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Cominimum Independence”

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An Axiomatization of Choquet Expected Utility with Cominimum Independence*

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Abstract

This paper proposes a class of independence axioms for simple acts. By introducing the $\mathcal{E}$ -*cominimum independence axiom* that is stronger than the comonotonic independence axiom but weaker than the independence axiom, we provide a new axiomatization theorem of simple acts within the framework of Choquet Expected Utility. Furthermore, in order to provide the axiomatization of simple acts, we generalize Kajii, Kojima, and Ui (2007, Journal of Mathematical Economics) into an infinite state space. Our axiomatization theorem relates Choquet Expected Utility to Multi-prior Expected Utility through the core of a capacity that is explicitly derived within our framework. Our result in this paper also derives Gilboa (1989, Econometrica), Eichberger and Kelsey (1999, Theory and Decision), and Rohde (2010, Social Choice and Welfare) as a corollary.

JEL Classification Numbers: C71; D81; D90

Key Words: Cominimum Additivity; Cominimum Independence; Choquet Expected Utility; Multi-prior Expected Utility; Core; E-Capacity Expected Utility

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1. Introduction

A lot of work has been done since Ellsberg (1961) casts doubts on the validity of the Subjective Expected Utility (henceforth, SEU) theory axiomatized by Savage (1954). One of the seminal works is Schmeidler (1989) that shows that the decision maker's beliefs are captured by a *capacity* and her preferences are represented by the *Choquet integral* if she satisfies a set of axioms. As a related paper, Schmeidler (1986) proposes the notion of *comonotonic additivity* and provides some characterization of comonotonic additive operators. The notions of *comonotonic independence* and comonotonic additivity¹ are a key to deriving the axiomatization theorem in Schmeidler (1989). The comonotonic independence axiom that is weaker than the independence axiom in Anscombe and Aumann (1963) enables us to overcome Ellsberg's paradox and to represent the decision maker's preferences in the abovementioned way.

As a generalization of Schmeidler (1986), Kajii, Kojima, and Ui (2007) propose the concept of *cominimum additive operator* that is weaker than the concept of additive operator but is stronger than the concept of comonotonic additive operator considered by Schmeidler (1986),² and provide some characterization of cominimum additive operators. However, two problems remain to be solved. First, Kajii, Kojima, and Ui (2007) do not provide axiomatizations of the decision maker's preferences. From a normative point of view, it is intriguing to shed some light on cominimum additivity, which illuminates the meaning of cominimum additivity in decision theory. Second, since Kajii, Kojima, and Ui (2007) analyze only a finite state space, the framework discussed in Kajii, Kojima, and Ui (2007) is not suitable for axiomatizations of simple acts comparable to the ones introduced by Schmeidler (1989). Moreover, analyses in an infinite state space are technically difficult. Therefore, the analysis of Kajii, Kojima, and Ui (2007) within an infinite state space is interesting not only from a mathematical viewpoint but also from the point of view of decision theory. The purpose of this paper is to generalize Kajii, Kojima, and Ui (2007) into an infinite state space, and to provide a new axiomatization of simple acts based on the notion of $\mathcal{E}$ -*cominimum independence* within the framework of the Choquet Expected Utility (henceforth, CEU) theory.³

Our axiomatization theorem can be also interpreted as an axiomatization theorem of totally monotone games. Although only a subset of the set of all totally monotone games is axiomatized, our axiomatization theorem enables us to derive Gilboa (1989) and Eichberger and Kelsey (1999) as a corollary. Furthermore, we show that our axiomatization

¹For the definitions of comonotonic independence and comonotonic additivity, see Section 3. See also Schmeidler (1986).

²For the definitions of cominimum operators and comonotonic operators, see Section 3.

³In Section 3, we provide the definition of a collection of sets $\mathcal{E}$ in detail.

theorem within CEU can be related to Multi-prior Expected Utility (or Maxmin Expected Utility; henceforth, MEU) through the core of a capacity.⁴ In addition to the existence of the core, our result clarifies how the core can be represented.

The organization of this paper is as follows. Section 2 provides motivations and examples. Section 3 provides the notions of cominimum additivity and cominimum functions. Section 4 presents the axioms and the main result of this paper. Section 5 extends the result of Section 4. Section 6 concludes this paper. Proofs of Lemmas 1 and 2 and of Theorem 3 are relegated to the Appendix.

