{-1}(\psi_L \mathcal{F} u).$$

We also set $Q_{\geq N} := \sum_{K \geq N} Q_K$ and $Q_{\leq N} := \sum_{K \leq N} Q_K$.

§ 2.2. Function spaces

For $1 \leq p \leq \infty$, $L^p(\mathbb{T})$ is the standard Lebesgue space with the norm $\|\cdot\|_{L^p}$. For $s \in \mathbb{R}$, $H^s(\mathbb{T})$ is the usual L^2 -based Sobolev space equipped with the norm

$$\|u\|_{H^s} := \left(\sum_N (1 \vee N)^{2s} \|P_N u\|_{L^2}^2 \right)^{1/2}.$$

If B_x is one of spaces defined above, for $1 \leq p \leq \infty$ and $T > 0$, we define the space-time spaces $L_t^p B_x := L^p(\mathbb{R}; B_x)$ and $L_T^p B_x := L^p([0, T]; B_x)$ equipped with the norms (with obvious modifications for $p = \infty$)

$$\|u\|_{L_t^p B_x} = \left(\int_{\mathbb{R}} \|u(t, \cdot)\|_{B_x}^p dt \right)^{1/p} \quad \text{and} \quad \|u\|_{L_T^p B_x} = \left(\int_0^T \|u(t, \cdot)\|_{B_x}^p dt \right)^{1/p},$$

respectively. For $s, b \in \mathbb{R}$, we introduce the Bourgain spaces $X^{s,b}$ associated to the operator $L_{\alpha+1}$ endowed with the norm

$$\|u\|_{X^{s,b}} = \left(\sum_{\xi=-\infty}^{\infty} \int_{-\infty}^{\infty} \langle \xi \rangle^{2s} \langle \tau - p_{\alpha+1}(\xi) \rangle^{2b} |\tilde{u}(\tau, \xi)|^2 d\tau \right)^{1/2}.$$

We define the function spaces Z^s with $s \in \mathbb{R}$, as $Z^s := L_t^\infty H^s \cap X^{s-1,1}$, endowed with the natural norm

$$\|u\|_{Z^s} = \|u\|_{L_t^\infty H^s} + \|u\|_{X^{s-1,1}}.$$

We also use the restriction in time versions of these spaces. Let $T > 0$ be a positive time and B be a normed space of space-time functions. The restriction space B_T will be the space of functions $u :]0, T[\times \mathbb{T} \rightarrow \mathbb{R}$ or $\mathbb{C}$ satisfying

$$\|u\|_{B_T} := \inf \{ \|v\|_B \mid v : \mathbb{R} \times \mathbb{T} \rightarrow \mathbb{R} \text{ or } \mathbb{C}, v = u \text{ on }]0, T[\times \mathbb{T} \} < \infty.$$

§ 2.3. Basic estimates

In this subsection, we collect some fundamental estimates. Well-known estimates are adapted for our setting $f(u)$.

Lemma 2.1 (Lemma 2.2 in [22]). *Let $s \geq 0$. Then we have the estimate*

$$(2.1) \quad \|uv\|_{H^s} \lesssim \|u\|_{H^s} \|v\|_{L^\infty} + \|u\|_{L^\infty} \|v\|_{H^s}.$$

In particular for any fixed analytic function f with $f(0) = 0$, there exists an analytic function $G = G[f]$ that is increasing nonnegative on $\mathbb{R}_+$ such that

$$(2.2) \quad \|f(u)\|_{H^s} \lesssim G(\|u\|_{L^\infty}) \|u\|_{H^s}.$$

We will frequently use the following lemma, which can be seen as a variant of the integration by parts.

Lemma 2.2 (Lemma 2.4 in [22]). *Let $N \in 2^{\mathbb{N}} \cup \{0\}$. Then,*

$$\left| \int_{\mathbb{T}} \Pi(u, v) w dx \right| \lesssim \|u\|_{L_x^2} \|v\|_{L_x^2} \|\partial_x w\|_{L_x^\infty},$$

where

$$(2.3) \quad \Pi(u, v) := v \partial_x P_N^2 u + u \partial_x P_N^2 v.$$

§ 3. Improved Strichartz estimates

In this section, we establish improved Strichartz estimates which play an important role in our estimates. Since we treat the general operator $L_{\alpha+1}$, we first have to restate the standard Strichartz estimate.

Definition 3.1. For $\alpha \in [1, 2]$, we define $\beta(\alpha) := 1/4(\alpha + 1)$.

We make use of the Strichartz estimate in the Bourgain space, which is originally obtained in [3].

Lemma 3.2. There exists $C > 0$ such that for any $v \in X^{0, b(\alpha)}$,

$$\|v\|_{L_{t,x}^4} \leq C \|v\|_{X^{0, \beta(\alpha)+1/4}}.$$

Lemma 3.2 enables to establish the following Strichartz estimate on a fixed time interval:

Lemma 3.3. Let $0 < T < 1$. Then there exists $C > 0$ such that for any $u \in L^2(\mathbb{T})$,

$$\|U_\alpha(t)u\|_{L_{T,x}^4} \leq CT^{1/4-\beta(\alpha)} \|u\|_{L_x^2}.$$

Proof. See Lemma 2.1 in [16]. □

Now we prove the improved Strichartz estimates with using Lemma 3.3. In this note, we claim them in a slightly generalized form, compared with Proposition 3.5 in [22]. In the proof of Proposition 4.7, we use (3.1) with $(p, q) = (4, 4), (3, \infty)$. We point out that it is crucial to keep the time integrability p in the right hand side of (3.1) so as to reach $s = s(\alpha)$ in Theorem 1.3. For that purpose, how to choose $c_{j,N}$ plays an important role in the proof below. This type of argument can be found in Lemma 2.4 in [23] for instance.

