

ADIABATIC LIMITS OF η -INVARIANTS AND THE MEYER FUNCTION FOR SMOOTH THETA DIVISORS

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1. Introduction

Let Σ_g be a closed oriented surface of genus g and let $\mathcal{M}_g$ be the mapping class group of genus g , namely the group of all isotopy classes of orientation-preserving diffeomorphisms of Σ_g . Meyer introduced a cocycle $\tau_g : \mathcal{M}_g \times \mathcal{M}_g \rightarrow \mathbb{Z}$, called the signature cocycle or the Meyer cocycle, and he gave a signature formula for the signature of surface bundles over surfaces ([21]). Let $[\tau_g] \in H^2(\mathcal{M}_g, \mathbb{Z})$ denotes the cohomology class of τ_g . When $g = 1$, since $\mathcal{M}_1 = SL_2(\mathbb{Z})$, $H^1(SL_2(\mathbb{Z}), \mathbb{Z}) = 0$ and $3[\tau_1] = 0$, there exists a unique 1-cocycle $\phi_1 : SL_2\mathbb{Z} \rightarrow \frac{1}{3}\mathbb{Z}$ such that cobounds τ_1 . The function ϕ_1 is called the Meyer function of genus one, which has the following property: Let $\pi : Z \rightarrow X$ be a Σ_1 -bundle over a compact oriented surface with boundary $\partial Z = c_1 \amalg \cdots \amalg c_k$. Let $A_1, \dots, A_k$ be the monodromies around each component of the boundary. Since the Picard-Lefschetz transformation along c_i is an automorphism of $H^1(\Sigma_1, \mathbb{Z})$ preserving the intersection form, one has $A_i \in SL_2(\mathbb{Z})$ by fixing a symplectic basis of $H^1(\Sigma_1, \mathbb{Z})$. Then the signature of Z , which is defined as the signature of the cup-product pairing on $H^2(Z, \partial Z, \mathbb{R})$, satisfies

$$(1) \quad \text{Sign}(Z) = \sum_{i=1}^k \phi_1(A_i).$$

The explicit formula of ϕ_1 was obtained by Meyer ([21]).

When $g = 2$, since $5[\tau_2] = 0 \in H^2(\mathcal{M}_2, \mathbb{Z}) \cong \mathbb{Z}/10\mathbb{Z}$ and $H^1(\mathcal{M}_2, \mathbb{Z}) = 0$, there exists a unique 1-cocycle $\phi_2 : \mathcal{M}_2 \rightarrow \frac{1}{5}\mathbb{Z}$ satisfying (1), for every Σ_2 -bundles over compact oriented surfaces. The function ϕ_2 is called the Meyer function of genus two.

In [1], Atiyah investigated the Meyer function ϕ_1 from the several view points. For an odd dimensional closed oriented Riemannian manifold M , let $\eta(M)$ be the η -invariant of M with respect to the signature operator of M [2]. For $\sigma \in SL_2\mathbb{Z}$, let $\pi : M_\sigma \rightarrow S^1$ be the mapping

torus associated with σ , i.e., Σ_1 -bundle over S^1 with monodromy σ . Then Atiyah showed the following identity, when M_σ is equipped with a certain metric:

$$\phi_1(\sigma) = \eta(M_\sigma)$$

Moreover, he gave several interpretation of ϕ_1 in terms of the following quantities: (1) Hirzebruch's signature defect; (2) the transformation laws of the logarithm of the Dedekind η -function; (3) the logarithm of the monodromy of Quillen's line bundle; (4) the special value of the Shimizu L -function at the origin.

In this note, we study an extension of the result of Atiyah to the case $g = 2$ and higher dimensional manifold. We shall construct a higher dimensional analogue of the Meyer function for smooth theta divisors of odd dimension.

Notation : For a complex manifold M , $T^{1,0}M$ (resp. $T^{0,1}M$) denotes the holomorphic (resp. anti-holomorphic) tangent bundle and TM denotes the real tangent bundle. We set $d^c := \frac{1}{4\pi\sqrt{-1}}(\partial - \bar{\partial})$. Hence $dd^c = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}$.

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2. Preliminaries from Riemannian geometry

In this section, we recall some results of Riemannian geometry which will be used in the proof of the main theorem. Following [10], we define connections of fiber bundles and the connection of relative tangent bundles. Let M be a manifold and let $\pi : Z \rightarrow B$ be a fiber bundle with typical fiber M .

The *relative tangent bundle* $T(Z/B)$ is the subbundle of TZ defined by

$$T(Z/B) := \text{Ker}\{\pi_* : TZ \rightarrow \pi^*TB\}.$$

A vector of $T(Z/B)$ is said to be *vertical*.

Definition 2.1. A subbundle $T_H Z \subset TZ$ with $TZ = T(Z/B) \oplus T_H Z$ is called a *connection* of the fiber bundle $\pi : Z \rightarrow B$.

For a connection, one has $T_H Z \cong \pi^*TB$ via the projection $\pi_* : TZ \rightarrow \pi^*TB$. A vector of $T_H Z$ is said to be *horizontal*.

When Z is trivial, i.e., $Z = M \times B$, TZ is naturally isomorphic to the direct sum $(\text{pr}_1)^*TM \oplus (\text{pr}_2)^*TB$. This connection is called the *trivial connection* of the trivial fiber bundle.

Given a connection, one can define the projection $P_Z : TZ \rightarrow T(Z/B)$ with kernel $T_H Z$. We often identify P_Z with the corresponding connection $T_H Z := \text{Ker}(P_Z)$. In the rest of Section 2, we fix a connection $T_H Z$, or equivalently P_Z . One can define the pull-back of a connection, as follows: Let B' be a manifold and let $h : B' \rightarrow B$ be a C^∞ map. The fiber product $Z' := Z \times_B B' = \{(x, b) \in Z \times B' \mid \pi(x) = h(b)\}$ satisfies the following commutative diagram:

$$\begin{array}{ccc} Z' & \xrightarrow{\tilde{h}} & Z \\ \pi' \downarrow & & \downarrow \pi \\ B' & \xrightarrow{h} & B \end{array} \quad \tilde{h} = \text{pr}_1, \pi' = \text{pr}_2.$$

Since the map $P_Z \circ \tilde{h}_* : TZ' \rightarrow h^*T(Z/B)$ is surjective, $\text{Ker}(P_Z \circ \tilde{h}_*)$ is a subbundle of TZ' . Since $T(Z'/B')$ is canonically isomorphic to $h^*T(Z/B)$, the map $P_Z \circ \tilde{h}_*$ is identified with a projection from TZ' to $T(Z'/B')$.

Definition 2.2. The connection of $\pi' : Z' \rightarrow B'$ induced from $T_H Z$ by h is defined by

$$T_H Z' := \text{Ker}(P_Z \circ \tilde{h}_* : T Z' \rightarrow T(Z/B)),$$

under the identification between $T(Z'/B')$ and $h^* T(Z/B)$. The projection corresponding to $T_H Z'$ is denoted by $h^* P_Z$.

We fix a metric $g^{Z/B}$ on the relative tangent bundle, a Riemannian metric g^B on B , and the connection $T_H Z$ and the corresponding projection P_Z . We define the Riemannian metric g^Z on the total space Z by

$$g^Z := g^{Z/B} \oplus \pi^* g^B,$$

under the isomorphism $TZ \cong T(Z/B) \oplus T_H Z \cong T(Z/B) \oplus \pi^* TB$. Let ∇^Z be the Levi-Civita connection of (Z, g^Z) . We define the connection $\nabla^{Z/B}$ on $T(Z/B)$ by

$$\nabla^{Z/B} := P_Z \circ \nabla^Z.$$

Then $\nabla^{Z/B}$ preserves the metric $g^{Z/B}$.

Lemma 2.3. The connection $\nabla^{Z/B}$ is independent of a choice of g^B

Proof. See [10, Proposition 10.2] □

Lemma 2.4. Let B' be a manifold and let $h : B' \rightarrow B$ be a C^∞ -map, and set $Z' := Z \times_B B'$. Let $g^{Z'/B'} = h^* g^{Z/B}$ be the metric on $T(Z'/B')$ induced from $g^{Z/B}$, and let $P_{Z'} = h^* P_Z$ be the connection of Z' induced from P_Z . Then $\nabla^{Z'/B'} = h^* \nabla^{Z/B}$.

Proof. See [15] □

With respect to the decomposition $TZ = T(Z/B) \oplus T_H Z$, We put for $\varepsilon \in \mathbb{R}^+$

$$g^{Z,\varepsilon} := g^{Z/B} \oplus \varepsilon^{-1} \pi^* g^B.$$

The Levi-Civita connections of $(Z, g^{Z,\varepsilon})$ and (B, g^B) are denoted by $\nabla^{Z,\varepsilon}$ and ∇^B , respectively. Let $R^{Z,\varepsilon}$ and R^B be the curvature of $\nabla^{Z,\varepsilon}$ and ∇^B , respectively. Then $g^Z := g^{Z,1}$ and $\nabla^Z := \nabla^{Z,1}$. We define another connection ∇ on Z by

$$\nabla := \nabla^{Z/B} \oplus \pi^* \nabla^B,$$

and we put

$$S^{(\varepsilon)} := \nabla^{Z,\varepsilon} - \nabla \in \mathcal{A}^1(\text{End}(TZ)), \quad S := S^{(1)}.$$

Then ∇ preserves the Riemannian metric $g^{Z,\varepsilon}$, and P_Z is parallel with respect to ∇ , i.e. $\nabla \circ P_Z - P_Z \circ \nabla = 0$.

Let $\{e_1, \dots, e_k\}$ be a local orthogonal framing for $(T(Z/B), g^{Z/B})$, and let $\{f_1, \dots, f_l\}$ be a local orthogonal framing for $(T_H Z, \pi^* g^B)$.

Proposition 2.5. With respect to the splitting $TZ = T(Z/B) \oplus T_H B$, the following identity holds:

$$\lim_{\varepsilon \rightarrow 0} R^{Z,\varepsilon} = \begin{pmatrix} R^{Z/B} & P_Z(\nabla S) \\ 0 & \pi^* R^B \end{pmatrix}.$$

Proof. See [7] (3.195). □

3. η -invariants

In this section, we recall the definition and some properties of η -invariants. Let (M, g^M) be a coled oriented Riemannian manifold of dimension $(2l - 1)$. Denote the space of C^∞ k -forms on M by $\mathcal{A}^k(M)$. Let $*$: $\mathcal{A}^k(M) \rightarrow \mathcal{A}^{2l-k-1}(M)$ be the Hodge star operation with respect to g^M . The signature operator $D : \oplus_{p \geq 0} \mathcal{A}^{2p}(M) \rightarrow \oplus_{p \geq 0} \mathcal{A}^{2p}(M)$ of M is defined by

$$D : \omega \mapsto (\sqrt{-1})^l (-1)^{p+1} (*d - d*)\omega, \quad \omega \in \mathcal{A}^{2p}(M).$$

Then D is an elliptic self-adjoint differential operator of first order acting on $\oplus_{p \geq 0} \mathcal{A}^{2p}(M)$. Let $\sigma(D)$ be the spectrum of D . The η -function of M is defined by

$$\eta(s) := \sum_{\lambda \in \sigma(D) \setminus \{0\}} \frac{\text{sign} \lambda}{\lambda^s},$$

for $s \in \mathbb{C}$ with $\text{Re}(s) \gg 0$. Then $\eta(s)$ extends meromorphically to $\mathbb{C}$ and is holomorphic at $s = 0$ by [2], [7].