2. Motivations and Examples

To provide our motivations for this paper, we explain Ellsberg's paradox, as pointed out by Ellsberg (1961). There are 90 balls in an urn. You are supposed to know that 30 balls are red and that the other 60 balls are blue or yellow. You have no additional information. You draw one ball from the urn. Suppose the following lotteries:

f_1 : \$100 if the ball is red, and \$0 if otherwise.

f_2 : \$100 if the ball is blue, and \$0 if otherwise.

f_3 : \$100 if the ball is red or yellow, and \$0 if otherwise.

f_4 : \$100 if the ball is blue or yellow, and \$0 if otherwise.

As pointed out by Ellsberg (1961), most people tend to prefer f_1 to f_2 and prefer f_4 to f_3 .

Eichberger and Kelsey (1999) provide an axiomatic foundation for a rational decision maker whose preferences are represented by Choquet integrals and whose beliefs are E-capacities.⁵ Their representation is called the *E-capacity expected utility*. To explain Eichberger and Kelsey's (1999) model, let $\Omega = \{R, B, Y\}$ be the state space and let $\{\{R\}, \{B, Y\}\}$ be a partition of Ω , where R , B , and Y denote the colors of the red, blue, and yellow balls, respectively. Eichberger and Kelsey (1999) call these events— $\{R\}$ and $\{B, Y\}$ —*unambiguous events* since their probabilities are known as $p(\{R\}) = 1/3$ and $p(\{B, Y\}) = 2/3$, but the events $\{B\}$, $\{Y\}$, $\{R, B\}$, and $\{R, Y\}$ are *ambiguous events* since their probabilities are not known.⁶ The E-capacity expected utility for this partition is as follows: for $\alpha \in (0, 1)$,

$$J(f) = \alpha \left(\frac{1}{3}f(R) + \frac{1}{3}f(B) + \frac{1}{3}f(Y) \right)$$

⁴In Section 5, we define the core of a capacity.

⁵In Section 5, we explain Eichberger and Kelsey's (1999) representation in detail. For the definition of E-capacities, see footnote 15.

⁶See Epstein (1999), Zhang (1999), or Asano (2006) that use a λ -system for a collection of unambiguous events.

$$+ (1 - \alpha) \left(\frac{1}{3} \min_{\omega \in \{R\}} f(\omega) + \frac{2}{3} \min_{\omega \in \{B,Y\}} f(\omega) \right).$$

This representation is consistent with the behaviors presented by Ellsberg (1961).

The E-capacity expected utility by Eichberger and Kelsey (1999) can capture decision makers' preferences when they are facing ambiguous events. However, it is plausible to analyze more complicated collections of events than partitions. For example, suppose that in an urn, there are 120 balls with six colors, white (W), blue (B), red (R), yellow (Y), green (G), and pink (P). You are supposed to know that 40 balls are either white or blue, 40 balls are either red or yellow, 40 balls are either yellow or green, and 40 balls are either green or pink. Suppose the following lotteries:

g_1 : \$100 if the ball is white or blue, and \$0 if otherwise.

g_2 : \$100 if the ball is red or green, and \$0 if otherwise.

Most people are expected to prefer g_1 to g_2 since event $\{W, B\}$ is unambiguous while event $\{R, G\}$ is ambiguous. Now, consider the following lottery h : \$100 if the ball is yellow, and \$0 if otherwise. Next, define the following compound lotteries: $g_3 = (1/2)g_1 + (1/2)h$ and $g_4 = (1/2)g_2 + (1/2)h$. Most people are expected to prefer g_4 to g_3 ; this is because there is only one unambiguous event— $\{W, B\}$ —on which g_3 takes positive values, and there are two unambiguous events— $\{R, Y\}$ and $\{Y, G\}$ —on which g_4 takes positive values. In this case, the collection of unambiguous events is $\{\{W, B\}, \{R, Y\}, \{Y, G\}, \{G, P\}\}$, which is not a partition of the state space, $\{W, B, R, Y, G, P\}$. However, we can express these preferences using the following representation: for $\alpha \in (0, 1)$,

$$J(f) = \alpha \left(\frac{1}{6}f(W) + \frac{1}{6}f(B) + \frac{1}{6}f(R) + \frac{1}{6}f(Y) + \frac{1}{6}f(G) + \frac{1}{6}f(P) \right) \\ + (1 - \alpha) \left(\frac{1}{4} \min_{\omega \in \{W,B\}} f(\omega) + \frac{1}{4} \min_{\omega \in \{R,Y\}} f(\omega) + \frac{1}{4} \min_{\omega \in \{Y,G\}} f(\omega) + \frac{1}{4} \min_{\omega \in \{G,P\}} f(\omega) \right).$$