Proposition 3.4. Let $2 \leq p \leq 4 \leq q \leq \infty$, $\alpha \in [1, 2]$, $N \geq 1$, $0 < T < 1$ and $f \in L^\infty([0, T]; L^2(\mathbb{T}))$. Suppose that $u \in C([0, T]; L^2(\mathbb{T}))$ satisfies

$$\partial_t u + L_{\alpha+1} u + \partial_x f = 0$$

on $]0, T[\times \mathbb{T}$ with $L_{\alpha+1}$ satisfying Hypothesis 1. Then,

$$(3.1) \quad \|P_N u\|_{L_T^p L_x^q} \lesssim T^{-\beta(\alpha)} N^{\beta(\alpha) + \frac{1}{4} - \frac{1}{q}} (\|P_N u\|_{L_T^p L_x^2} + \|P_N f\|_{L_T^p L_x^2}).$$

Proof. We divide the interval in small intervals of length $\sim TN^{-1}$. In other words, we define $\{I_{j,N}\}_{j \in J_N}$ so that $\bigcup_{j \in J_N} I_{j,N} = [0, T]$, $|I_{j,N}| \sim TN^{-1}$ and $\#J_N \sim N$. By the hypothesis, we see that $\|P_N u(t)\|_{L_x^q}^q \in C([0, T])$. For $j \in J_N$, we choose $c_{j,N} \in I_{j,N}$ at which $\|P_N u(t)\|_{L_x^q}^q$ attains its minimum on $I_{j,N}$. Observe that

$$P_N u(t) = e^{-(t-c_{j,N})L_{\alpha+1}} P_N u(c_{j,N}) + \int_{c_{j,N}}^t e^{-(t-t')L_{\alpha+1}} P_N \partial_x f(t') dt'$$

for $t \in I_{j,N}$. We see from the hypothesis $2 \leq p \leq 4 \leq q \leq \infty$, the Hölder inequality (in time) and the Bernstein inequality (in space) that

$$\left(\sum_{j \in J_N} \|P_N u\|_{L^p(I_{j,N}; L_x^q)}^p \right)^{1/p} \lesssim N^{\frac{1}{4} - \frac{1}{q}} \left(\sum_{j \in J_N} |I_{j,N}|^{1 - \frac{p}{4}} \|P_N u\|_{L^4(I_{j,N}; L_x^4)}^p \right)^{1/p}.$$

Lemma 3.3 and the definition of $c_{j,N}$ show that

$$\begin{aligned} & \left(\sum_{j \in J_N} |I_{j,N}|^{1 - \frac{p}{4}} \|e^{-(t-c_{j,N})L_{\alpha+1}} P_N u(c_{j,N})\|_{L^4(I_{j,N}; L_x^4)}^p \right)^{1/p} \\ & \lesssim \left(\sum_{j \in J_N} |I_{j,N}|^{1 - p\beta(\alpha)} \|P_N u(c_{j,N})\|_{L_x^2}^p \right)^{1/p} \\ & \lesssim T^{-\beta(\alpha)} N^{\beta(\alpha)} \left(\sum_{j \in J_N} \int_{I_{j,N}} \|P_N u(t)\|_{L_x^2}^p dt \right)^{1/p} = T^{-\beta(\alpha)} N^{\beta(\alpha)} \|P_N u\|_{L_T^p L_x^2}. \end{aligned}$$

Next, we estimate the contribution of the Duhamel term. Lemma 3.3 and the Hölder inequality in time imply that

$$\begin{aligned} & \left(\sum_{j \in J_N} |I_{j,N}|^{1 - \frac{p}{4}} \left\| \int_{c_{j,N}}^t e^{-(t-t')L_{\alpha+1}} P_N \partial_x f(t') dt' \right\|_{L^4(I_{j,N}; L_x^4)}^p \right)^{1/p} \\ & \lesssim \left(\sum_{j \in J_N} |I_{j,N}|^{1 - \frac{p}{4}} \left(N |I_{j,N}|^{\frac{1}{4} - \beta(\alpha)} \int_{I_{j,N}} \|P_N f(t')\|_{L_x^2}^p dt' \right)^p \right)^{1/p} \\ & \lesssim T^{1 - \beta(\alpha)} N^{\beta(\alpha)} \left(\sum_{j \in J_N} \int_{I_{j,N}} \|P_N f(t')\|_{L_x^2}^p dt' \right)^{1/p} = T^{1 - \beta(\alpha)} N^{\beta(\alpha)} \|P_N f\|_{L_T^p L_x^2}. \end{aligned}$$

Then, the Minkowski inequality shows (3.1), which completes the proof. $\square$

Remark 3.1. In the right hand side of (3.1), there is a negative power of T . This is cancelled out by a trivial inequality $\|u\|_{L_T^p H_x^s} \lesssim T^{1/p} \|u\|_{L_T^\infty H_x^s}$ after summing over N . The reason why we keep the time integrability at this point is that we would like to change the summation order of ℓ_N^p and L_T^p without a loss. Since $p \leq 4$, we note that $-\beta(\alpha) + 1/p \geq 1/8 > 0$. In the proof of well-posedness, we take T sufficiently small, so it is important to have a positive power of T in the last step.

§ 4. A priori estimate

In this section, we discuss how to build the proof of a priori estimate (Proposition 4.7) for the solution of (1.1). As explained in Section 1, we have to recover the loss of one derivative coming from the nonlinear term $\partial_x(f(u))$. Because of the periodic setting, there is no linear smoothing effect. Since we do not make use of a nonlinear transformation, we have either to exploit some nonlinear smoothing effect or to share this derivative on different high frequency functions in the case there are many of them. For that purpose, it is helpful for us to look into the resonant function (Ω_{k+1} defined as below). If the resonant function is small, this corresponds to the (nearly) resonance whereas if the resonant function is large, this corresponds to the nonresonance. In the nonresonant case, we exploit Bourgain type estimates to recover the lost derivative while in the resonant case we share the derivative on different high frequency functions by exploiting improved Strichartz estimate (3.1). At this stage, one of the fundamental well-known observations is that for dispersive operator satisfying Hypothesis 1, nonlinear terms of the form $u_1 u_2 \dots u_k$, $k \geq 2$, are nonresonant in the case of a single high frequency function together with much lower frequency functions. By Bourgain type estimates, we mean $\|Q_{\geq L} u\|_{L^2_{t,x}} \lesssim L^{-1} \|u\|_{X^{0,1}}$ for $L \geq 1$. Recall that Q_L is a projection operator restricting $|\tau - p_\alpha(\xi)| \sim L$. If L is sufficiently large, we can recover the derivative loss (see Case 2 in Subsection 4.3). Such estimates are available when the nonlinear interaction is nonresonant. However, there is a delicate problem in using these estimates. In order to use Bourgain type estimates, we first multiply $u \in C([0, T]; H^s(\mathbb{T}))$ by $\mathbf{1}_{[0, T]}$ to expand the integral interval from $[0, T]$ to $\mathbb{R}$. Since the multiplication is not bounded in $X^{s-1,1}$, following [17], we decompose $\mathbf{1}_{[0, T]}$ into two functions: a high (temporal) frequency part that is small enough in L^1 to recover the derivative loss (see (4.2)) and a low (temporal) frequency part which commutes to Q_L in $L^2_{t,x}$ (see (4.3)).