Definition 3.1. The real number $\eta(0)$ is called the η -invariant of (M, g^M) and is denoted by $\eta(M, g^M)$.

Let (X, g^X) be a $4k$ -dimensional, oriented, compact, Riemannian manifold with boundary Y . Put $g^Y := g^X|_Y$ and fix a color neighborhood $U \supset Y$ such that $U \cong Y \times [0, 1)$. Assume that $g^X|_U = g^Y \oplus dt^2$ under the above isomorphism. Let ∇^L be the Levi-Civita connection of (X, g^X) .

Theorem 3.2 (Atiyah-Patodi-Singer [2]). *The following equation holds:*

$$\text{Sign}(X) = \int_X L(TX, \nabla^L) - \eta(Y, g^Y)$$

Here L denotes the Hirzebruch L -polynomial, which is a multiplicative genus associated with the power series: $L(x) := x / \tanh(x)$.

Let X, B and M be closed oriented manifolds. Let $\pi : X \rightarrow B$ be a C^∞ -submersion, whose fibers are isomorphic to M . Assume that $\dim X = 4k$. Let $g^{X/B}$ be a metric on $T(X/B)$ and let g^B be a metric on TB . Let $T_H X \subset TX$ be a connection. We identify $T_H X$ with $\pi^* TB$ via π . With respect to the decomposition $TX = T(X/B) \oplus \pi^* TB$, we define the metric on X by $g^X := g^{X/B} \oplus \pi^* g^B$ and we consider the one parameter family of metrics on X defined by

$$g^{X, \varepsilon} := g^{X/B} \oplus \varepsilon^{-1} \pi^* g^B, \quad \varepsilon \in \mathbb{R}^+.$$

Theorem 3.3 (Bismut-Cheeger, [6]). *The limit $\lim_{\varepsilon \rightarrow 0} \eta(X, g^{X, \varepsilon})$ exists.*

The limit $\lim_{\varepsilon \rightarrow 0} \eta(X, g^{X, \varepsilon})$ is called the *adiabatic limit of the η -invariants* and is denoted by $\eta^0(X)$. By definition, $\eta^0(X, g^X)$ depends on the three data: $g^{X/B}$, g^B and $T_H X$.

4. Family of smooth theta divisors

We fix the following notation. Let $\mathfrak{S}_g$ be the Siegel upper-half space of degree g and let Γ_g be the integral symplectic group, i.e.,

$$\begin{aligned} \mathfrak{S}_g &:= \{\tau \in M(g, \mathbb{C}) \mid {}^t \tau = \tau, \text{Im} \tau > 0\} \\ \Gamma_g &:= \{\gamma \in GL(2g, \mathbb{Z}) \mid \gamma J_g {}^t \gamma = J_g\}, \end{aligned}$$

where $J_g = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}$ and 1_g denotes the $g \times g$ identity matrix. Γ_g acts on $\mathfrak{S}_g$ by

$$\gamma \cdot \tau := (A\tau + B)(C\tau + D)^{-1}, \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g, \quad \tau \in \mathfrak{S}_g.$$

For $\tau \in \mathfrak{S}_g$, write $\tau = {}^t(\tau_1, \dots, \tau_g)$ and set

$$\Lambda_\tau := \mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_g \oplus \mathbb{Z}\tau_1 \oplus \dots \oplus \mathbb{Z}\tau_g \subset \mathbb{C}^g$$

where $1_g = {}^t(e_1, \dots, e_g)$ and $\tau = {}^t(\tau_1, \dots, \tau_g) \in \mathfrak{S}_g$. Define the $\mathbb{Z}^{2g}$ -action on $\mathbb{C}^g \times \mathfrak{S}_g$ by

$$(m, n) \cdot (z, \tau) := (z + m\tau + n, \tau), \quad (z, \tau) \in \mathbb{C}^g \times \mathfrak{S}_g, \quad m, n \in \mathbb{Z}^{2g}.$$

Then

$$f : \mathbb{A}_g := (\mathbb{C}^g \times \mathfrak{S}_g) / \mathbb{Z}^{2g} \rightarrow \mathfrak{S}_g$$

is the universal family of principally polarized Abelian varieties over $\mathfrak{S}_g$, whose fiber at τ is $A_\tau := \mathbb{C}^g / \Lambda_\tau$. For $(a, b) \in \mathbb{R}^{2g}$, $z \in \mathbb{C}^g$ and $\tau \in \mathfrak{S}_g$ we define the theta function with characteristic by

$$\vartheta_{a,b}(z, \tau) := \sum_{n \in \mathbb{Z}^{2g}} e\left(\frac{1}{2}(n+a)\tau^t(n+a) + (n+a)^t(z+b)\right),$$

where $e(t) = \exp(2\pi\sqrt{-1}t)$. Let

$$f : \Theta_{a,b} := \{(z, \tau) \in \mathbb{A}_g \mid \vartheta_{a,b}(z, \tau) = 0\} \rightarrow \mathfrak{S}_g.$$

be the *universal family of theta divisors*. For simplicity we write ϑ for $\vartheta_{0,0}$ and set $\Theta = \Theta_{0,0}$. On $\mathbb{A}_g$, Γ_g acts by

$$\gamma \cdot (z, \tau) := (z(C\tau + D)^{-1}, (A\tau + B)(C\tau + D)^{-1}), \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g, \quad z \in \mathbb{C}^g, \quad \tau \in \mathfrak{S}_g.$$

For any $(m, n) \in \mathbb{R}^{2g}$, we define an automorphism $t_{m,n} : \mathbb{A}_g \rightarrow \mathbb{A}_g$ by

$$(z, \tau) := (z + m\tau + n, \tau).$$

Then $t_{(m,n)}$ has no fixed points when $(m, n) \in \mathbb{R}^{2g} \setminus \mathbb{Z}^{2g}$ and the subgroup $\mathbb{Z}^{2g} \subset \mathbb{R}^{2g}$ acts trivially on $\mathbb{A}_g$. For $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we define

$$\tilde{\gamma} := t_{(m,n)} \circ \gamma \in \text{Aut}(\mathbb{A}_g), \quad (m, n) := \frac{1}{2}((C^t D)_0, (A^t B)_0).$$

Then $\tilde{\gamma}$ preserves the family $f : \Theta \rightarrow \mathfrak{S}_g$.

Proposition 4.1. *For any $\gamma_1, \gamma_2 \in \Gamma_g$,*

$$\tilde{\gamma}_1 \circ \tilde{\gamma}_2 = \widetilde{\gamma_1 \gamma_2}$$

Proof. See [15] □

We set

$$g^{\mathbb{A}_g/\mathfrak{S}_g} := dz \cdot (\text{Im}\tau)^{-1} \cdot {}^t d\bar{z}.$$

Then $g^{\mathbb{A}_g/\mathfrak{S}_g}$ is a Γ_g -invariant Hermitian metric on the relative tangent bundle $T(\mathbb{A}_g/\mathfrak{S}_g)$. The next purpose of this section is to construct a Γ_g -invariant Kähler metric on $T\mathbb{A}_g$ such that $g^{\mathbb{A}_g}|_{A_\tau} = dz \cdot (\text{Im}\tau)^{-1} \cdot {}^t d\bar{z}$ for all $\tau \in \mathfrak{S}_g$.

Put $T^{2g} := \mathbb{R}^{2g} / \mathbb{Z}^{2g}$. Define a $\mathbb{Z}^{2g}$ -action on $\mathbb{R}^{2g} \times \mathfrak{S}_g$ by $(m, n) \cdot (x, y, \tau) := (x + m, y + n, \tau)$ for $(m, n) \in \mathbb{Z}^{2g}$, $(x, y) \in \mathbb{R}^{2g}$, $\tau \in \mathfrak{S}_g$. Then $(\mathbb{R}^{2g} \times \mathfrak{S}_g) / \mathbb{Z}^{2g}$ is the trivial T^{2g} -bundle $T^{2g} \times \mathfrak{S}_g$. We define a C^∞ -map $\tilde{\rho} : \mathbb{R}^{2g} \times \mathfrak{S}_g \rightarrow \mathbb{C}^g \times \mathfrak{S}_g$ by

$$\tilde{\rho}((x, y), \tau) := (x\tau + y, \tau), \quad x, y \in \mathbb{R}^g, \quad \tau \in \mathfrak{S}_g.$$

Since $\tilde{\rho}$ is a $\mathbb{Z}^{2g}$ -equivariant map, $\tilde{\rho}$ induces a C^∞ -isomorphism $\rho : T^{2g} \times \mathfrak{S}_g \rightarrow \mathbb{A}_g$ as T^2 -bundles over $\mathfrak{S}_g$. Define a Γ_g -action on $T^{2g} \times \mathfrak{S}_g$ by

$$\gamma \cdot ((x, y), \tau) := ((x, y)\gamma^{-1}, \gamma \cdot \tau), \quad \gamma \in \Gamma_g.$$

Then for any $\gamma \in \Gamma_g$, the following diagram is commutative.

$$\begin{array}{ccc} T^{2g} \times \mathfrak{S}_g & \xrightarrow{\rho} & \mathbb{A}_g \\ \gamma \downarrow & & \downarrow \gamma \\ T^{2g} \times \mathfrak{S}_g & \xrightarrow[\rho]{} & \mathbb{A}_g \end{array}$$

Since the trivial connection on $T^{2g} \times \mathfrak{S}_g$ is Γ_g -invariant, $\mathbb{A}_g$ has the induced Γ_g -invariant connection $T_H \mathbb{A}_g \subset T \mathbb{A}_g$ via the Γ_g -equivariant isomorphism ρ . We denote the Γ_g -equivariant projection corresponding to $T_H \mathbb{A}_g$ by P_ρ . Let $P_\rho^{\mathbb{C}} : T \mathbb{A}_g \otimes \mathbb{C} \rightarrow T(\mathbb{A}_g/\mathfrak{S}_g) \otimes \mathbb{C}$ be the complexification of P_ρ . Then $P_\rho^{\mathbb{C}}$ is also Γ_g -equivariant.