The example mentioned above can be fully explained if we provide an axiomatic foundation with representations such as

$$\sum_{\omega \in \Omega} \beta_{\omega} f(\omega) + \sum_{T \in \mathcal{E}} \beta_T \min_{\omega \in T} f(\omega), \quad (1)$$

where Ω is a state space, $\mathcal{E}$ is a collection of events of Ω , T is an event of Ω , β_{ω} and β_T are constants, and $f : \Omega \rightarrow \mathbb{R}$ is a function. By obtaining such a representation as (1), our representation theorem in this paper derives Gilboa (1989), Eichberger and Kelsey (1999), and Rohde (2010) as a corollary. In a multiperiod decision model, Gilboa (1989) axiomatizes the following:

$$\sum_{i=1}^t p_i x_i + \sum_{i=2}^t \delta_i |x_i - x_{i-1}|,$$

where $p_1, \dots, p_t$ are positive constants and $\delta_2, \dots, \delta_t$ are negative constants, and x_i denotes the income for time $i = 1, 2, \dots, t$. By letting $\Omega = \{1, 2, \dots, t\}$ denote a set of time periods and letting $\mathcal{E} = \{\{1, 2\}, \{2, 3\}, \dots, \{t-1, t\}\}$ denote the collection of adjacent time periods, Gilboa's (1989) representation can be rewritten as

$$\sum_{i=1}^t \beta_i x_i + \sum_{i=2}^t \beta_{\{i-1, i\}} \min\{x_{i-1}, x_i\},$$

where $\beta_i = p_i + \delta_i$ for $i = 1, n$, $\beta_i = p_i + \delta_i + \delta_{i+1}$ for $i \in \{2, \dots, n-1\}$, and $\beta_{\{i-1, i\}} = -2\delta_i$ for $i \in \{2, \dots, n\}$.⁷ Rohde (2010) provides a preference foundation for Fehr and Schmidt (1999) that introduce a social utility function for individuals capturing concerns about fairness in the sense of inequality aversion. Rohde (2010) axiomatizes the following:

$$x_0 - \alpha \sum_{i=1}^n \max\{x_i - x_0, 0\} - \beta \sum_{i=1}^n \max\{x_0 - x_i, 0\},$$

where α and β are positive constants, and x_i denotes individual i 's payoff for $i = 0, 1, \dots, n$. By letting $\Omega = \{0, 1, \dots, n\}$ denote a set of individuals and letting $\mathcal{E} = \{\{0, 1\}, \{0, 2\}, \{0, 3\}, \dots, \{0, n\}\}$ denote the collection of individuals, Rohde's (2010) representation can be rewritten as

$$x_0 - n\beta x_0 - \sum_{k=1}^n \alpha x_k + \sum_{k=1}^n (\alpha + \beta) \min\{x_k, x_0\}.$$

3. Choquet Integrals and Cominimum Additive Operators

Let Ω be a nonempty finite or infinite set, and let Σ denote a nonempty algebra of subsets of Ω . A usual interpretation is that a generic element $\omega \in \Omega$ denotes a state of the world and that a generic element $E \in \Sigma$ denotes an event. Let $\mathbb{R}^\Omega = \{x | x : \Omega \rightarrow \mathbb{R}\}$ denote the set of all real valued functions on Ω , and let Φ_Σ be the set of functions from Ω to $\mathbb{R}$, which are constant on each element in some finite measurable partition of Ω ; that is, the set of all finite step functions (simple functions) from Ω to $\mathbb{R}$. Let $1_A \in \mathbb{R}^\Omega$ be the indicator function of an event $A \in \Sigma$. Let I denote an operator from Φ_Σ to $\mathbb{R}$. Let $\mathcal{F}$ be the collection of all nonempty subsets of Ω . Let $\mathcal{F}_k$ be the collection of subsets with k elements. For ease of notation, we write $\min_E x = \min_{\omega \in E} x(\omega)$, $\operatorname{argmin}_E x = \operatorname{argmin}_{\omega \in E} x(\omega)$, $\max_E x = \max_{\omega \in E} x(\omega)$, and $\operatorname{argmax}_E x = \operatorname{argmax}_{\omega \in E} x(\omega)$ for $E \in \mathcal{F}$ and $x \in \mathbb{R}^\Omega$.

A set function $v : 2^\Omega \rightarrow \mathbb{R}$ with $v(\emptyset) = 0$ is called a *game* or a *non-additive signed measure*; further, the set function is *monotone* if $E \subseteq F$ implies $v(E) \leq v(F)$ for all $E, F \in 2^\Omega$, *finitely additive* if $v(E \cup F) = v(E) + v(F)$ for all $E, F \in 2^\Omega$ with

⁷See the proof of Proposition 3 in Kajii, Kojima, and Ui (2007).