§ 4.1. Preliminary technical estimates

Let us denote by $\mathbf{1}_T$ the characteristic function of the interval $]0, T[$. As discussed above, $\mathbf{1}_T$ does not commute with Q_L . To avoid this difficulty, following [17], we further decompose $\mathbf{1}_T$ as

$$(4.1) \quad \mathbf{1}_T = \mathbf{1}_{T,R}^{\text{low}} + \mathbf{1}_{T,R}^{\text{high}}, \quad \text{with} \quad \mathcal{F}_t(\mathbf{1}_{T,R}^{\text{low}})(\tau) = \chi(\tau/R) \mathcal{F}_t(\mathbf{1}_T)(\tau),$$

for some $R > 0$ to be fixed later.

Lemma 4.1 (Lemma 3.6 in [20]). *For any $R > 0$ and $T > 0$, it holds*

$$(4.2) \quad \|\mathbf{1}_{T,R}^{\text{high}}\|_{L^1} \lesssim T \wedge R^{-1},$$

and

$$\|\mathbf{1}_{T,R}^{\text{high}}\|_{L^\infty} + \|\mathbf{1}_{T,R}^{\text{low}}\|_{L^\infty} \lesssim 1.$$

Lemma 4.2 (Lemma 3.7 in [20]). *Assume that $T > 0$, $R > 0$, and $L \gg R$. Then, it holds*

$$(4.3) \quad \|Q_L(\mathbf{1}_{T,R}^{\text{low}}u)\|_{L_{t,x}^2} \lesssim \|Q_{\sim L}u\|_{L_{t,x}^2},$$

for all $u \in L^2(\mathbb{R}_t \times \mathbb{T}_x)$.

Now, we define the resonant function Ω_j .

Definition 4.3. *Let $j \in \mathbb{N}$. We define $\Omega_j(\xi_1, \dots, \xi_{j+1}) : \mathbb{Z}^{j+1} \rightarrow \mathbb{R}$ as*

$$\Omega_j(\xi_1, \dots, \xi_{j+1}) := \sum_{n=1}^{j+1} p_{\alpha+1}(\xi_n)$$

for $(\xi_1, \dots, \xi_{j+1}) \in \mathbb{Z}^{j+1}$, where $p_{\alpha+1}$ satisfies Hypothesis 1.

In the proof of Proposition 4.7, we make use of the following two types of nonresonant interactions. Note however that on the real line, if we restrict ourselves to the purely dispersive operator $L_{\alpha+1} = \partial_x D_x^\alpha$, Lemma 4.4 is enough to close the proof since we can use a stronger estimate (Proposition 5.2) than Proposition 3.4 for the resonant case. See Section 5 for details.

Lemma 4.4 (Lemma 4.4 in [22]). *Let $k \geq 1$ and $(\xi_1, \dots, \xi_{k+2}) \in \mathbb{Z}^{k+2}$ satisfy $\sum_{j=1}^{k+2} \xi_j = 0$. Assume that $|\xi_1| \sim |\xi_2| \gtrsim |\xi_3|$ if $k = 1$ or $|\xi_1| \sim |\xi_2| \gtrsim |\xi_3| \gg k \max_{j \geq 4} |\xi_j|$ if $k \geq 2$. Then,*

$$|\Omega_{k+1}(\xi_1, \dots, \xi_{k+2})| \gtrsim |\xi_3| |\xi_1|^\alpha$$

for $|\xi_1| \gg (\max_{\xi \in [0, \xi_0]} |p'_{\alpha+1}(\xi)|)^{1/\alpha}$.

Lemma 4.5 (Lemma 4.5 in [22]). *Let $k \geq 2$ and $(\xi_1, \dots, \xi_{k+2}) \in \mathbb{Z}^{k+2}$ satisfy $\sum_{j=1}^{k+2} \xi_j = 0$. Assume that $|\xi_1| \sim |\xi_2| \gg |\xi_3| \gtrsim |\xi_4|$ if $k = 2$ or $|\xi_1| \sim |\xi_2| \gg |\xi_3| \gtrsim |\xi_4|$ with $|\xi_3 + \xi_4| \gg k \max_{j \geq 5} |\xi_j|$ if $k \geq 3$. Then,*

$$|\Omega_{k+1}(\xi_1, \dots, \xi_{k+2})| \gtrsim |\xi_3 + \xi_4| |\xi_1|^\alpha$$

for $|\xi_1| \gg (\max_{\xi \in [0, \xi_0]} |p'_{\alpha+1}(\xi)|)^{1/\alpha}$.

Let $(\xi_1, \dots, \xi_{k+2}) \in \mathbb{Z}^{k+2}$ and $(\tau_1, \dots, \tau_{k+2}) \in \mathbb{R}^{k+2}$ satisfy $\sum_{j=1}^{k+2} \xi_j = \sum_{j=1}^{k+2} \tau_j = 0$. Set $\sigma(\tau, \xi) := \tau - p_{\alpha+1}(\xi)$. Assume that $|\Omega_{k+1}(\xi_1, \dots, \xi_{k+2})| \gtrsim L$ for some $L \geq 1$.

It follows from the definition that $\Omega_{k+1}(\xi_1, \dots, \xi_{k+2}) = \sum_{j=1}^{k+2} \sigma(\tau_j, \xi_j)$. Then, by the triangle inequality we have

$$(4.4) \quad \max\{|\sigma(\tau_j, \xi_j)| : j = 1, \dots, k+2\} \geq \frac{1}{k+2} |\Omega_{k+1}(\xi_1, \dots, \xi_{k+2})| \gtrsim \frac{L}{k}.$$

Recall that Q_L is a projector on $|\tau - p_{\alpha+1}(\xi)| \sim L$. Therefore, when $(\xi_1, \dots, \xi_{k+2}) \in \mathbb{Z}^{k+2}$ satisfies the assumption of either Lemma 4.4 or Lemma 4.5, at least one of $\{\sigma(\tau_j, \xi_j)\}_j$ has a high modulation, i.e., $\gtrsim L/k$. This property is used to recover the derivative loss. See Case 2 in Proposition 4.7 for example. See also Lemma 2.2 in [17] for details.