Under the projection, the horizontal lift of a $(1, 0)$ (resp. $(1, 0)$) tangent vector is a $(1, 0)$ (resp. $(1, 0)$) tangent vector. Therefore the extension $P_\rho^{\mathbb{C}} : T \mathbb{A}_g \otimes \mathbb{C} \rightarrow T(\mathbb{A}_g/\mathfrak{S}_g) \otimes \mathbb{C}$ decomposes

$$(2) \quad P_\rho^{\mathbb{C}} = P_\rho^{1,0} \oplus P_\rho^{0,1},$$

under the isomorphism $T \mathbb{A}_g \otimes \mathbb{C} = T^{1,0} \mathbb{A}_g \oplus T^{0,1} \mathbb{A}_g$ and $T(\mathbb{A}_g/\mathfrak{S}_g) \otimes \mathbb{C} = T^{1,0}(\mathbb{A}_g/\mathfrak{S}_g) \oplus T^{0,1}(\mathbb{A}_g/\mathfrak{S}_g)$. Hence P_ρ induces a Γ_g -equivariant C^∞ -isomorphism

$$(3) \quad T^{1,0} \mathbb{A}_g \cong T^{1,0}(\mathbb{A}_g/\mathfrak{S}_g) \oplus f^* T^{1,0} \mathfrak{S}_g.$$

Let $g^{\mathfrak{S}_g}$ be the Bergman metric on $\mathfrak{S}_g$ with Kähler form

$$(4) \quad \omega_{\mathfrak{S}_g} = -2\sqrt{-1} \partial \bar{\partial} \log \det \text{Im} \tau.$$

Then $g^{\mathfrak{S}_g}$ is Γ_g -invariant. Using the Γ_g -equivariant isomorphism (3), we define the Γ_g -invariant Hermitian metric $g^{\mathbb{A}_g}$ on $T \mathbb{A}_g$ by

$$g^{\mathbb{A}_g} := g^{\mathbb{A}_g/\mathfrak{S}_g} \oplus f^* g^{\mathfrak{S}_g}.$$

Theorem 4.2. *The Hermitian metric $g^{\mathbb{A}_g}$ is Kähler.*

Proof. See [15] □

We put

$$A_k(\Gamma_g, \chi) = \{f \in \mathcal{O}(\mathfrak{S}_g) \mid f(\gamma \cdot \tau) = j(\tau, \gamma)^k \chi(\gamma) f(\tau), \gamma \in \Gamma_g\}$$

where χ is a character of Γ_g and $j(\tau, \gamma) = \det(C\tau + D)$ for $\gamma \in \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. An element of $A_k(\Gamma_g, \chi)$ is called a *Siegel modular form* of weight k with character χ . In particular, an element of $A_k(\Gamma_g, 1)$ is called a Siegel modular form. Let $\mathcal{F}_g^k := \mathfrak{S}_g \times \mathbb{C}^g$ be the trivial holomorphic line bundle over $\mathfrak{S}_g$ with the Γ_g -action

$$\gamma \cdot (\tau, \xi) := (\gamma \cdot \tau, j(\gamma, \tau)^k \xi).$$

A Siegel modular form of weight k is regarded as a Γ_g -invariant holomorphic section of $\mathcal{F}_g^k$. Define the *Peterson metric* on $\mathcal{F}_g^k$ by

$$\|\xi\|_{\mathcal{F}_g^k}^2 := (\det \text{Im} \tau)^k |\xi|^2, \quad (\tau, \xi) \in \mathcal{F}_g^k.$$

By the automorphic property of $\det \text{Im}(\gamma \cdot \tau) = |j(\tau, \gamma)|^{-2} \det \text{Im} \tau$, we see that $\|\cdot\|_{\mathcal{F}_g^k}$ is Γ_g -invariant.

Let $\mathcal{N}_g := \{\tau \in \mathfrak{S}_g \mid \text{Sing} \Theta_\tau \neq \emptyset\}$ be the Andreotti-Mayer locus, which is the locus of Abelian varieties whose theta divisors is singular. The followings are known for the locus $\mathcal{N}_g$.

Theorem 4.3 ([12]). $\mathcal{N}_g$ is a divisor of $\mathfrak{S}_g$, consisting of two irreducible components as a divisor of the modular variety $\Gamma_g \backslash \mathfrak{S}_g$:

$$\mathcal{N}_g = \theta_{\text{null},g} + 2\mathcal{N}'_g.$$

Here $\theta_{\text{null},g}$ is the zero divisor of Igusa's modular form $\chi_g(\tau)$ which is the Siegel modular form of weight $2^{g-2}(2^g + 1)$ defined as the product of all even theta constants and $\mathcal{N}'_g = \emptyset$ for $g = 2, 3$. For a generic point $\tau \in \theta_{\text{null},g}$, $\text{Sing}(\Theta_\tau)$ consists of one ordinary double point.

Theorem 4.4 ([25]). There is a Siegel cusp form $\Delta_g(\tau)$ of weight $\frac{(g+3) \cdot g!}{2}$ with zero divisor $\mathcal{N}_g$. By the Proposition 4.3, this implies that there exists $J_g(\tau)$ which is a Siegel modular form of weight $\frac{(g+3) \cdot g!}{4} - 2^{g-3}(2^g + 1)$ with zero divisor $\mathcal{N}'_g$ such that

$$\Delta_g := \chi_g(\tau) J_g(\tau)^2.$$

We put $\mathfrak{S}'_g := \mathfrak{S}_g - \mathcal{N}_g$, $\Theta'_g := \Theta|_{\mathfrak{S}'_g}$. Then $f : \Theta' \rightarrow \mathfrak{S}'_g$ is a family of smooth theta divisors. Endow $T^{1,0}(\Theta'/\mathfrak{S}'_g)$ the Hermitian metric $g^{\Theta'/\mathfrak{S}'_g} := g^{\mathfrak{A}_g/\mathfrak{S}_g}|_{\Theta'}$. Let $g^{\Theta'} := g^{\mathfrak{A}_g}|_{\Theta'}$ be the restriction of the Kähler metric $g^{\mathfrak{A}_g}$. Consider $g^{\Theta'/\mathfrak{S}'_g}$ and $g^{\Theta'}$ as Riemannian metric on $T(\Theta'/\mathfrak{S}'_g)$ and $T\Theta'$. Let

$$T_H\Theta' := (T(\Theta'/\mathfrak{S}'_g))^\perp$$

be the orthogonal complement of $T(\Theta'/\mathfrak{S}'_g)$ in $T\Theta'$, which induces a connection $P_{\Theta'} : T\Theta' \rightarrow T\Theta'/\mathfrak{S}'_g$. Hence we obtain the connection $\nabla^{\Theta'/\mathfrak{S}'_g}$ on $T(\Theta'/\mathfrak{S}'_g)$ by using $g^{\Theta'/\mathfrak{S}'_g}$ and $P_{\Theta'}$ as in Section 2.2. Let ∇^h be the holomorphic Hermitian connection on $T^{1,0}(\Theta'/\mathfrak{S}'_g)$ with respect to the Hermitian metric $g^{\Theta'/\mathfrak{S}'_g}$.

Lemma 4.5. Under the C^∞ -isomorphism $T(\Theta'/\mathfrak{S}'_g) \otimes \mathbb{C} \cong T^{1,0}(\Theta'/\mathfrak{S}'_g) \oplus T^{0,1}(\Theta'/\mathfrak{S}'_g)$, the following equality of connections holds.

$$\nabla^{\Theta'/\mathfrak{S}'_g} \otimes \mathbb{C} = \nabla^h \oplus \bar{\nabla}^h$$

Proof. Let ∇^L be the Levi-Civita connection on $T\mathfrak{A}_g$ and let ∇^H be the holomorphic Hermitian connection on $T^{1,0}\mathfrak{A}_g$. Since $g^{\mathfrak{A}_g}$ is Kähler, the following equality holds ([18])

$$\nabla^L \otimes \mathbb{C} = \nabla^H \oplus \bar{\nabla}^H$$

under the isomorphism $T\mathfrak{A}_g \otimes \mathbb{C} = T^{1,0}\mathfrak{A}_g \oplus T^{0,1}\mathfrak{A}_g$. By (2), we get

$$\begin{aligned} \nabla^{\Theta'/\mathfrak{S}'_g} \otimes \mathbb{C} &= (P_\rho \nabla^L P_\rho) \otimes \mathbb{C} \\ &= P_\rho^C (\nabla^L \otimes \mathbb{C}) P_\rho^C \\ &= P_\rho^{1,0} \nabla^H P_\rho^{1,0} \oplus P_\rho^{0,1} \bar{\nabla}^H P_\rho^{0,1}. \end{aligned}$$

Since $P_\rho^{1,0} \nabla^H P_\rho^{1,0} = \nabla^h$ (see [18] Chapter I, Section 6), we get the result. $\square$

Let g_{1_g} be the restriction of the Hermitian metric $|dz|^2$ on $T\mathfrak{A}_g/\mathfrak{S}_g$ to the relative tangent bundle $T\Theta'/\mathfrak{S}'_g$. Let $F(T\Theta'/\mathfrak{S}'_g, g_{1_g})$ be the corresponding Chern-Weil form for $F(x)$ and the holomorphic Hermitian connection of $(T\Theta'/\mathfrak{S}'_g, g_{1_g})$.

Proposition 4.6 ([24], Proposition 2.1). The following equality holds:

$$[F(T\Theta'/\mathfrak{S}'_g, g_{1_g})]^{(g,g)} \equiv 0.$$

In particular one has

$$[f_* F(T\Theta'/\mathfrak{S}'_g, g_{1_g})]^{(1,1)} \equiv 0.$$

Let $\|\Delta_{2g}(\tau)\|^2 := (\det \text{Im} \tau)^{\frac{(2g+3) \cdot (2g)!}{2}} |\Delta_{2g}(\tau)|^2$ denote the Peterson norm of the Siegel modular form $\Delta_{2g}(\tau)$ and let B_k be the k -th Bernoulli number, i.e.,

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} B_k \frac{x^{2k}}{(2k)!}.$$

Theorem 4.7. *The followig equality holds:*

$$\begin{aligned} \left[f_* L(T(\Theta' / \mathfrak{S}'_{2g}), \nabla^{\Theta' / \mathfrak{S}'_{2g}}) \right]^{(2)} &= \frac{(-1)^g 2^{2g+1} (2^{2g+2} - 1)}{(2g+1)(g+1)} B_{g+1} dd^c \log \det \text{Im} \tau \\ &= \frac{(-1)^g 2^{2g+3} (2^{2g+2} - 1)}{(2g+3)!} B_{g+1} dd^c \log \|\Delta_{2g}(\tau)\|^2. \end{aligned}$$

By Lemma 4.5 and the fact that $(\nabla^h)^2$ is a $(1, 1)$ -form, we see that the left-hand side is equal to $\left[f_* L(T^{1,0}(\Theta' / \mathfrak{S}'_{2g}), \nabla^h) \right]^{(1,1)}$. By Proposition 4.6 we obtain

$$\left[L(T^{1,0}(\Theta' / \mathfrak{S}'_{2g}), \nabla^h) \right]^{(1,1)} = -dd^c f_* [\tilde{L}(T^{1,0}(\Theta' / \mathfrak{S}'_{2g}), g_{1g}, g^{\Theta' / \mathfrak{S}'_{2g}})]^{(2g-1, 2g-1)}.$$

Hence we deduced the proof to the computation of the Bott-Chern form and we can compute it by using the same idea in [25]. Since this is rather complicated, we omit the proof.