$E \cap F = \emptyset$, *convex* if $v(E \cup F) + v(E \cap F) \geq v(E) + v(F)$ for all $E, F \in 2^\Omega$, *normalized* if $v(\Omega) = 1$, and a *non-additive measure* if it is monotone. A normalized non-additive measure is called a *capacity*. Furthermore, v is a *finitely additive measure* if it is monotone and finitely additive, and a normalized finitely additive measure is called a *finitely additive probability measure*.

For $x \in \mathbb{R}^\Omega$ and a game v , the *Choquet integral* of x is defined as $\int_\Omega x dv = \int_0^\infty v(x \geq \alpha) d\alpha + \int_{-\infty}^0 (v(x \geq \alpha) - 1) d\alpha$, where $v(x \geq \alpha) = v(\{\omega \in \Omega \mid x(\omega) \geq \alpha\})$. Two functions $x, y \in \mathbb{R}^\Omega$ are *comonotonic* if $(x(\omega) - x(\omega'))(y(\omega) - y(\omega')) \geq 0$ for all $\omega, \omega' \in \Omega$. It can be shown that these two functions are comonotonic if and only if $\operatorname{argmin}_E x \cap \operatorname{argmin}_E y \neq \emptyset$ for all $E \in \mathcal{F}$. Note that for $x, y \in \mathbb{R}^\Omega$, if $\operatorname{argmin}_E x \cap \operatorname{argmin}_E y \neq \emptyset$, then $\min_E (x + y) = \min_E x + \min_E y$. It can be proved that for all comonotonic functions $x, y \in \mathbb{R}^\Omega$, $\int (x + y) dv = \int x dv + \int y dv$.⁸ This is called a *comonotonic additivity* of Choquet integrals. Kajii, Kojima, and Ui (2007) extend the notion of comonotonic additivity by replacing $\mathcal{F}$ with a collection of events $\mathcal{E} \subseteq \mathcal{F}$ where Ω is a finite set. Kajii, Kojima, and Ui (2007) propose the notion of *$\mathcal{E}$ -cominimum functions*, which is stronger than that of comonotonic functions. Let $\mathcal{E} \subseteq \mathcal{F}$ be a collection of events. Two functions $x, y \in \mathbb{R}^\Omega$ are said to be *$\mathcal{E}$ -cominimum* if $\operatorname{argmin}_E x \cap \operatorname{argmin}_E y \neq \emptyset$ for all $E \in \mathcal{E}$.

Based on the notion of $\mathcal{E}$ -cominimum functions, Kajii, Kojima, and Ui (2007) consider the following class of weak additivity concepts for an operator $I : \mathbb{R}^\Omega \rightarrow \mathbb{R}$ that includes additivity and comonotonic additivity as extreme cases. An operator $I : \mathbb{R}^\Omega \rightarrow \mathbb{R}$ is *$\mathcal{E}$ -cominimum additive* if $I(x + y) = I(x) + I(y)$ for any $\mathcal{E}$ -cominimum functions $x, y \in \mathbb{R}^\Omega$. If $\mathcal{E} = \mathcal{F}$, then a pair of two functions x and y are comonotonic if and only if they are $\mathcal{E}$ -cominimum. Therefore, if $\mathcal{E} = \mathcal{F}$, then $\mathcal{E}$ -cominimum additivity is equivalent to comonotonic additivity. An operator $I : \mathbb{R}^\Omega \rightarrow \mathbb{R}$ is *positively homogeneous of degree one* (henceforth, *homogeneous*) if $I(\alpha x) = \alpha I(x)$ for any $\alpha > 0$ and any $x \in \mathbb{R}^\Omega$. Kajii, Kojima, and Ui (2007) show that an operator $I : \mathbb{R}^\Omega \rightarrow \mathbb{R}$ is homogeneous and $\mathcal{E}$ -cominimum additive for some $\mathcal{E} \subseteq \mathcal{F}$ if and only if $I(x) = \int x dv$ for all $x \in \mathbb{R}^\Omega$ where $v \in \mathbb{R}^\mathcal{F}$ is defined by the rule $v(E) = I(1_E)$. Therefore, a homogeneous $\mathcal{E}$ -cominimum additive operator is associated with a game v . Then, the following definition is in order.

Definition 1. A game $v \in \mathbb{R}^\mathcal{F}$ is *$\mathcal{E}$ -cominimum additive* if $\int (x + y) dv = \int x dv + \int y dv$ whenever x and y are $\mathcal{E}$ -cominimum functions.

Let $\mathcal{E} \subseteq \mathcal{F}$ be a collection of events. Kajii, Kojima, and Ui (2007) introduce the notion of *$\mathcal{E}$ -completeness* for their characterization theorem (see Theorem 4 in Appendix A).