Remark 4.1. In the proof of Proposition 4.8 in [22], we used the following estimate:

$$\begin{aligned} J_{\infty,3}^{(2)} &\leq J_{\infty}^{(2)}(P_{N_1} Q_{\gtrsim L}(\mathbf{1}_{t,R}^{\text{low}} \check{u}), P_{N_2} \mathbf{1}_{t,R}^{\text{low}} \check{u}, P_{N_3} \check{u}, \dots, P_{N_{k+2}} \check{u}) \\ &\quad + J_{\infty}^{(2)}(P_{N_1} Q_{\ll L}(\mathbf{1}_{t,R}^{\text{low}} \check{u}), P_{N_2} Q_{\gtrsim L}(\mathbf{1}_{t,R}^{\text{low}} \check{u}), P_{N_3} \check{u}, \dots, P_{N_{k+2}} \check{u}) \\ &\quad + J_{\infty}^{(2)}(P_{N_1} Q_{\ll L}(\mathbf{1}_{t,R}^{\text{low}} \check{u}), P_{N_2} Q_{\ll L}(\mathbf{1}_{t,R}^{\text{low}} \check{u}), P_{N_3} Q_{\gtrsim L} \check{u}, \dots, P_{N_{k+2}} \check{u}) \\ &\quad + \dots + J_{\infty}^{(2)}(P_{N_1} Q_{\ll L}(\mathbf{1}_{t,R}^{\text{low}} \check{u}), P_{N_2} Q_{\ll L}(\mathbf{1}_{t,R}^{\text{low}} \check{u}), P_{N_3} Q_{\ll L} \check{u}, \dots, P_{N_{k+2}} Q_{\gtrsim L} \check{u}), \end{aligned}$$

where $L = N_3 N_1^{\alpha}$ and $R = N_1^{1/3} N_3^{4/3}$. See Case 2 in the proof of Proposition 4.8 in [22] for details. However, there are errors in this estimate and the subsequent argument. As observed in (4.4), L (resp. R) should be defined as $L := N_3 N_1^{\alpha}/k$ (resp. $R := N_1^{1/3} N_3^{4/3}/k$), and then we obtain $\|P_{N_1} Q_{\gtrsim L}(\mathbf{1}_{t,R}^{\text{low}} \check{u})\|_{L_{t,x}^2} \lesssim \|P_{N_1} Q_{\gtrsim L} \check{u}\|_{L_{t,x}^2} \lesssim k N_3^{-1} N_1^{-\alpha} \|P_{N_1} \check{u}\|_{X^{0,1}}$ for instance. A factor of k also appears in the estimate of $\mathbf{1}_{t,R}^{\text{high}}$. Thanks to Hypothesis 1, such an additional factor k does not cause any problems. A similar comment applies to Case 3 in the proof of Proposition 4.8 and Subcase 3.1 and Subcase 3.2 in the proof of Proposition 5.1 in [22].

§ 4.2. Estimates on solutions to (1.1)

For nonresonant interactions, we use either Lemma 4.4 or Lemma 4.5 to recover the derivative loss. For that purpose, we put the solution u in $X^{s-1,1}$. Indeed, if $u \in L_T^{\infty} H_x^s$ satisfies (1.1), it easily follows that $u \in X_T^{s-1,1}$ as soon as $s > 1/2$ (see Lemma 4.6). Here, the restriction $s > 1/2$ comes from the Sobolev embedding $H^{1/2+}(\mathbb{T}) \hookrightarrow L^{\infty}(\mathbb{T})$. In [17], the first author and Vento put u in the space $Y^s := X^{s-\frac{3}{2},1} + X^{s-\frac{3}{8},\frac{5}{8}}$ for the KdV equation so as to show the unconditional local well-posedness in $H^s(\mathcal{M})$ for $s \geq 0$ and $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$. They used the nonresonant structure of the nonlinearity twice to put u in Y^s -norm in a priori estimate and to control Y^s -norm. This is reminiscent of the second normal form reduction conducted in [2]. See also [18, 19, 20] for well-posedness results below $s \leq 1/2$ with using this argument.

Lemma 4.6 (Lemma 4.7 in [22]). *Let $0 < T < 1$, $s > 1/2$ and $u \in L_T^\infty H_x^s$ be a solution to (1.1) associated with an initial datum $u_0 \in H^s(\mathbb{T})$. Then $u \in Z_T^s$ and it holds*

$$\|u\|_{Z_T^s} \lesssim \|u\|_{L_T^\infty H_x^s} + G(\|u\|_{L_{T,x}^\infty}) \|u\|_{L_T^\infty H_x^s}.$$

Moreover, for any couple $(u, v) \in (L_T^\infty H_x^s)^2$ of solutions to (1.1) associated with a couple of initial data $(u_0, v_0) \in (H^s(\mathbb{T}))^2$ it holds

$$\|u - v\|_{Z_T^{s-1}} \lesssim \|u - v\|_{L_T^\infty H_x^{s-1}} + G(\|u\|_{L_T^\infty H_x^s} + \|v\|_{L_T^\infty H_x^s}) \|u - v\|_{L_T^\infty H_x^{s-1}}.$$

The following proposition is one of main estimates in [22].

Proposition 4.7 (A priori estimate). *Let $0 < T < 1$, $\alpha \in [1, 2]$ and $2 \geq s \geq s(\alpha) := 1/2 + 2\beta(\alpha)$. Let $u_0 \in H^s(\mathbb{T})$ and let $u \in L^\infty(0, T; H^s(\mathbb{T}))$ be a solution to (1.1)–(1.2) on $[0, T]$. Then there exists an analytic function $G = G[f]$ that is increasing and nonnegative on $\mathbb{R}_+$ such that*

$$(4.5) \quad \|u\|_{L_T^\infty H_x^s}^2 \leq \|u_0\|_{H^s}^2 + T^{1/4} G(\|u\|_{Z_T^{s(\alpha)}}) \|u\|_{Z_T^s} \|u\|_{L_T^\infty H_x^s}.$$

Remark 4.2. There are typos in the statement of Proposition 4.8 in [22]. If we state the estimate with the frequency envelope, we need to assume ‘ $u_0 \in H_\omega^s(\mathbb{T})$ and $u \in L^\infty(0, T; H_\omega^s(\mathbb{T}))$ ’ not ‘ $u_0 \in H^s(\mathbb{T})$ and $u \in L^\infty(0, T; H^s(\mathbb{T}))$ ’.

§ 4.3. Strategy for the proof of Proposition 4.7

In this subsection, we discuss how to recover the derivative loss coming from the nonlinearity $\partial_x(f(u))$. For the complete proof, see Proposition 4.8 in [22]. The main goal is to show that for any $k \geq 1$ it holds

$$(4.6) \quad I_{k+1}^t \leq C^k T^{1/4} G(C_0) C_0^k (\|u\|_{X_T^{s-1,1}} + \|u\|_{L_T^\infty H_x^s}) \|u\|_{L_T^\infty H_x^s}.$$

where $C_0 := \|u\|_{Z_T^{s(\alpha)}}$ and

$$I_k^t := \sum_{N \geq 1} N^{2s} \left| \int_0^t \int_{\mathbb{T}} u^k P_N^2 \partial_x u \, dx dt' \right|.$$

Once we establish (4.6), we obtain (4.5) by the Fubini-Lebesgue theorem, Hypothesis 2 and the standard argument of the energy method.