Remark 4.8. In Section 5, it will be crucial that $dd^c \log \|\Delta_g(\tau)\|^2$ is Γ_g -invariant and that $dd^c \log \|\Delta_g(\tau)\|^2$ is an exact form as a 2-form on $\Gamma_g \backslash \mathfrak{S}'_g$.

5. The signature cocycle for smooth theta divisors

Since Γ_g acts on $\mathfrak{S}'_g$ properly discontinuously the space $\Gamma_g \backslash \mathfrak{S}'_g$ has naturally orbifold structure and can be regarded as the moduli space of smooth theta divisors. We shall consider the orbifold fundamental group of $\Gamma_g \backslash \mathfrak{S}'_g$ and construct a 2-cocycle of this group.

In the rest of this section we fix a generic base point $* \in \mathfrak{S}'_g$, i.e., $*$ satisfies $\{\gamma \in \Gamma_g \mid \gamma * = *\} = \{\pm 1_{2g}\}$. Let (B, b) be a topological space with a base point and let $\pi : \tilde{B} \rightarrow B$ be the universal covering. Then the fundamental group $\pi_1(B, b)$ acts on $\tilde{B}$ as the deck transformation. Fix a lift $\tilde{b} \in \tilde{B}$ of $b \in B$. We set

$$[B, \Gamma_g \backslash \mathfrak{S}'_g]^{\text{orb}} := \{(p, \beta) \mid p : \tilde{B} \rightarrow \mathfrak{S}'_g, \beta : \pi_1(B, b) \rightarrow \Gamma_g, \text{ s.t. } p(\tilde{b}) = *, p(\gamma \cdot x) = \beta(\gamma) \cdot p(x)\} / \sim.$$

Here the relation $(p_0, \beta_0) \sim (p_1, \beta_1)$ holds if and only if $\beta_0 = \beta_1$ and there is a map $\tilde{p} : \tilde{B} \times [0, 1] \rightarrow \mathfrak{S}'_g$ such that $\tilde{p}(x, 0) = p_0$, $\tilde{p}(x, 1) = p_1$ and $\tilde{p}(\gamma \cdot x, t) = \beta(\gamma) \cdot \tilde{p}(x, t)$ for any $\gamma \in \Gamma_g$, $x \in \tilde{B}$, $t \in [0, 1]$.

Definition 5.1. We define the *orbifold fundamental group* of $\Gamma_g \backslash \mathfrak{S}'_g$ by

$$\begin{aligned} S_g &:= [S^1, \Gamma_g \backslash \mathfrak{S}'_g]^{\text{orb}} \\ &= \{(\alpha, \gamma) \mid \gamma \in \Gamma_g, \alpha : \mathbb{R} \rightarrow \mathfrak{S}'_g, \text{ s.t. } \alpha(0) = *, \alpha(t) = \gamma \cdot \alpha(t+1), t \in \mathbb{R}\} / \sim. \end{aligned}$$

Then

$$S_g = \{(\alpha, \gamma) \mid \gamma \in \Gamma_g, \alpha : [0, 1] \rightarrow \mathfrak{S}'_g, \text{ s.t. } \alpha(0) = \gamma \cdot \alpha(1) = *\} / \sim.$$

Here $(\alpha_0, \gamma_0) \sim (\alpha_1, \gamma_1)$ if and only if $\gamma_0 = \gamma_1$ and there exists a homotopy $\alpha(s, t) : [0, 1] \times [0, 1] \rightarrow \mathfrak{S}'_g$ connecting α_0 and α_1 , such that $\alpha(s, 0) = \gamma_0 \cdot \alpha(s, 1) = *$ for $s \in [0, 1]$.

The group law of S_g is defined as follows. Let $[(\alpha_1, \gamma_1)], [(\alpha_2, \gamma_2)] \in S_g$. Then $\gamma_2^{-1} \cdot \alpha_1$ is a path from $\gamma_2^{-1} \cdot *$ to $(\gamma_1 \gamma_2)^{-1} \cdot *$. We define the new path $\alpha : [0, 1] \rightarrow \mathfrak{S}'_g$ by $\alpha(t) := \alpha_2(2t)$ for $0 \leq t \leq \frac{1}{2}$, $\alpha(t) := \gamma_2^{-1} \cdot \alpha_1(2t-1)$ for $\frac{1}{2} \leq t \leq 1$. Then we define $[(\alpha_1, \gamma_1)] \cdot [(\alpha_2, \gamma_2)] := [(\alpha, \gamma_1 \gamma_2)] \in S_g$.

Let $p : S_g \rightarrow \Gamma_g$ be the projection to the second factor. Since the kernel of p is isomorphic to $\pi_1(\mathfrak{S}'_g, *)$, we have an exact sequence

$$(5) \quad 1 \rightarrow \pi_1(\mathfrak{S}'_g, *) \rightarrow S_g \rightarrow \Gamma_g \rightarrow 1.$$

Remark 5.2. When $g = 1$, $\Gamma_1 \backslash \mathfrak{S}'_1 = SL_2\mathbb{Z} \backslash \mathfrak{S}_1$ is the moduli space of curves of genus 1 and $S_1 = \mathcal{M}_1$. When $g = 2$, $\Gamma_2 \backslash \mathfrak{S}'_2$ is the moduli space of curves of genus 2 by the Torelli theorem and $S_2 = \mathcal{M}_2$.

Recall that a $\pi_1(B, b)$ -equivariant map $f : (\tilde{B}, \tilde{b}) \rightarrow (\mathfrak{S}'_g, *)$ induces the homomorphism of groups $f_* : \pi_1(B, b) \rightarrow S_g$ such that $f_*([c]) = [f \circ c]$ for $[c] \in \pi_1(B, b)$.

Proposition 5.3. *Let (B, b) be a compact oriented surface with base point and with non empty boundary. Then the map*

$$[B, \Gamma_g \backslash \mathfrak{S}'_g]^{orb} \ni [f] \mapsto f_* \in \text{Hom}(\pi_1(B, b), S_g).$$

is a bijection.

Proof. It is known that B is homotopy equivalent to an n -bouquet $\vee_{k=1}^n S_k^1$ for some n and the fundamental group $\pi_1(B, b) \cong \pi_1(\vee_{k=1}^n S_k^1, o)$ is isomorphic to the free group of rank n . Hence we get

$$[B, \Gamma_g \backslash \mathfrak{S}'_g]^{orb} \simeq [\vee_{k=1}^n S_k^1, \Gamma_g \backslash \mathfrak{S}'_g]^{orb} \simeq \text{Hom}(\pi_1(\vee_{k=1}^n S_k^1, o), S_g) \simeq \text{Hom}(\pi_1(B, b), S_g).$$

which completes the proof. $\square$

In the rest of this section we assume that $B = S^2 - \Pi_{k=1}^3 D_k$, where D_1, D_2, D_3 are mutually disjoint open discs. Since B is homotopy equivalent to a 2-bouquet $\pi_1(B, b)$ is the free group of rank 2. Let g_1, g_2 be generators of $\pi_1(B, b)$ represented by the loops which are mutually homotopy equivalent to $\partial D_1, \partial D_2$. By Proposition 5.3 we have a bijection

$$(6) \quad [B, \Gamma_g \backslash \mathfrak{S}'_g]^{orb} \simeq S_g \times S_g,$$

which is given by $[f] \mapsto (f_*(g_1), f_*(g_2)) \in S_g \times S_g$ for $[f] \in [B, \Gamma_g \backslash \mathfrak{S}'_g]^{orb}$.

For $[f] \in [B, \Gamma_g \backslash \mathfrak{S}'_g]^{orb}$ the fiber product $\pi : \tilde{B} \times_f \Theta \rightarrow \tilde{B}$ is a $\pi_1(B, b)$ -equivariant fiber bundle because $f : \tilde{B} \rightarrow \mathfrak{S}'_g$ is a $\pi_1(B, b)$ -equivariant map. We get the fiber bundle $\pi : (\tilde{B} \times_f \Theta) / \pi_1(B, b) \rightarrow B$, which is uniquely determined by $[f] \in [B, \Gamma_g \backslash \mathfrak{S}'_g]^{orb}$ up to an isomorphism and which is $2g$ -dimensional compact oriented manifold with boundary. For $(\sigma_1, \sigma_2) \in S_g \times S_g$, Let $\pi : X(\sigma_1, \sigma_2) \rightarrow B$ denote the corresponding fiber bundle under the isomorphism (6).

Definition 5.4. Define the map $c_{2g} : S_{2g} \times S_{2g} \rightarrow \mathbb{Z}$ by

$$c_{2g}(\sigma_1, \sigma_2) := \text{Sign}(X(\sigma_1, \sigma_2)).$$

We call c_{2g} the *signature cocycle for smooth theta divisors*.

Remark 5.5. We only consider the case of an even genus because in the case of an odd genus $\text{Sign}(X(\sigma_1, \sigma_2))$ always vanishes.

Lemma 5.6. *The following relation holds:*

$$c_{2g}(\sigma_1, \sigma_2) + c_{2g}(\sigma_1\sigma_2, \sigma_3) = c_{2g}(\sigma_2, \sigma_3) + c_{2g}(\sigma_2\sigma_3, \sigma_1),$$

for any $\sigma_1, \sigma_2, \sigma_3 \in S_{2g}$. In particular, c_{2g} is a 2-cocycle of the group S_{2g} ([11]).

Proof. By the same argument in [1], we obtain the assertion. $\square$

Let $[c_{2g}] \in H^2(S_{2g}, \mathbb{Z})$ be the cohomology class of c_{2g} . When $g = 1$, c_2 is the Meyer cocycle.