⁸For example, see Schmeidler (1986).

Definition 2. Let $\mathcal{E} \subseteq \mathcal{F}$ be a collection of events. An event $T \in \mathcal{F}$ is $\mathcal{E}$ -complete if, for any two distinct points ω_1 and ω_2 in T , there exists a set $E \in \mathcal{E}$ such that $\{\omega_1, \omega_2\} \subseteq E \subseteq T$. The collection of all $\mathcal{E}$ -complete events is called the $\mathcal{E}$ -complete collection and is denoted by $\Upsilon(\mathcal{E})$. Moreover, $\mathcal{E}$ is said to be *complete* if all $\mathcal{E}$ -complete subsets belong to $\mathcal{E}$, that is, $\mathcal{E} = \Upsilon(\mathcal{E})$.

Note that a singleton set, $\mathcal{F}_1$, is $\mathcal{E}$ -complete, and so is any $E \in \mathcal{E}$.⁹ Moreover, if $\mathcal{E}$ is complete, then $\mathcal{E}$ contains all singleton sets automatically. The completeness is a technical condition. However, expanding a collection of events $\mathcal{E}$ into larger ones enables us to obtain important representation theorems in the literature, for example, Gilboa (1989), Eichberger and Kelsey (1999), and Rohde (2010). This is one reason why we introduce the notion of completeness.

4. Main Result

In this section, we provide the main result of this paper. Let X be the nonempty set of all deterministic outcomes, and Y be the set of all distributions over X with finite supports, that is, $Y = \{y : X \rightarrow [0, 1] \mid y(x) \neq 0 \text{ for finitely many } x \in X \text{ and } \sum_{x \in X} y(x) = 1\}$. We call an element of Y a *lottery*. For notational simplicity, we identify $x \in X$ with the Dirac measure $\delta_x \in Y$: δ_x is the probability measure that assigns probability one to $\{x\}$. The set of all Σ -measurable finite step functions from Ω to Y is denoted by L_0 , and elements of L_0 are called *simple lottery acts* or *acts*. The set of all constant functions in L_0 is denoted by L_c , and elements of L_c are called *constant acts*. For all $f, g \in L_0$ and $\lambda \in [0, 1]$, the compound lottery is defined by $(\lambda f + (1 - \lambda)g)(\omega) \equiv \lambda f(\omega) + (1 - \lambda)g(\omega)$ for all $\omega \in \Omega$. We assume that the decision maker's preferences are captured by a binary relation $\succeq$ on L_0 .¹⁰ A binary relation $\succeq$ on Y is defined by restricting $\succeq$ on L_c , and it is denoted by the same symbol $\succeq$. That is, for all $y, z \in Y$, $y \succeq z$ if and only if $y^\Omega \succeq z^\Omega$ where y^Ω and z^Ω denote constant functions on Ω . The following definition provides the comonotonicity of acts with respect to the binary relation.

Definition 3. For two acts $f, g \in L_0$, f and g are said to be *comonotonic* if for all $\omega, \omega' \in \Omega$, $f(\omega) \succeq f(\omega') \Leftrightarrow g(\omega) \succeq g(\omega')$.¹¹

Schmeidler (1989) axiomatizes the CEU within the framework of Anscombe and Aumann (1963).

⁹In Kajii, Kojima, and Ui (2007), the term “complete” is adopted from an analogy to a complete graph. For $T \in \mathcal{F}$, let us consider an undirected graph with a vertex set T where $\{\omega, \omega'\} \subseteq T$ is an edge if there exists $E \in \mathcal{E}$ such that $\{\omega, \omega'\} \subseteq E \subseteq T$. Then, this is a complete graph if and only if T is $\mathcal{E}$ -complete. See Kajii, Kojima, and Ui (2007).

¹⁰The asymmetric ($\succ$) and symmetric ($\sim$) parts of $\succeq$ are defined as usual. For details, see Kreps (1988).

¹¹Equivalently, for two acts $f, g \in L_0$, f and g are said to be *comonotonic* if there are no ω and ω' such that $f(\omega) \succ f(\omega')$ and $g(\omega') \succ g(\omega)$. See Schmeidler (1989).

Axiom 1 (Weak Order). (a) For all $f, g \in L_0$, $f \succeq g$ or $g \succeq f$.

(b) For all $f, g, h \in L_0$, if $f \succeq g$ and $g \succeq h$, then $f \succeq h$.¹²

Axiom 2 (Comonotonic Independence). For every pairwise comonotonic $f, g, h \in L_0$ and all $\alpha \in (0, 1)$, $f \succeq g$ implies $\alpha f + (1 - \alpha)h \succeq \alpha g + (1 - \alpha)h$.