In the sequel we fix $k \geq 1$. For simplicity, for any positive numbers a and b , the notation $a \lesssim_k b$ means there exists a positive constant $C > 0$ independent of k such that $a \leq C^k b$. Remark that $a \leq k^m b$ for $m \in \mathbb{N}$ can be expressed by $a \lesssim_k b$ too since an elementary calculation shows $k^m \leq m! e^k$ for $m \in \mathbb{N}$. Here, e is Napier’s constant.

We also focus on the sum over $N \gg 1$ since the sum over $N \lesssim 1$ can be treated easily. By the symmetrization argument, I_k^t is estimated by

(4.7)

$$\begin{aligned}
I_{k+1}^t &\lesssim_k \sum_{N \gg 1} \sum_{\substack{N_1, \dots, N_{k+2}, \\ N_1 \geq N_2 \geq N_3 \geq N_4 \geq N_5 = \max_{j \geq 5} N_j}} N^{2s} \left| \int_0^t \int_{\mathbb{T}} \Pi(P_{N_1} u, P_{N_2} u) \prod_{j=3}^{k+2} P_{N_j} u dx dt' \right| \\
&\quad + \sum_{l=3}^{k+2} \sum_{N \gg 1} \sum_{\substack{N_1, \dots, N_{k+2}, \\ N_1 \geq N_2 \geq N_3 \geq N_4 \geq N_5 = \max_{j \geq 5} N_j}} N^{2s} \left| \int_0^t \int_{\mathbb{T}} \partial_x P_N^2 P_{N_l} u \prod_{j=1, j \neq l}^{k+2} P_{N_j} u dx dt' \right| \\
&=: J_t + \tilde{J}_t,
\end{aligned}$$

where $\Pi(\cdot, \cdot)$ is defined in (2.3). The reason why we sum over $N_1 \geq N_2 \geq N_3 \geq N_4 \geq N_5 = \max_{j \geq 5} N_j$ not simply $N_1 \geq \dots \geq N_{k+2}$ is that we would like to make the cost of choice less than $(k+2)^4$ to be able to sum over k . Note that Hypothesis 1 ensures that the Taylor expansion of f at the origin has infinite radius of convergence. See Remark 1.3 in [22]. The most difficult term in (4.7) is the first term since the derivative lands on either N_1 or N_2 , and we only treat J_t . In this case, we use improved Strichartz estimates (3.1) or Bourgain type estimates (Lemmas 4.4 and 4.5) depending on whether the nonlinear interaction is resonant or nonresonant. We divide the contribution J_t in (4.7) into three cases: 1. $N_2 \lesssim N_3 \lesssim kN_4$, 2. $N_3 \gg kN_4$ or $k = 1$, 3. $N_2 \gg N_3$.

It is worth noticing that the frequency projection operator P_N ensures that the contribution of $N_1 \leq N/4$ does vanish. We thus can assume that $N_1 \geq N/2$ and $N_2 \gtrsim N_1/k$ with $N_2 \geq 1$.

Case 1: $N_2 \lesssim N_3 \lesssim kN_4$. It holds that $N_1 \lesssim kN_2 \lesssim k^2N_4$. So, in this configuration, we can distribute derivatives to four functions, i.e., we obtain $N_1^{2s+1} \lesssim k^{2(2s+1)} N_4^{2s+1} \leq k^{10} N_4^{2s+1}$ since $s \leq 2$. When $N_4 \lesssim 1$, we can easily close the estimate. Thus, we may assume that $N_4 \gg 1$. We notice that $N_2, N_3 \geq 1$. By the Hölder inequality, we put $P_{N_j} u$ in $L_{T,x}^4$ norm for $j = 1, 2, 3, 4$, and put other terms $P_{N_j} u$ in $L_{T,x}^\infty$ norm for $j \geq 5$. Then, we use (3.1), the Young inequality and (2.2):

$$\begin{aligned}
&\sum_{N_1 \geq N_2 \geq N_3 \geq N_4 \gg 1} N_4^{2s+1} \prod_{j=1}^4 \|P_{N_j} u\|_{L_{T,x}^4} \\
&\lesssim T^{-4\beta(\alpha)} \sum_{N_1 \geq N_2 \geq N_3 \geq N_4 \gg 1} N_4^{2s+1} \prod_{j=1}^4 N_j^{\beta(\alpha)} U_{N_j}^{0,4} \\
&\lesssim T^{-4\beta(\alpha)} \sum_{N_1, N_2, N_3 \geq N_4 \geq 1} \left(\frac{N_4}{N_1}\right)^{s-\beta(\alpha)} \left(\frac{N_4}{N_2}\right)^{s-\beta(\alpha)} \left(\frac{N_4}{N_3}\right)^{\beta(\alpha)+1/2} U_{N_1}^{s,4} U_{N_2}^{s,4} U_{N_3}^{s(\alpha),4} U_{N_4}^{s(\alpha),4} \\
&\lesssim T^{1-4\beta(\alpha)} G(C_0) \|u\|_{L_T^\infty H_x^s}^2 \|u\|_{L_T^\infty H_x^{s(\alpha)}}^2,
\end{aligned}$$

where $U_K^{s,p} := \|P_K u\|_{L_T^p H_x^s} + \|P_K f(u)\|_{L_T^p H_x^s}$ for $s \in \mathbb{R}$, $1 \leq p \leq \infty$ and $K \geq 1$. Remark that if we merely use the Sobolev embedding $H^{1/4}(\mathbb{T}) \hookrightarrow L^4(\mathbb{T})$, we only get $2s + 1 \leq 4(s - \frac{1}{4}) \Leftrightarrow s \geq 1$.