6. Construction of the Meyer function

Let $\sigma = [(\alpha, \gamma)]$ be an element of S_{2g} , where $\alpha : \mathbb{R} \rightarrow \mathfrak{S}'_{2g}$ and $\gamma \in \Gamma_{2g}$. Let $\mathbb{R} \times_{\alpha} \Theta'$ be the fiber product, which has a natural $\pi_1(S^1)$ -action. We define the *mapping torus* M_{σ} for σ by

$$\pi : M_{\sigma} := (\mathbb{R} \times_{\alpha} \Theta') / \pi_1(S^1) \rightarrow S^1.$$

Since the metric $g^{\Theta'/\mathfrak{S}'_{2g}}$ on $T(\Theta'/\mathfrak{S}'_{2g})$ and the connection $P_{\Theta'}$ on Θ' are Γ_{2g} -invariant and the map $p : \widetilde{S^1} = \mathbb{R} \rightarrow \mathfrak{S}'_{2g}$ is $\pi_1(S^1)$ -equivariant, the metric g^{M_{σ}/S^1} on $T(M_{\sigma}/S^1)$ and the connection P_{σ} on M_{σ} are naturally induced via the map p . Using the connection P_{σ} we define the 1-parameter family of Riemannian metrics $\{g^{M_{\sigma}, \varepsilon}\}_{\varepsilon > 0}$ on M_{σ} by

$$g^{M_{\sigma}, \varepsilon} := g^{M_{\sigma}/S^1} \oplus \varepsilon^{-1} \pi^* dt^2, \quad \varepsilon \in \mathbb{R}_{>0}.$$

Here we regard S^1 as $\mathbb{R}/\mathbb{Z}$ and $t \in \mathbb{R}$ as a coordinate of S^1 . By the theorem 3.3, the adiabatic limit

$$\eta^0(M_{\sigma}, g^{M_{\sigma}, \varepsilon}) := \lim_{\varepsilon \rightarrow 0} \eta(M_{\sigma}, g^{M_{\sigma}, \varepsilon})$$

exists. Recall that the Siegel modular form $\Delta_{2g}(\tau)$ with zero divisors $\mathcal{N}_{2g}$ (see Section 3.3.). Since the 1-form $d^c \log \|\Delta_{2g}(\tau)\|^2$ is Γ_{2g} -invariant the pull-back $p^* d^c \log \|\Delta_{2g}(\tau)\|^2$ can be regarded as a 1-form on S^1 .

Definition 6.1. For $\sigma \in S_{2g}$ we fix (p, γ) which represents $\sigma = [(p, \gamma)]$, where $\gamma \in \Gamma_{2g}$ and $p : \mathbb{R} \rightarrow \mathfrak{S}'_{2g}$. we set

$$\Phi_{2g}(p, \gamma) := \eta^0(M_{\sigma}, g^{M_{\sigma}}) + \frac{(-1)^g 2^{2g+3} (2^{2g+2} - 1) B_{g+1}}{(2g+3)!} \int_{S^1} p^* d^c \log \|\Delta_{2g}(\tau)\|^2.$$

The following theorem is the main result of this paper.

Theorem 6.2. (a) The value $\Phi_{2g}(p, \gamma)$ is independent of a choice of (p, γ) which represents $\sigma \in S_{2g}$. In particular Φ_{2g} is a function on S_{2g} .

(b) The cocycle $-c_{2g}$ is the coboundary of the function Φ_{2g} . In particular $[c_{2g}] \otimes \mathbb{Q} = 0 \in H^2(S_{2g}, \mathbb{Z})$.

As a corollary of the Theorem 6.2, it follows that $\phi_2 = \Phi_2$ by the uniqueness of Meyer's function of genus 2. On the other hand, $\Delta_2(\tau)$ coincides with the Igusa's modular form $\chi_2(\tau)$ ([25]), which is the product of all even theta constants. Then we can derive the following formula:

Corollary 6.3 ([15]). Let $\sigma = [(p, \gamma)]$ be an element of $S_2 = \mathcal{M}_2$ as before. Then we have

$$\phi_2(\sigma) = \eta^0(M_{\sigma}, g^{M_{\sigma}, \varepsilon}) - \frac{2}{15} \int_{S^1} p^* d^c \log \|\chi_2(\tau)\|^2.$$

Proof of Theorem 6.2. (a) Assume that (p_0, γ) and (p_1, γ) represents the same element $\sigma \in S_{2g}$. Put $I := [0, 1]$. There is a map

$$\tilde{p} : I \times \mathbb{R} \rightarrow \mathfrak{S}'_{2g}$$

which satisfies $\tilde{p}(s, 0) = *$ for $s \in I$ and $\tilde{p}(s, t) = \gamma \cdot \tilde{p}(s, t+1)$ for $(s, t) \in I \times \mathbb{R}$ and the following condition

$$(7) \quad \tilde{p}(s, t) = p_0(t), \quad s \in [0, \frac{1}{3}) \quad \text{and} \quad \tilde{p}(s, t) = p_1(t), \quad s \in (\frac{2}{3}, 1].$$

Since $\tilde{p}$ is $\pi_1(I \times \mathbb{R})$ -equivariant, the fiber product $(I \times \mathbb{R}) \times_{\tilde{p}} \Theta'$ has the $\pi_1(I \times S^1)$ -action and the quotient space

$$\bar{\pi} : \bar{M}_{\sigma} := (I \times \mathbb{R}) \times_{\tilde{p}} \Theta' / \pi_1(I \times S^1) \rightarrow I \times S^1$$

has the induced metric $g^{\bar{M}_\sigma/I \times S^1}$ on $T(\bar{M}_\sigma/I \times S^1)$ from the metric $g^{\Theta'}/\mathfrak{S}'_2$ and the connection $\bar{P}_\sigma$ on $\bar{M}_\sigma$ from the connection $P_{\Theta'}$ mutually via the map p . Using the connection $\bar{P}_\sigma$ we set

$$g^{\bar{M}_\sigma, \varepsilon} := g^{\bar{M}_\sigma/I \times S^1} \oplus \varepsilon^{-1} \pi^*(ds^2 \oplus dt^2), \quad \varepsilon \in \mathbb{R}_{>0}.$$

Let $g_i^{M_\sigma, \varepsilon}$ be the metrics on M_{σ_i} induced from the map p_i for $i = 0, 1$ as above. The condition (7) implies that

$$g^{\bar{M}_\sigma, \varepsilon}|_{[0, \frac{1}{3}) \times S^1} = g_0^{M_\sigma, \varepsilon} \oplus \varepsilon^{-1} dt^2, \quad g^{\bar{M}_\sigma, \varepsilon}|_{(\frac{2}{3}, 1] \times S^1} = g_1^{M_\sigma, \varepsilon} \oplus \varepsilon^{-1} dt^2.$$

Then we can apply the Atiyah-Patodi-Singer's index theorem to $(\bar{M}_\sigma, g^{\bar{M}_\sigma, \varepsilon})$:

$$(8) \quad \text{Sign}(\bar{M}_\sigma) = \int_{I \times S^1} \bar{\pi}_* L(T\bar{M}_\sigma, g^{\bar{M}_\sigma, \varepsilon}) - (\eta(M_\sigma, g_0^{M_\sigma, \varepsilon}) - \eta(M_\sigma, g_1^{M_\sigma, \varepsilon})).$$

Since $\bar{M}_\sigma$ is isomorphic to the product $M_\sigma \times I$, we have (see [3]),

$$(9) \quad \text{Sign}(\bar{M}_\sigma) = \text{Sign}(M_\sigma) \times \text{Sign}(I) = 0.$$

By Proposition 2.4 and the Proposition 2.5, we get

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{I \times S^1} \bar{\pi}_* L(T\bar{M}_\sigma, g^{\bar{M}_\sigma, \varepsilon}) &= \int_{I \times S^1} \bar{\pi}_* \left(L(T(\bar{M}_\sigma/(I \times S^1))) \cdot \bar{\pi}^* L(T(I \times S^1)) \right) \\ &= \int_{I \times S^1} \left(\bar{\pi}_* L(T(\bar{M}_\sigma/(I \times S^1)), \nabla^{\bar{M}_\sigma/(I \times S^1)}) \right)^{(2)} \\ &= \int_{I \times S^1} \left[\bar{\pi}_* \bar{p}^* L(T(\Theta'/\mathfrak{S}'_2), \nabla^{\Theta'/\mathfrak{S}'_2}) \right]^{(2)} \\ (10) \quad &= \int_{I \times S^1} \bar{p}^* \left[\bar{\pi}_* L(T(\Theta'/\mathfrak{S}'_2), \nabla^{\Theta'/\mathfrak{S}'_2}) \right]^{(2)} \end{aligned}$$

where $\nabla^{\bar{M}_\sigma/(S^1 \times I)}$ is the connection on the relative tangent bundle $T(\bar{M}_\sigma/(S^1 \times I))$ associated with $g^{\bar{M}_\sigma/(S^1 \times I)}$ and $\bar{P}_\sigma$ and we used the commutativity of fiber integrals and base changes in the last equality. By the Proposition 4.7, we have

$$\begin{aligned} (11) \quad & \int_{I \times S^1} \bar{p}^* \left[\bar{\pi}_* L(T(\Theta'/\mathfrak{S}'_2), \nabla^{\Theta'/\mathfrak{S}'_2}) \right]^{(2)} \\ &= -\frac{2^{2g+3}(2^{2g+2}-1)B_{2g+2}}{(2g+3)!} \int_{I \times S^1} \bar{p}^* d^c \log \|\Delta_{2g}(\tau)\|^2 \\ &= -\frac{2^{2g+3}(2^{2g+2}-1)B_{2g+2}}{(2g+3)!} \left(\int_{\{1\} \times S^1} p_1^* d^c \log \|\Delta_{2g}(\tau)\|^2 - \int_{\{0\} \times S^1} p_0^* d^c \log \|\Delta_{2g}(\tau)\|^2 \right), \end{aligned}$$

where we used the Γ_{2g} -invariance of the 1-form $d^c \log \|\Delta_{2g}(\tau)\|^2$ in the last equality. By (25) ~ (12) and the Definition 6.1, we obtain

$$0 = \Phi_{2g}(p_1, \gamma) - \Phi_{2g}(p_0, \gamma),$$

which completes the proof of (a).