Axiom 3 (Continuity). For all $f, g, h \in L_0$, if $f \succeq g$ and $g \succeq h$, then there exist α and $\beta \in (0, 1)$ such that $\alpha f + (1 - \alpha)h \succeq g$ and $g \succeq \beta f + (1 - \beta)h$.

Axiom 4 (Monotonicity). For all $f, g \in L_0$, if $f(\omega) \succeq g(\omega)$ on Ω , then $f \succeq g$.

Axiom 5 (Nondegeneracy). Not for all $f, g \in L_0$, $f \succeq g$.

Under the above axioms, Schmeidler (1989) proves the following theorem.

Theorem 1. (Schmeidler (1989)) *A binary relation $\succeq$ defined on L_0 satisfies A1, A2, A3, A4, and A5 if and only if there exist a unique capacity v on Σ and an affine real valued function u on Y such that for all f and g in L_0 ,*

$$f \succeq g \Leftrightarrow \int_{\Omega} u(f(\omega)) dv(\omega) \geq \int_{\Omega} u(g(\omega)) dv(\omega).$$

We provide our representation theorem within the framework of Anscombe and Aumann (1963). Our axioms are common, except for Axiom 6. Our new axiom (Axiom 6) requires that independence should hold true for every pairwise $\mathcal{E}$ -cominimum act defined as follows.

Definition 4. Let $\mathcal{E} = \{E_1, E_2, \dots, E_n\} \subseteq \mathcal{F}$ be a collection of events. Two acts $f, g \in L_0$ are said to be $\mathcal{E}$ -cominimum if $\{\omega \in E_i \mid f(\omega') \succeq f(\omega) \text{ for all } \omega' \in E_i\} \cap \{\omega \in E_i \mid g(\omega') \succeq g(\omega) \text{ for all } \omega' \in E_i\} \neq \emptyset$ for all $i = 1, \dots, n$.

Note that for every $E_i \in \mathcal{E}$, there exists an $\omega \in E_i$ such that $f(\omega') \succeq f(\omega)$ for all $\omega' \in E_i$ since f is a simple act.

Axiom 6 ($\mathcal{E}$ -Cominimum Independence). For every pairwise $\mathcal{E}$ -cominimum $f, g, h \in L_0$ and $\alpha \in (0, 1)$, $f \succeq g$ implies $\alpha f + (1 - \alpha)h \succeq \alpha g + (1 - \alpha)h$.

Some remarks are in order here. First, as we explain above, the notion of $\mathcal{E}$ -cominimum acts is stronger than that of comonotonic acts, and the former is equivalent to the latter if $\mathcal{E} = \mathcal{F}$. Axiom 2 (Comonotonic Independence) states that if two acts never disagree with the ranking of any two states, then independence holds for the two acts.

¹²A binary relation $\succeq$ is a weak order if and only if $\succ$ is asymmetric and negatively transitive, whereas a binary relation $\succ$ is *asymmetric* if for all $f, g \in L_0$, $f \succ g \Rightarrow g \not\succ f$ and it is *negatively transitive* if for all $f, g, h \in L_0$, $f \not\succ g$ and $g \not\succ h \Rightarrow f \not\succ h$.

On the other hand, Axiom 6 ($\mathcal{E}$ -Cominimum Independence) states that if two acts assign the worst ranking to the same state, then independence holds for the two acts. Second, $\mathcal{E}$ -cominimum independence states that independence should hold true for every pairwise $\mathcal{E}$ -cominimum act. Therefore, it can be easily checked that Axiom 6 implies Axiom 2. Indeed, assume Axiom 6. Moreover, suppose that f, g, h are pairwise comonotonic. If f, g, h are pairwise comonotonic, then they are pairwise $\mathcal{E}$ -cominimum. Therefore, $f \succeq g$ implies $\alpha f + (1 - \alpha)h \succeq \alpha g + (1 - \alpha)h$ by Axiom 6. Thus, Axiom 2 holds.

Thus, now we are in a position to present the main result of this paper.

Theorem 2. *Let $\mathcal{E} = \{E_1, E_2, \dots, E_n\} \subseteq \mathcal{F}$ with $|E_i| \geq 2$ for $i = 1, \dots, n$ be a collection of events, and assume that $\mathcal{E} \cup \mathcal{F}_1$ is complete. A binary relation $\succeq$ defined on L_0 satisfies A1, A3, A4, A5, and A6 if and only if there exist a unique capacity v on (Ω, Σ) , a unique finitely additive measure μ on (Ω, Σ) , an affine function u and a set of coefficients $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ such that*

$$J(f) = \int_{\Omega} u(f(\omega)) d\mu(\omega) + \sum_{i=1}^n \varepsilon_i \min_{\omega \in E_i} u(f(\omega)) \quad (2)$$

is the CEU with respect to capacity v satisfying $f \succeq g \Leftrightarrow J(f) \geq J(g)$.