Case 2: $N_3 \gg kN_4$ or $k = 1$. The case $k = 1$ is treated in [17]. So we focus on the case $N_3 \gg kN_4$. Notice that $N_1 \lesssim N_2 + N_3 + (k-1)N_4 \lesssim N_2$, so we have $N_1 \sim N_2$. Since $N_1 \gg N$ is not possible, we have $N \sim N_1 \sim N_2$. In this configuration, we use Lemma 4.4. For simplicity, with a slight abuse of notation, we define the following functional:

$$J_\infty^{(2)}(u_1, \dots, u_{k+2}) := \sum_{N \gg 1} \sum_{\substack{N_1, \dots, N_{k+2}, \\ N_1 \sim N_2 \sim N, N_3 \gg kN_4, \\ N_4 \geq N_5 = \max_{j \geq 5} N_j}} N^{2s} \left| \int_{\mathbb{R}} \int_{\mathbb{T}} \Pi(u_1, u_2) \prod_{j=3}^{k+2} u_j dx dt' \right|.$$

Setting $R = N_1^{1/3} N_3^{4/3} / k$, we split J_t as

$$\begin{aligned} J_t &\leq J_\infty^{(2)}(P_{N_1} \mathbf{1}_{t,R}^{\text{high}} \tilde{u}, P_{N_2} \mathbf{1}_t \tilde{u}, P_{N_3} \tilde{u}, \dots, P_{N_{k+2}} \tilde{u}) \\ &\quad + J_\infty^{(2)}(P_{N_1} \mathbf{1}_{t,R}^{\text{low}} \tilde{u}, P_{N_2} \mathbf{1}_{t,R}^{\text{high}} \tilde{u}, P_{N_3} \tilde{u}, \dots, P_{N_{k+2}} \tilde{u}) \\ &\quad + J_\infty^{(2)}(P_{N_1} \mathbf{1}_{t,R}^{\text{low}} \tilde{u}, P_{N_2} \mathbf{1}_{t,R}^{\text{low}} \tilde{u}, P_{N_3} \tilde{u}, \dots, P_{N_{k+2}} \tilde{u}) =: J_{\infty,1}^{(2)} + J_{\infty,2}^{(2)} + J_{\infty,3}^{(2)}, \end{aligned}$$

where $\tilde{u}$ is a suitable extension of u (see (2.1) and (4.1) in [22] for precise definitions). Since $J_{\infty,1}^{(2)}$ and $J_{\infty,2}^{(2)}$ have $\mathbf{1}_{t,R}^{\text{high}}$, we can use (4.2): $\|\mathbf{1}_{t,R}^{\text{high}}\|_{L^1} \lesssim T^{1/4} R^{-3/4} = k^{3/4} T^{1/4} N_1^{-1/4} N_3^{-1}$, which is enough to close. On the other hand, thanks to Lemma 4.4, we further decompose $J_{\infty,3}^{(2)}$. Putting $L := N_3 N_1^\alpha / k$, we obtain

$$\begin{aligned} J_\infty &\lesssim J_\infty^{(2)}(P_{N_1} Q_{\gtrsim L}(\mathbf{1}_{t,R}^{\text{low}} \tilde{u}), P_{N_2}(\mathbf{1}_{t,R}^{\text{low}} \tilde{u}), P_{N_3} \tilde{u}, \dots, P_{N_{k+2}} \tilde{u}) \\ &\quad + J_\infty^{(2)}(P_{N_1} Q_{\ll L}(\mathbf{1}_{t,R}^{\text{low}} \tilde{u}), P_{N_2} Q_{\gtrsim L}(\mathbf{1}_{t,R}^{\text{low}} \tilde{u}), P_{N_3} \tilde{u}, \dots, P_{N_{k+2}} \tilde{u}) \\ &\quad + J_\infty^{(2)}(P_{N_1} Q_{\ll L}(\mathbf{1}_{t,R}^{\text{low}} \tilde{u}), P_{N_2} Q_{\ll L}(\mathbf{1}_{t,R}^{\text{low}} \tilde{u}), P_{N_3} Q_{\gtrsim L} \tilde{u}, \dots, P_{N_{k+2}} \tilde{u}) \\ &\quad + \dots + J_\infty^{(2)}(P_{N_1} Q_{\ll L}(\mathbf{1}_{t,R}^{\text{low}} \tilde{u}), P_{N_2} Q_{\ll L}(\mathbf{1}_{t,R}^{\text{low}} \tilde{u}), P_{N_3} Q_{\ll L} \tilde{u}, \dots, P_{N_{k+2}} Q_{\gtrsim L} \tilde{u}) \\ &=: J_{\infty,1}^{(2)} + \dots + J_{\infty,k+2}^{(2)}. \end{aligned}$$

Note that each $J_{\infty,l}^{(2)}$ for $l = 1, \dots, k+2$ has a high modulation term $\gtrsim L$, and we can get $\|Q_{\gtrsim L} \tilde{u}\|_{L_{t,x}^2} \lesssim L^{-1} \|\tilde{u}\|_{X^{0,1}} = k N_3^{-1} N_1^{-\alpha} \|\tilde{u}\|_{X^{0,1}}$, which allows us to recover the derivative loss N_3 with using Lemma 2.2. The reason why we put $\mathbf{1}_{[0,t]}$ on two functions not one function is that we would like to get the positive power of T (i.e. $T^{1/4}$ in this case). See Remark 4.1 and (4.18) in [22].

Case 3: $N_2 \gg N_3$. In this case, roughly speaking, we compare $|\xi_3 + \xi_4|$ and $k|\xi_5|$ where ξ_i is the i -th largest frequency. If $|\xi_3 + \xi_4| \gg k|\xi_5|$, we have a suitable nonresonance relation (Lemma 4.5) whereas otherwise we can share the lost derivative between three

functions. We split J_t as

$$J_t \leq \sum N^{2s} \left| \int_0^t \int_{\mathbb{T}} \Pi(P_{N_1} u, P_{N_2} u) P_{\lesssim k N_5} \left(\prod_{j=3}^{k+2} P_{N_j} u \right) dx dt' \right| \\ + \sum_{k N_5 \ll M \lesssim N_3} \sum N^{2s} \left| \int_0^t \int_{\mathbb{T}} \Pi(P_{N_1} u, P_{N_2} u) P_M \left(\prod_{j=3}^{k+2} P_{N_j} u \right) dx dt' \right| =: I_t^{(3)} + J_t^{(3)},$$

where $\sum$ denotes $\sum_{N \gg 1} \sum_{\substack{N_1, \dots, N_{k+2}, \\ N_1 \geq N_2 \geq N_3 \geq N_4 \geq N_5 = \max_{j \geq 5} N_j}}$ for the sake of simplicity. Remark that if $k = 2$, then the term $I_t^{(3)}$ does not appear, and it holds that $J_t \leq J_t^{(3)}$ with the summation $\sum_{M \lesssim N_3}$ instead of $\sum_{k N_5 \ll M \lesssim N_3}$. Note also that the contribution in respectively $I_t^{(3)}$ and $J_t^{(3)}$ of respectively $N_5 = 0$ and $N_5 = M = 0$ does vanish by integration by parts. Therefore we can always assume that $N_5 \geq 1$ in $I_t^{(3)}$ and that $M \geq 1$ in $J_t^{(3)}$. For $I_t^{(3)}$, we put $L_T^\infty L_x^2$ (resp. $L_T^3 L_x^\infty$) norm on $P_{N_1} u$ and $P_{N_2} u$ (resp. $P_{N_j} u$ for $j = 3, 4, 5$). We put $L_{T,x}^\infty$ norm on other terms $P_{N_j} u$ for $j = 6, \dots, k+2$. Then, we use (3.1) (and the Young inequality and (2.2)) to $\|P_{N_j} u\|_{L_T^3 L_x^\infty}$ for $j = 3, 4, 5$:

$$\sum_{N_3 \geq N_4 \geq N_5 \geq 1} N_5 \prod_{j=3}^5 \|P_{N_j} u\|_{L_T^3 L_x^\infty} \\ \lesssim T^{-3\beta(\alpha)} \sum_{N_3 \geq N_4 \geq N_5 \geq 1} N_5 \prod_{j=3}^5 N_j^{\beta(\alpha)+1/4} U_{N_j}^{0,3} \\ \lesssim T^{-3\beta(\alpha)} \sum_{N_3 \geq N_4 \geq N_5 \geq 1} N_5^{-3\beta(\alpha)+1/4} \left(\frac{N_5}{N_3} \right)^{\beta(\alpha)+1/4} \left(\frac{N_5}{N_4} \right)^{\beta(\alpha)+1/4} U_{N_3}^{s(\alpha),3} U_{N_4}^{s(\alpha),3} U_{N_5}^{s(\alpha),3} \\ \lesssim T^{-3\beta(\alpha)} \sum_{K \geq 1} (U_{N_3}^{s(\alpha),3})^3 \lesssim_k T^{1-3\beta(\alpha)} G(C_0) \|u\|_{L_T^\infty H_x^{s(\alpha)}}^3$$

since $-3\beta(\alpha) + 1/4 \leq 0$ for $\alpha \in [1, 2]$.

For $J_t^{(3)}$, we can argue similarly to Case 2 with Lemma 4.5 instead of Lemma 4.4. In this case, we decompose u into $\tilde{u} = Q_{\gtrsim M N_1^\alpha/k} \tilde{u} + Q_{\ll M N_1^\alpha/k} \tilde{u}$ after taking a suitable extension $\tilde{u}$ of u .

§ 4.4. Estimate for the difference

We provide the estimate (at the regularity $s-1$) for the difference w of two solutions u, v of (1.1).

Proposition 4.8 (Proposition 5.1 in [22]). *Let $0 < T < 1$, $\alpha \in [1, 2]$ and $2 \geq s \geq s(\alpha) := 1/2 + 2\beta(\alpha)$. Let u and v be two solutions of (1.1) belonging to Z_T^s and associated with (1.2) and $v_0 \in H^s(\mathbb{T})$, respectively. Then there exists an analytic*

function $G = G[f]$ that is increasing and nonnegative on $\mathbb{R}_+$ such that

$$\|w\|_{L_T^\infty H_x^{s-1}}^2 \leq \|u_0 - v_0\|_{H_x^{s-1}}^2 + T^{1/4} G(\|u\|_{Z_T^s} + \|v\|_{Z_T^s}) \|w\|_{Z_T^{s-1}} \|w\|_{L_T^\infty H_x^{s-1}},$$

where we set $w = u - v$.

The strategy for the proof of Proposition 4.8 is essentially the same as that of Proposition 4.7. Remark that the equation for $w = u - v$ has fewer symmetries than the original equation, so the case-by-case analysis becomes slightly complicated.

§ 5. On the real line case

In this section, we briefly describe how we can obtain a priori estimate on $\mathcal{M} = \mathbb{R}$ corresponding to Proposition 4.7. Once we establish a priori estimate (and the difference estimate corresponding to Proposition 4.8), the unconditional local well-posedness follows from a standard argument (see Section 6 in [22]). We remark that estimates in Subsections 2.3 and 4.1 and Lemma 4.6 still hold on the real line.

As in Proposition 4.7, we have to recover the derivative loss. On the real line, we can utilize the Strichartz estimate proved in [10] instead of Lemma 3.3 if we take $L_{\alpha+1}$ as a purely dispersive operator: $L_{\alpha+1} = \partial_x D_x^\alpha$ for $\alpha \in [1, 2]$. Actually the proof of a priori estimate in this case becomes simpler than that of Proposition 4.7 and mainly follows the arguments used in [24]. Recall that the Bourgain type estimates allow us to close nonresonant interactions for $s > 1/2$. So, we estimate the resonant interactions with some s lower than $s(\alpha)$, which turns out to be $s \geq 1 - \frac{\alpha}{4}$ with $s > 1/2$. See Remark 1.3.

Lemma 5.1 (Theorem 2.1 in [10]). *Let $u \in L^2(\mathbb{R})$ and $\alpha \in [1, 2]$. Then,*

$$\|D_x^{(\alpha-1)/4} U_\alpha(t) u\|_{L_t^4 L_x^\infty} \lesssim \|u\|_{L^2}.$$

By using Lemma 5.1, we can obtain the real line counterpart of Proposition 3.4.

Proposition 5.2. *Let $\alpha \in [1, 2]$, $N \geq 1$, $0 < T < 1$ and $f \in L^\infty([0, T]; L^2(\mathbb{R}))$. Suppose that $u \in C([0, T]; L^2(\mathbb{R}))$ satisfies*

$$\partial_t u + \partial_x D_x^\alpha u + \partial_x f = 0$$

on $]0, T[\times \mathbb{R}$. Then,

$$\|P_N u\|_{L_T^2 L_x^\infty} \lesssim T^{-\frac{1}{4}} N^{\frac{1}{2} - \frac{\alpha}{4}} (\|P_N u\|_{L_{T,x}^2} + \|P_N f\|_{L_{T,x}^2}).$$

Proof. The proof is the same as that of Proposition 3.4 with Lemma 5.1 instead of Lemma 3.3. $\square$

Recall that for nonresonant interactions, we can close the estimate for $s > 1/2$. With using Proposition 5.2 for resonant interactions, we can obtain the following.