(b) Let $\sigma_1 = [(p_1, \gamma_1)]$, $\sigma_2 = [(p_2, \gamma_2)]$, $\sigma_3 := (\sigma_1 \sigma_2)^{-1} = (p_3, (\gamma_1 \gamma_2)^{-1}) \in S_{2g}$. Set $B := S^2 - \coprod_{k=1}^3 D_k$. Recall that the fiber bundle $\pi : X(\sigma_1, \sigma_2) \rightarrow B$ for σ_1, σ_2 defined at the Section 3.2. By the definition of Φ_{2g} , we have $\Phi_{2g}(\sigma^{-1}) = -\Phi_{2g}(\sigma)$ for any $\sigma \in S_{2g}$. Therefore to show that $-c_{2g}$ is the coboundary of Φ , we have to show that

$$(12) \quad \text{Sign}(X(\sigma_1, \sigma_2)) = -\sum_{i=1}^3 \Phi_{2g}(\sigma_i)$$

Let U_i be the neighborhood of ∂D_i in B such that $U_i \cong [0, 1) \times \partial D_i$. Let $\beta_i : \tilde{U}_i \cong [0, 1) \times \mathbb{R} \rightarrow \tilde{B}$ be the lift of the map $U_i \hookrightarrow B$. Let $g_1, g_2 \in \pi_1(B, b)$ be the generators represented by the loops $\partial D_1, \partial D_2$. Let $[(p, \alpha)] \in [B, \Gamma_{2g} \setminus \mathfrak{S}'_{2g}]^{orb}$ be the corresponding element for $(\sigma_1, \sigma_2) \in S_{2g} \times S_{2g}$ under the isomorphism (6) where $\alpha : \pi_1(B, b) \rightarrow \Gamma_{2g}$ is a group homomorphism and $p : \tilde{B} \rightarrow \mathfrak{S}'_{2g}$ is a $\pi_1(B, b)$ -equivariant homomorphism preserving the basepoint. Since $\partial D_1, \partial D_2$ and ∂D_3 are homotopy equivalent to the loops which represent g_1, g_2 and $(g_1 g_2)^{-1} \in \pi_1(B, b)$ we can assume that

$$(13) \quad p \circ \beta_i|_{\tilde{U}_i}(s_i, t) = p_i(t), \quad (s_i, t) \in \tilde{U}_i \cong [0, 1) \times \mathbb{R}, \quad i = 1 \sim 3.$$

Let $g^{X(\sigma_1, \sigma_2)/B}$ and $P_{X(\sigma_1, \sigma_2)}$ be the metric on $TX(\sigma_1, \sigma_2)$ and the connection on $X(\sigma_1, \sigma_2)$ induced from the metric $g^{\Theta'/\mathfrak{S}'_{2g}}$ and the connection $P_{\Theta'}$ via the map p . Let g^B be the metric on TB such that $g^B|_{U_i} = ds_i^2 \oplus dt^2$. Using the connection $P_{X(\sigma_1, \sigma_2)}$ we define the metric on $TX(\sigma_1, \sigma_2)$ by

$$g^{X(\sigma_1, \sigma_2), \varepsilon} := g^{X(\sigma_1, \sigma_2)/B} \oplus \varepsilon^{-1} \pi^* g^B, \quad \varepsilon \in \mathbb{R}_{>0}.$$

Let $g^{M_{\sigma_i}, \varepsilon}$ be the metric on M_{σ_i} induced from p_i for $i = 1 \sim 3$ as above. Let $\nabla^{X(\sigma_1, \sigma_2)/B}$ be the connection on $T(X(\sigma_1, \sigma_2))$ defined by the metric $g^{X(\sigma_1, \sigma_2)/B}$ and the connection $P_{X(\sigma_1, \sigma_2)}$. Since the condition (13) implies that the metric $g^{X(\sigma_1, \sigma_2), \varepsilon}$ is a product metric near the boundary of $X(\sigma_1, \sigma_2)$ we can apply the Atiyah-Patodi-Singer's index theorem to $(X(\sigma_1, \sigma_2), g^{X(\sigma_1, \sigma_2), \varepsilon})$:

$$\begin{aligned} \text{Sign}(X(\sigma_1, \sigma_2)) &= \int_{X(\sigma_1, \sigma_2)} L(TX(\sigma_1, \sigma_2), g^{X(\sigma_1, \sigma_2), \varepsilon}) - \sum_{i=1}^3 \eta(M_{\sigma_i}, g^{M_{\sigma_i}, \varepsilon}) \\ &= \int_B \pi_* L(T(X(\sigma_1, \sigma_2)/B), \nabla^{X(\sigma_1, \sigma_2)/B}) - \sum_{i=1}^3 \eta^0(M_{\sigma_i}, g^{M_{\sigma_i}, \varepsilon}) \\ &= \int_B p^* \left[f_* L(T(\Theta'/\mathfrak{S}'_{2g}), \nabla^{\Theta'/\mathfrak{S}'_{2g}}) \right]^{(2)} - \sum_{i=1}^3 \eta^0(M_{\sigma_i}, g^{M_{\sigma_i}, \varepsilon}) \\ &= - \sum_{i=1}^3 \int_{\partial D_i} \frac{2^{2g+3}(2^{2g+2} - 1)B_{2g+2}}{(2g+3)!} d^c \log \|\Delta_{2g}(\tau)\|^2 \\ &\quad - \sum_{i=1}^3 \eta^0(M_{\sigma_i}, g^{M_{\sigma_i}, \varepsilon}) \\ &= - \sum_{i=1}^3 \Phi_{2g}(\sigma_i) \end{aligned}$$

which completes the proof of (b). $\square$

7. The first cohomology of S_g

The uniqueness of a 1-cocycle that cobounds the 2-cocycle c_{2g} is equivalent to the vanishing of $H^1(S_{2g}, \mathbb{Z})$. In deed, if there is another 1-cocycle $\Phi'_{2g} : S_{2g} \rightarrow \mathbb{R}$ that cobounds c_{2g} , the difference $\Phi_{2g} - \Phi'_{2g}$ is an element of $\text{Hom}(S_{2g}, \mathbb{R}) \cong H^1(S_{2g}, \mathbb{R})$. While $H^1(S_1, \mathbb{Z}) = H^1(S_2, \mathbb{Z}) = 0$, the uniqueness no longer valid for higher genus.

Theorem 7.1. *The following holds:*

$$H^1(S_g, \mathbb{Z}) = \begin{cases} 0 & 1 \leq g \leq 3, \\ \mathbb{Z} & g \geq 4. \end{cases}$$

In particular, the cochain cobounding the signature cocycle c_{2g} is not unique when $g \geq 2$.

By (5) and [11], we have the 5-term exact sequence

$$(14) \quad 1 \rightarrow H^1(\Gamma_g, \mathbb{Z}) \rightarrow H^1(S_g, \mathbb{Z}) \rightarrow H^1(\pi_1(\mathfrak{S}'_g, *), \mathbb{Z})^{\Gamma_g} \xrightarrow{\delta} H^2(\Gamma_g, \mathbb{Z}) \rightarrow H^2(S_g, \mathbb{Z}).$$

We have $H^1(\Gamma_g, \mathbb{Z}) = 0$ for $g \geq 1$ and $H^2(\Gamma_g, \mathbb{Z}) = \mathbb{Z}$ for $g \geq 3$. By the Hurwitz theorem we see that

$$(15) \quad H^1(\pi_1(\mathfrak{S}'_g, *), \mathbb{Z}) \cong H^1(\mathfrak{S}'_g, \mathbb{Z}).$$

Lemma 7.2. *Let X be a connected complex manifold of $\dim_{\mathbb{C}} X \geq 2$. Assume that*

$$(16) \quad H^1(X, \mathbb{Z}) = H^2(X, \mathbb{Z}) = 0.$$

Let $D = \sum_{\lambda \in \Lambda} n_{\lambda} D_{\lambda}$ be a divisor on X such that $n_{\lambda} \neq 0$ and D_{λ} is irreducible for all $\lambda \in \Lambda$. Then

$$H^1(X - D, \mathbb{Z}) \cong \mathbb{Z}^{\Lambda}.$$

The generator of the cohomology $H^1(X - D, \mathbb{Z})$ corresponding to $\lambda \in \Lambda$ is represented by the map $l_{\lambda} \mapsto 1$ and $l_{\mu} \mapsto 0$ for $\mu \neq \lambda \in \Lambda$, where l_{μ} denotes the loop around a small disk and intersecting D_{μ} transversally.

Proof. Since the real codimension of $\text{Sing} D$ in X is greater than or equal to 4, we have $\pi_k(X, X - \text{Sing} D, *) = 0$ for $1 \leq k \leq 3$. The relative Hurwitz theorem asserts that $H_k(X, X - \text{Sing} D, \mathbb{Z}) = 0$ for $k \leq 3$. Hence $H^k(X, X - \text{Sing} D, \mathbb{Z}) = 0$ for $k \leq 3$, which together with the cohomology exact sequence for the triple $(X, X - \text{Sing} D, X - D)$, yields that

$$(17) \quad H^2(X, X - D, \mathbb{Z}) \cong H^2(X - \text{Sing} D, X - D, \mathbb{Z}).$$

By the cohomology exact sequence for the pair $(X, X - D)$ and (16), we obtain

$$(18) \quad H^1(X - D, \mathbb{Z}) \cong H^2(X, X - D, \mathbb{Z}).$$

Since $D - \text{Sing} D$ is a closed submanifold in $X - \text{Sing} D$ and $X - D = (X - \text{Sing} D) - (D - \text{Sing} D)$, the Thom isomorphism asserts that

$$(19) \quad H^2(X - \text{Sing} D, X - D, \mathbb{Z}) \cong H^0(D - \text{Sing} D, \mathbb{Z}).$$

By the irreducibility of D_{λ} , $D_{\lambda} - \text{Sing} D_{\lambda}$ is path connected so that

$$(20) \quad H^0(D - \text{Sing} D, \mathbb{Z}) \cong \mathbb{Z}^{\Lambda}.$$

The result follows from (17)~(20). □

Lemma 7.3. *The following holds:*

$$H^1(\pi_1(\mathfrak{S}'_g, *), \mathbb{Z})^{\Gamma_g} = \begin{cases} \mathbb{Z} & 1 \leq g \leq 3 \\ \mathbb{Z}^{\oplus 2} & g \geq 4. \end{cases}$$

By regarding $H^1(\mathfrak{S}'_g, \mathbb{C})$ as the de Rham cohomology group, the image of the generators under the natural map $H^1(\mathfrak{S}'_g, \mathbb{Z}) \rightarrow H^1(\mathfrak{S}'_g, \mathbb{C})$ are represented by the 1-forms $\frac{1}{2\pi\sqrt{-1}} d\log \chi_g(\tau)$ and $\frac{1}{2\pi\sqrt{-1}} d\log J_g(\tau)$. Here $J_g(\tau) \equiv 1$ and hence $d\log J_g(\tau) = 0$ for $1 \leq g \leq 3$.