Some comments are in order. First, collection $\mathcal{E}$ is exogenously given and the completeness of $\mathcal{E}$ is assumed. Second, if Axiom 6 is replaced with Axiom 2, or if we set $\mathcal{E} = \mathcal{F}$, then Schmeidler's (1989) representation theorem can be obtained. Third, by letting $\mathcal{E} = \{\{W, B\}, \{R, Y\}, \{Y, G\}, \{Y, P\}\}$ for state space $\Omega = \{W, B, R, Y, G, P\}$, the example mentioned in Section 2 can be explained. Moreover, by letting $\mathcal{E}$ be a partition of $\Omega = \{1, 2, \dots, n\}$, letting $\mathcal{E} = \{\{1, 2\}, \{2, 3\}, \dots, \{t-1, t\}\}$ of $\Omega = \{1, 2, \dots, t\}$, and letting $\mathcal{E} = \{\{0, 1\}, \{0, 2\}, \{0, 3\}, \dots, \{0, n\}\}$ of $\Omega = \{0, 1, \dots, n\}$, Eichberger and Kelsey (1999), Gilboa (1989), and Rohde (2010) follow from our results in this paper, respectively.¹³ Finally, to prove this theorem within the framework of simple acts, we must generalize Kajii, Kojima, and Ui (2007) that analyze the case of a finite state space.

To prove our result, the following two lemmas need to be shown.

Lemma 1. *Suppose that $I : \mathbb{R}^{\Omega} \rightarrow \mathbb{R}$ is an operator such that $I(\lambda 1_{\Omega}) = \lambda$. For any pairwise $\mathcal{E}$ -cominimum functions, $a, b, c \in \Phi_{\Sigma}$, and any $\alpha \in (0, 1)$, if $I(a) \geq I(b)$ implies $I(\alpha a + (1 - \alpha)c) \geq I(\alpha b + (1 - \alpha)c)$, then I is $\mathcal{E}$ -cominimum additive.*

Proof. See the Appendix. □

As mentioned above, the following lemma is a generalization of Kajii, Kojima, and Ui (2007) into the case of an infinite state space Ω .

¹³To be precise, the notion of being simple-complete should be introduced. See Section 5 (Definition 5 and Theorem 3) for details.

Lemma 2. Let $\mathcal{E} = \{E_1, E_2, \dots, E_n\} \subseteq \mathcal{F}$ with $|E_i| \geq 2$ for $i = 1, \dots, n$ be a collection of events, and assume that $\mathcal{E} \cup \mathcal{F}_1$ is complete. Let $I : \Phi_\Sigma \rightarrow \mathbb{R}$ be the Choquet integral with respect to a capacity v on (Ω, Σ) . Then, the following two conditions are equivalent:

- (i) I is $\mathcal{E}$ -cominimum additive.
- (ii) There exist a finitely additive measure μ and a set of coefficients $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ such that for every $a \in \Phi_\Sigma$,

$$I(a) = \int_{\Omega} a(\omega) d\mu(\omega) + \sum_{i=1}^n \varepsilon_i \min_{\omega \in E_i} a(\omega). \quad (3)$$

Note that $\operatorname{argmin}_{E_i} a(\omega) \neq \emptyset$ for every E_i since $a(\omega)$ is a finite step function.

Proof. (ii) $\Rightarrow$ (i): Suppose that $a, b \in \Phi_\Sigma$ are $\mathcal{E}$ -cominimum. Then, for every $E_i \in \mathcal{E}$, $\min_{\omega \in E_i} (a(\omega) + b(\omega)) = \min_{\omega \in E_i} a(\omega) + \min_{\omega \in E_i} b(\omega)$ since $\operatorname{argmin}_{\omega \in E_i} a(\omega) \cap \operatorname{argmin}_{\omega \in E_i} b(\omega) \neq \emptyset$. Thus, it follows that

$$\begin{aligned} & I(a + b) \\ &= \int_{\Omega} (a(\omega) + b(\omega)) d\mu(\omega) + \sum_{i=1}^n \varepsilon_i \min_{\omega \in E_i} (a(\omega) + b(\omega)) \\ &= \int_{\Omega} a(\omega) d\mu(\omega) + \int_{\Omega} b(\omega) d\mu(\omega) + \sum_{i=1}^n \varepsilon_i \min_{\omega \in E_i} a(\omega) + \sum_{i=1}^n \varepsilon_i \min_{\omega \in E_i} b(\omega) \\ &= I(a) + I(b), \end{aligned}$$

which shows that I is $\mathcal{E}$ -cominimum additive.