Proposition 5.3 (A priori estimate). *Let $0 < T < 1$, $\alpha \in [1, 2]$ and $2 \geq s \geq s_{\text{line}}(\alpha) := 1 - \frac{\alpha}{4}$ with $s > 1/2$. Let $u_0 \in H^s(\mathbb{R})$ and let $u \in L^\infty([0, T]; H^s(\mathbb{R}))$ be a solution to (1.1)–(1.2) on $[0, T]$. Then there exists an analytic function $G = G[f]$ that is increasing and nonnegative on $\mathbb{R}_+$ such that*

$$(5.1) \quad \|u\|_{L_T^\infty H_x^s}^2 \leq \|u_0\|_{H_x^s}^2 + T^{1/4} G(\|u\|_{Z_T^{s_{\text{line}}(\alpha)}} + \|u\|_{Z_T^{\frac{1}{2}+}}) \|u\|_{Z_T^s} \|u\|_{L_T^\infty H_x^s}.$$

The proof is essentially the same as that of Proposition 4.7. However, thanks to Proposition 5.2, we do not have to compare $|\xi_3 + \xi_4|$ and $k|\xi_5|$ as in Case 3 in the proof of Proposition 4.7. We use Lemma 4.4 while Lemma 4.5 is not necessary. The case-by-case analysis becomes simpler than that of Proposition 4.7: 1. $N_3 \gg (kN_4 \vee 1)$ or $k = 1$ and 2. $N_3 \lesssim (kN_4 \vee 1)$, provided $N_1 \geq N_2 \geq N_3 \geq N_4 = \max_{j \geq 4} N_j$ for $k \geq 2$. The case $N_3 \gg (kN_4 \vee 1)$ is exactly the same as Case 2 in Proposition 4.7. In the case $N_3 \lesssim (kN_4 \vee 1)$, it is enough to distribute one derivative coming from the integration by parts (Lemma 2.2) to two functions $P_{N_3}u$ and $P_{N_4}u$. More precisely, when $N_1 \sim N_2 \geq N_3 \geq N_4$ and $(kN_4 \vee 1) \gtrsim N_3$, we see from Lemma 2.2, the Hölder inequality, Proposition 5.2 and (2.2) that

$$\begin{aligned} & \sum_{N \gg 1} \sum_{\substack{N_1, \dots, N_{k+2}, \\ N_1 \geq N_2 \geq N_3 \geq N_4 = \max_{j \geq 4} N_j, \\ N_1 \sim N_2, N_3 \lesssim (kN_4 \vee 1)}} N^{2s} \left| \int_0^t \int_{\mathbb{R}} \Pi(P_{N_1}u, P_{N_2}u) \prod_{j=3}^{k+2} P_{N_j}u dx dt' \right| \\ & \lesssim_k \|u\|_{L_T^\infty H_x^{s_0}}^{k-2} \sum_{\substack{N_1 \sim N_2 \geq N_3 \geq N_4, \\ N_3 \lesssim (kN_4 \vee 1)}} N_1^{2s} \langle N_3 \rangle \int_0^t \|P_{N_1}u\|_{L_x^2} \|P_{N_2}u\|_{L_x^2} \|P_{N_3}u\|_{L_x^\infty} \|P_{N_4}u\|_{L_x^\infty} dt' \\ & \lesssim_k \sqrt{k} \|u\|_{L_T^\infty H_x^s}^2 \|u\|_{L_T^\infty H_x^{s_0}}^{k-2} \sum_{N_4 \leq N_3 \lesssim (kN_4 \vee 1)} \langle N_3 \rangle^{1/2} \langle N_4 \rangle^{1/2} \|P_{N_3}u\|_{L_T^2 L_x^\infty} \|P_{N_4}u\|_{L_T^2 L_x^\infty} \\ & \lesssim_k T^{-1/2} \|u\|_{L_T^\infty H_x^s}^2 \|u\|_{L_T^\infty H_x^{s_0}}^{k-2} \sum_{N_3} (U_{0, \sim N_3}^{s_{\text{line}}(\alpha), 2})^2 \lesssim_k T^{1/2} C(\|u\|_{L_T^\infty H_x^{s_0}}) \|u\|_{L_T^\infty H_x^s}^2, \end{aligned}$$

where $s_{\text{line}}(\alpha) = 1 - \frac{\alpha}{4}$ and $s_0 = \max(s_{\text{line}}(\alpha), 1/2+)$. We can also prove the difference estimate corresponding to Proposition 4.8 on the real line with the assumption $s \geq s_{\text{line}}(\alpha) = 1 - \frac{\alpha}{4}$ with $s > 1/2$. See also Proposition 3.2 in [24]. Therefore, the unconditional local well-posedness for (1.1)–(1.2) with $L_{\alpha+1} = \partial_x D_x^\alpha$ is obtained in $H^s(\mathbb{R})$ for $s \geq s_{\text{line}}(\alpha)$ with $s > 1/2$.

Remark 5.1. One may try to put four functions in $L_{T,x}^4$ as in Case 1 in Proposition 4.7, but one obtains the same restriction on s as described above. To see this, we use the Strichartz estimate with different indices, i.e., (see Theorem 2.1 in [10])

$$\|D_x^{(\alpha-1)/8} U_\alpha(t) u_0\|_{L_t^s L_x^4} \lesssim \|u_0\|_{L_x^2}.$$

Following the proof of Proposition 3.4, we obtain

$$\|P_N u\|_{L^4_{T,x}} \lesssim T^{-\frac{1}{8}} N^{\frac{2-\alpha}{8}} (\|P_N u\|_{L^4_T L^2_x} + \|P_N f\|_{L^4_T L^2_x}).$$

So, we can recover the derivative loss if $2s + 1 \leq 4(s - \frac{2-\alpha}{8}) \Leftrightarrow s \geq 1 - \frac{\alpha}{4} = s_{\text{line}}(\alpha)$.

Remark 5.2. Recently, in [21], the authors improved Theorem 1.3 in terms of regularity. They proved the unconditional well-posedness for (1.1) under Hypotheses 1–2 in $H^s(\mathbb{T})$ for $s \geq 1 - \frac{\alpha}{4}$ with $s > 1/2$. This result proved in [21] also holds on the real line with exactly the same proof. See Remark 1.2 in [21]. In other words, they treated more general operators than $L_{\alpha+1} = \partial_x D_x^\alpha$ on $\mathbb{R}$. For this, they introduced the bilinear version of the linear improved Strichartz estimates (See Proposition 3.1 in [21]).

Acknowledgements

The authors express deep gratitude to the anonymous referee for his/her valuable remarks and suggestions which improved this manuscript. The second author was supported by JSPS KAKENHI Grant Number JP20J12750, JSPS Overseas Challenge Program for Young Researchers and Iizuka Takeshi Scholarship Foundation, and partially supported by the EPSRC New Investigator Award (grant no. EP/V003178/1). The result obtained in [22] was conducted during a visit of the second author at Institut Denis Poisson (IDP) of Université de Tours in France. The second author is deeply grateful to IDP for its kind hospitality.

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