Proof. By Proposition 4.3, Proposition 4.4, the isomorphism (15) and Lemma 7.2, we get the assertion. □

Recall that the automorphic factor $j(\tau, \gamma)$ is a nowhere vanishing holomorphic function on $\mathfrak{S}_g$. Since $\mathfrak{S}_g$ is simply connected, the logarithm of $j(\tau, \gamma)$ makes sense. Choose a branch of the logarithm of $j(\tau, \gamma)$ and denote it by $\log_\sigma j(\tau, \gamma)$ for $\gamma \in \Gamma_g$. Define the function $\lambda_\sigma : \Gamma_g \times \Gamma_g \rightarrow \mathbb{Z}$ by

$$(21) \quad \lambda_\sigma(A, B) := \frac{1}{2\pi\sqrt{-1}} \{ \log_\sigma j(\tau, AB) - \log_\sigma j(B \cdot \tau, A) - \log_\sigma j(\tau, B) \}, \quad (A, B) \in \Gamma_g \times \Gamma_g.$$

Lemma 7.4. *The function λ_σ is a 2-cocycle of Γ_g , whose cohomology class generates $H^2(\Gamma_g, \mathbb{Z})$.*

Proof. For $g = 1$ see [4]. When $g \geq 1$, we follow [4]. Let $G := Sp(2g, \mathbb{R})$ be the symplectic group and let G^δ be the same group endowed with the discrete topology. Let $u \in H^2(G^\delta, \mathbb{Z})$ be the cohomology class corresponding to the universal covering

$$0 \rightarrow \mathbb{Z} \rightarrow \tilde{G} \rightarrow G \rightarrow 1.$$

We choose the branch $\log_\sigma j(\tau, \gamma)$ satisfying

$$(22) \quad \text{Im } \log_\sigma j(\sqrt{-1} \cdot 1_{2g}, \gamma) \in [0, 2\pi).$$

Since the function $\bar{\lambda}_\sigma$ is measurable, the cohomology class $[\bar{\lambda}_\sigma]$ is a constant multiple of u by [20]. Therefore it suffices to determine the restriction of the cohomology class $[\bar{\lambda}_\sigma]$ to the maximal compact subgroup of G . We shall identify the unitary group $U(g)$ with the maximal compact subgroup of G by the inclusion map defined as

$$\iota : U(g) \ni Z \mapsto \begin{pmatrix} \text{Re } Z & \text{Im } Z \\ -\text{Im } Z & \text{Re } Z \end{pmatrix} \in G, \quad Z \in U(g).$$

Since $j(\sqrt{-1} \cdot 1_{2g}, \iota(Z)) = \det(Z)^{-1}$ for $Z \in U(g)$ and the isotropy subgroup at $\sqrt{-1} \cdot 1_{2g} \in \mathfrak{S}_g$ is just $U(g)$, we have

$$(23) \quad 2\pi\sqrt{-1}\bar{\lambda}_\sigma(Z_1, Z_2) = -\log_\sigma \det(Z_1 Z_2) + \log_\sigma \det(Z_1) + \log_\sigma \det(Z_2)$$

for $(Z_1, Z_2) \in U(g) \times U(g)$. By (23), the restriction of the cohomology class $[\bar{\lambda}_\sigma]$ to $U(g)$ is the pull-back of the cohomology class corresponding to the universal covering $0 \rightarrow \mathbb{Z} \rightarrow \tilde{U}(1) \cong \mathbb{R} \rightarrow U(1) \rightarrow 1$, via the map $\det : U(g) \rightarrow U(1)$. Since the induced map $(\det)_* : \pi_1(U(g)) \rightarrow \pi_1(U(1))$ is an isomorphism, we obtain $[\bar{\lambda}_\sigma] = u$. Since the cohomology class $[\bar{\lambda}_\sigma]$ is independent of the choice of the branch of $\log_\sigma j(\tau, \gamma)$ and since the restriction of u to Γ_g is the generator of the cohomology $H^2(\Gamma_g, \mathbb{Z})$ we obtain the assertion. $\square$

Lemma 7.5. *Let $g \geq 2$. The map $\delta : H^1(\pi_1(\mathfrak{S}'_g, *), \mathbb{Z}) \rightarrow H^2(\Gamma_g, \mathbb{Z})$ is given by*

$$(m, n) \mapsto (k_1(g)m + k_2(g)n) \in H^2(\Gamma_g, \mathbb{Z}) \cong \mathbb{Z}$$

for $(m, n) \in H^1(\pi_1(\mathfrak{S}'_g, *), \mathbb{Z}) \cong \mathbb{Z}^{\oplus 2}$. Here,

$$k_1(g) = 2^{g-2}(2^g + 1), \quad k_2(g) = \frac{(g+3) \cdot g!}{4} - 2^{g-3}(2^g + 1)$$

are the weights of Siegel modular forms $\chi_g(\tau), J_g(\tau)$, respectively.

Proof. Let $\sigma : \Gamma_g \rightarrow S_g$ be a section, and write $\sigma(\gamma) = [(l_\gamma, \gamma)] \in S_g$ for $\gamma \in \Gamma_g$. We can assume that $l_{\gamma^{-1}} = -\gamma \cdot l_\gamma$, where $-l(t) := l(1-t)$, $t \in [0, 1]$ for a path $l(t)$. Hence $\sigma(\gamma^{-1}) = \sigma(\gamma)^{-1}$. Let α be an element of $H^1(\pi_1(\mathfrak{S}'_g, *), \mathbb{Z})^{\Gamma_g} \cong \text{Hom}(\pi_1(\mathfrak{S}'_g, *), \mathbb{Z})^{\Gamma_g}$. Then $\delta(\alpha) : \Gamma_g \times \Gamma_g \rightarrow \mathbb{Z}$ is given by

$$(A, B) \mapsto \alpha(\sigma(A)\sigma(B)\sigma(AB)^{-1}) \in \mathbb{Z}, \quad (A, B) \in \Gamma_g \times \Gamma_g,$$

where we identify $\sigma(A)\sigma(B)\sigma(AB)^{-1} \in \text{Im}\{\pi_1(\mathfrak{S}'_g, *) \rightarrow S_g\}$ with the corresponding preimage of $\pi_1(\mathfrak{S}'_g, *)$ under the inclusion $\pi_1(\mathfrak{S}'_g, *) \hookrightarrow S_g$. Write $\sigma(A)\sigma(B)\sigma(AB)^{-1} = [(l_{(A,B)}, 1)] \in \pi_1(\mathfrak{S}'_g, *)$.

Here $l_{(A,B)}$ is a loop on $\mathfrak{S}'_g$, which is the composition of the paths l_B , $B^{-1} \cdot l_A$ and $-l_{AB}$. Under the identification $H^1(\pi_1(\mathfrak{S}'_g, *), \mathbb{Z})^{\Gamma_g} \cong \mathbb{Z}^{\oplus 2}$ given in Lemma 7.3, the cochain $\delta(m, n)$ is given by

$$\delta(m, n)(A, B) = \frac{1}{2\pi\sqrt{-1}} \int_{l_{(A,B)}} d\log\chi_g(\tau)^m J_g(\tau)^n \in \mathbb{Z}, \quad (A, B) \in \Gamma_g \times \Gamma_g,$$

for $(m, n) \in H^1(\pi_1(\mathfrak{S}'_g, *), \mathbb{Z})^{\Gamma_g} \cong \mathbb{Z}^{\oplus 2}$. Using σ , we choose the branch $\log_\sigma j(\tau, \gamma)$ for $\gamma \in \Gamma_g$ such that

$$\log_\sigma j(*, \gamma) := \frac{1}{k_1(g)} \int_{l_{\gamma^{-1}}} d\log\chi_g(\tau).$$

Then we get

$$\begin{aligned} 2\pi\sqrt{-1}\delta(1, 0)(A, B) &= \int_{l_{(A,B)}} d\log\chi_g(\tau) \\ &= \int_{AB \cdot l_{(A,B)}} d\log\chi_g(AB \cdot \tau) \\ &= \int_{AB \cdot l_{(A,B)}} [k_1(g)d\log j(\tau, AB) + d\log\chi_g(\tau)] \\ &= \int_{AB \cdot l_B} d\log\chi_g(\tau) + \int_{A \cdot l_A} d\log\chi_g(\tau) - \int_{AB \cdot l_{AB}} d\log\chi_g(\tau) \\ &= - \int_{l_{B^{-1}}} d\log\chi_g(A \cdot \tau) - \int_{l_{A^{-1}}} d\log\chi_g(\tau) + \int_{l_{(AB)^{-1}}} d\log\chi_g(\tau) \\ &= - \int_{l_{B^{-1}}} [k_1(g)d\log j(\tau, A) + d\log\chi_g(\tau)] \\ &\quad - k_1(g)\log_\sigma j(*, A) + k_1(g)\log_\sigma j(*, AB) \\ &= k_1(g)[- \log_\sigma j(B \cdot *, A) + \log_\sigma j(*, A) - \log_\sigma j(*, B) \\ &\quad - \log_\sigma j(*, A) + \log_\sigma j(*, AB)] \\ &= k_1(g)[\log_\sigma j(*, AB) - \log_\sigma j(B \cdot *, A) - \log_\sigma j(*, B)]. \end{aligned}$$

By Lemma 7.4 we get $\delta(1, 0) = k_1(g) \in H^2(\Gamma_g, \mathbb{Z}) \cong \mathbb{Z}$. Similarly, $\delta(0, 1) = k_2(g) \in H^2(\Gamma_g, \mathbb{Z}) \cong \mathbb{Z}$. This completes the proof. $\square$

Proof of Theorem 7.1. Since $H^1(\Gamma_g, \mathbb{Z})$ in the exact sequence (5), we get $H^1(S_g, \mathbb{Z}) = \ker \delta$. By Lemma 7.5, we get $\ker \delta = 0$ for $1 \leq g \leq 3$ and $\ker \delta \cong \mathbb{Z}$ for $g \geq 4$. This completes the proof of Theorem 7.1. $\square$

8. The value for the Dehn twist

In this section, we shall compute the value of Φ_{2g} for the *Dehn twist*, which is defined as follows (cf. [16]). Let $\Delta \subset \mathbb{C}$ be the unit disk. Recall that the Andreotti-Mayer locus $\mathcal{N}_{2g}$ has two irreducible components $\theta_{null, 2g}$ and $\mathcal{N}'_{2g}$ by Theorem 4.3. Let $\rho : \Delta \rightarrow \mathfrak{S}_{2g}$ be a C^∞ -map such that $\rho(0) \in \theta_{null, 2g}$ is a generic point, $\rho(z) \notin \mathcal{N}_{2g}$ for $z \in \Delta \setminus \{0\}$ and $\rho(\Delta)$ intersects with $\theta_{null, 2g}$ at $\rho(0)$ transversally. For simplicity we assume that the base point $*$ lies in $\rho(\partial\Delta)$ and we denote the monodromy corresponding to the loop $\rho|_{\partial\Delta} : \partial\Delta \rightarrow \mathfrak{S}'_{2g}$ by $\sigma_{2g} \in S_{2g}$. The element σ_{2g} is called the Dehn twist. We put

$$\varpi : X_{2g} := \Delta \times_\rho \Theta \rightarrow \Delta,$$

which is smooth family of theta divisors over Δ induced from the universal family $\pi : \Theta \rightarrow \mathfrak{S}_{2g}$ by ρ . Let $\tilde{\rho} : X_{2g} \rightarrow \Theta$ be the lift of the map ρ defined as the projection to the second factor. By

the assumption of ρ and the Theorem 4.3, $\text{Sing}(\varpi^{-1}(0))$ consists of one ordinary double point and $\varpi^{-1}(z)$ is a smooth theta divisor for $z \in \Delta \setminus \{0\}$. Notice that ∂X_{2g} endowed with the orientation induced from X_{2g} is diffeomorphic to the mapping torus $M_{\sigma_{2g}^{-1}}$ endowed with the natural orientation, i.e., $\partial X_{2g} = -M_{\sigma_{2g}}$.

Theorem 8.1. *The following equality holds:*

$$\Phi_{2g}(\sigma_{2g}) = \begin{cases} -\frac{4}{5} & \text{if } g = 1, \\ (-1)^{g+1} \frac{(2g+1)2^{2g+2}(2^{2g+2}-1)}{(2g+3)!} B_{g+1} & \text{if } g > 1. \end{cases}$$

Proof. Put $\Delta_r := \{z \in \Delta \mid |z| < r\} \subset \Delta$ for $0 < r < 1$. We choose ρ such that the restriction $\rho|_{\Delta_{1/3}} : \Delta_{1/3} \rightarrow \rho(\Delta_{1/3}) \subset \mathbb{S}_{2g}$ is a holomorphic embedding that

$$(24) \quad \rho(re^{\sqrt{-1}\theta}) = \rho\left(\frac{2}{3}e^{\sqrt{-1}\theta}\right), \quad \frac{2}{3} < r \leq 1, \quad 0 \leq \theta < 2\pi.$$

Let g^Δ be the metric on $T\Delta$ which is a product metric near the boundary $\partial\Delta$ and coincides with the metric $\rho^*g^{\mathbb{S}_g}$ on $\Delta_{1/3}$. Let $p \in X_{2g}$ be the unique singular point on the singular fiber X_0 . Let $g^{X_{2g}/\Delta}$ be the metric on $T(X_{2g}/\Delta)|_{X_{2g}-\{p\}}$ induced from the metric $g^{\Theta/\mathbb{S}_g}$ via the map ρ . Let $g^{X_{2g}}$ be the metric on TX_{2g} which coincides with $g^{X_{2g}/\Delta} \oplus \varpi^*g^\Delta$, where we used the connection induced from the connection $P_{\Theta'}$ on Θ' via the map ρ , on $X_{2g} - \{p\}$ and coincides with the metric induced from the metric $g^{\Theta'}$ via the map $\tilde{\rho}$ on a neighbourhood of p . Set

$$g^{X_{2g},\varepsilon} := g^{X_{2g}} \oplus \varepsilon^{-1} \varpi^* g^\Delta, \quad \varepsilon \in \mathbb{R}_{>0}.$$

By the assumption of g^Δ and the condition (24), $g^{X_{2g},\varepsilon}$ is the product metric near the boundary ∂X_{2g} for $\varepsilon \in \mathbb{R}_{>0}$. By the Atiyah-Patodi-Singer index theorem,

$$(25) \quad \text{Sign}(X_{2g}) = \int_{X_{2g}} L(TX_{2g}, g^{X_{2g},\varepsilon}) + \eta(M_{\sigma_{2g}}, g^{M_{\sigma_{2g}},\varepsilon}).$$

Here ∂X_{2g} is identified with $-M_{\sigma_{2g}}$, and $g^{M_{\sigma_{2g}},\varepsilon}$ is the restriction of $g^{X_{2g},\varepsilon}$ to the boundary $\partial X_{2g} \cong -M_{\sigma_{2g}}$. By the formula in [26], the first term of the right-hand side of (25):

$$(26) \quad \lim_{\varepsilon \rightarrow 0} L(TX_{2g}, g^{X_{2g},\varepsilon}) = L(T(X_{2g}/\Delta), \nabla^{X_{2g}/\Delta}) + P(-t, \dots, (-t)^{2g})|_{t^{2g}} \cdot \mu(p) \delta_p$$

Here $L(T(X_{2g}/\Delta), \nabla^{X_{2g}/\Delta})$ is only defined on $X_{2g} - \{p\}$ but has the natural smooth extension on whole X_{2g} . The constant $\mu(p)$ is the Milnor number of the singular point p , δ_p is the Dirac delta current supported at p and $P(x_1, \dots, x_{2g}) \in \mathbb{C}[[x_1, \dots, x_{2g}]]$ is defined by

$$\prod_{k=1}^{2g} L(x_k) = P(\sigma_1, \dots, \sigma_{2g}),$$

where $L(x) = x/\tanh(x)$ and $\sigma_1 = \sum_k x_k$, $\sigma_2 = \sum_{i>j} x_i x_j$, $\dots$, $\sigma_{2g} = \prod_k x_k$ are the fundamental symmetric polynomials. Notice that

$$P(-t, \dots, (-t)^{2g})|_{t^{2g}} = L^{-1}(t)|_{t^{2g}}.$$

Since p is a non-degenerate critical point of $\pi : X \rightarrow \Delta$, we get $\mu(p) = 1$, which together with (25), (26) and Theorem 4.7, yields that

$$(27) \quad \begin{aligned} \text{Sign}(X_{2g}) &= \frac{(-1)^g 2^{2g+1} (2^{2g+2} - 1)}{(g+1)(2g+1)} B_{g+1} \int_{\Delta} \rho^* dd^c \log \det \text{Im} \tau \\ &+ \frac{(-1)^g 2^{2g+2} (2^{2g+2} - 1)}{(2g+2)!} B_{g+1} + \eta^0(M_{\sigma_{2g}}, g^{M_{\sigma_{2g}},\varepsilon}). \end{aligned}$$

By (27) and Definition 6.1, we get

$$\begin{aligned}
\Phi_{2g}(\sigma_{2g}) &= \eta^0(M_{\sigma_{2g}}, g^{M_{\sigma_{2g}}, \epsilon}) \\
&+ \frac{(-1)^g 2^{2g+3} (2^{2g+2} - 1)}{(2g+3)!} B_{g+1} \int_{\partial \Delta} p^* d^c \left(\log |\Delta_{2g}(\tau)|^2 (\det \operatorname{Im} \tau)^{\frac{(2g+3) \cdot (2g)!}{2}} \right) \\
&= - \frac{(-1)^g 2^{2g+2} (2^{2g+2} - 1)}{(2g+2)!} B_{g+1} + \operatorname{Sign}(X_{2g}) \\
&+ \frac{(-1)^g 2^{2g+3} (2^{2g+2} - 1)}{(2g+3)!} B_{g+1} \int_{\Delta} p^* dd^c \log |\Delta_{2g}(\tau)|^2 \\
&= \frac{(-1)^{g+1} (2g+1) 2^{2g+2} (2^{2g+2} - 1)}{(2g+3)!} B_{g+1} + \operatorname{Sign}(X_{2g}),
\end{aligned}$$

where we used the Poincaré-Lelong formula and Theorem 4.4 to get the last equality. When $g = 1$, since the singular fiber has two irreducible components and $\operatorname{Sign}(X_2) = -1$, we obtain the proof for the case $g = 1$. We complete the proof by the following Lemma in the case $g > 1$. $\square$

Lemma 8.2. *Let $\pi : \mathfrak{X} \rightarrow \Delta$ be a Lefschetz degeneration of relative dimension $2n - 1$, i.e., π is a proper holomorphic surjective map from a $2n$ -dimensional complex manifold $\mathfrak{X}$ to the unit disk Δ and there is a point $p \in \mathfrak{X}_0$ and an open neighbourhood $p \in U \cong \{(z_1, \dots, z_{2n}) \in \mathbb{C}^{2n} \mid \sum_{k=1}^{2n} |z_k|^2 < 1\}$ such that*

$$\pi(z_1, \dots, z_{2n}) = \sum_{k=1}^{2n} z_k^2, \quad (z_1, \dots, z_{2n}) \in U$$

and π_* has maximal rank on $\mathfrak{X} \setminus p$. Assume that $n > 1$. Then $\operatorname{Sign}(\mathfrak{X}) = 0$.

Proof. For $t \in \Delta$, we set $U_t := \mathfrak{X}_t \cap U$. Then a sequence of inclusions

$$\mathfrak{X}_0 \setminus U_0 \subset \mathfrak{X}_0 \setminus \{p\} \subset \mathfrak{X}_0 \subset \mathfrak{X}$$

induces a sequence of isomorphisms:

$$(28) \quad H_{2n}(\mathfrak{X}_0 \setminus U_0, \mathbb{Z}) \cong H_{2n}(\mathfrak{X}_0 \setminus \{p\}, \mathbb{Z}) \cong H_{2n}(\mathfrak{X}_0, \mathbb{Z}) \cong H_{2n}(\mathfrak{X}, \mathbb{Z}).$$

Here the first isomorphism follows from the homotopy equivalence of $\mathfrak{X}_0 \setminus U_0$ and $\mathfrak{X}_0 \setminus \{p\}$, the second isomorphism follows from the fact $\operatorname{codim}_{\mathbb{R}} \{p\} / \mathfrak{X}_0 = 4n - 2 > 2n + 1$, and the third isomorphism follows from the fact that the inclusion $\mathfrak{X}_0 \hookrightarrow \mathfrak{X}$ is a deformation retraction. By Ehresman's Theorem, $\mathfrak{X} \setminus U$ is diffeomorphic to $(\mathfrak{X}_0 \setminus U_0) \times \Delta$ as a fiber bundle over Δ . Since Δ is contractible, the inclusion $\mathfrak{X}_t \setminus U_t \hookrightarrow \mathfrak{X} \setminus U$ induces an isomorphism $H_{2n}(\mathfrak{X}_t \setminus U_t, \mathbb{Z}) \cong H_{2n}(\mathfrak{X} \setminus U, \mathbb{Z})$. By (28), the inclusion $\mathfrak{X}_t \setminus U_t \hookrightarrow \mathfrak{X}$ induces an isomorphism $H_{2n}(\mathfrak{X}_t \setminus U_t, \mathbb{Z}) \rightarrow H_{2n}(\mathfrak{X}, \mathbb{Z})$. Hence, for any $t \in \Delta$, any element of $H_{2n}(\mathfrak{X}, \mathbb{Z})$ can be represented by a cycle contained in $\mathfrak{X}_t$. Therefore the intersection matrix of $H_{2n}(\mathfrak{X}, \mathbb{Z})$ is trivial and $\operatorname{Sign}(\mathfrak{X}) = 0$. This completes the proof. $\square$

Remark 8.3. When $g = 1$, $\sigma_2 \in \mathcal{M}_2$ is the Dehn twist along a separating simple closed curve on a Riemann surface of genus two. Since $\operatorname{Sign}(X_2) = -1$ and $B_2 = \frac{1}{30}$, we obtain $\phi_2(\sigma_2) = \Phi_2(\sigma_2) = -\frac{4}{5}$, which confirms a result of Matsumoto ([19]).

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