(i) $\Rightarrow$ (ii): Only an outline of the proof is provided; the complete proof is provided in the Appendix. Let Ω be an arbitrary infinite state space. Let $\mathcal{E} = \{E_1, E_2, \dots, E_n\}$ with $|E_i| \geq 2$ for $i = 1, \dots, n$, and assume that $\mathcal{E} \cup \mathcal{F}_1$ is complete. Let $\mathcal{P}$ be a partition of Ω . Moreover, let $\Omega_{\mathcal{P}, \mathcal{E}}$ be the partition generated by $\mathcal{P}$ and $\mathcal{E}$. Choose one element arbitrarily from each set in $\Omega_{\mathcal{P}, \mathcal{E}}$. Denote by $\Omega_{\mathcal{P}, \mathcal{E}}^*$ the finite set of these elements. Since we assume that $\mathcal{E} \cup \mathcal{F}_1$ is complete, we can adopt Kajii, Kojima, and Ui's (2007) result that is proved in the framework of a finite state space. Then, expression (3) is obtained with respect to each partition $\mathcal{P}$. In order to complete the proof, we need to show that (3) does not depend on the choice of partitions. $\square$

Proof of Theorem 2. (if part) A1, A3, A4, and A5 hold by Theorem 1 since $J(f)$ is the CEU with respect to capacity v . We prove only Axioms 6. Denote $J(f) = \int_{\Omega} u(f(\omega)) d\mu(\omega) + \sum_{i=1}^n \varepsilon_i \min_{\omega \in E_i} u(f(\omega))$. Suppose that $f, g, h \in L_0$ are pairwise $\mathcal{E}$ -cominimum, and that $f \succeq g$, that is, $J(f) \geq J(g)$. Note that $\min_{\omega \in E_i} u(\alpha f(\omega) + (1-\alpha)h(\omega)) = \min_{\omega \in E_i} \{u(\alpha f(\omega)) +$

$u((1 - \alpha)h(\omega))\} = \min_{\omega \in E_i} u(\alpha f(\omega)) + \min_{\omega \in E_i} u((1 - \alpha)h(\omega))$ since f and h are $\mathcal{E}$ -cominimum and u is an affine function. Then, it follows that

$$\begin{aligned}
& J(\alpha f + (1 - \alpha)h) - J(\alpha g + (1 - \alpha)h) \\
&= \int_{\Omega} u(\alpha f(\omega) + (1 - \alpha)h(\omega)) d\mu(\omega) + \sum_{i=1}^n \varepsilon_i \min_{\omega \in E_i} u(\alpha f(\omega) + (1 - \alpha)h(\omega)) \\
&\quad - \int_{\Omega} u(\alpha g(\omega) + (1 - \alpha)h(\omega)) d\mu(\omega) - \sum_{i=1}^n \varepsilon_i \min_{\omega \in E_i} u(\alpha g(\omega) + (1 - \alpha)h(\omega)) \\
&= \int_{\Omega} u(\alpha f(\omega)) d\mu(\omega) + \sum_{i=1}^n \varepsilon_i \min_{\omega \in E_i} u(\alpha f(\omega)) - \int_{\Omega} u(\alpha g(\omega)) d\mu(\omega) - \sum_{i=1}^n \varepsilon_i \min_{\omega \in \Omega} u(\alpha g(\omega)) \\
&= \alpha(J(f) - J(g)) \geq 0.
\end{aligned}$$

Thus, Axiom 6 holds.

(only if part) Note that Axiom 6 implies Axiom 2. Therefore, we can apply Theorem 1. By the proof of Theorem 1, we can assume the following properties:

- (1) There exist $y^*, y_* \in L_c$ such that $y^* \succ y_*$ and there exists an affine function $u : L_c \rightarrow \mathbb{R}$ that represents the decision maker's preference on L_c such that $u(y^*) = 1$ and $u(y_*) = -1$.
- (2) There exists a function $J : L_0 \rightarrow \mathbb{R}$ that represents the decision maker's preference on L_0 and coincides with u on L_c .
- (3) We can construct an operator $I : B_0(K) \rightarrow \mathbb{R}$.

Let $K \equiv u(Y)$. By property (1), K is a convex subset of the real line including interval $[-1, 1]$. Let $B_0(K)$ denote the set of all Σ -measurable, K -valued finite step functions on Ω , and define $U : L_0 \rightarrow B_0(K)$ as follows: for $f \in L_0$,

$$(\forall \omega \in \Omega) \quad U(f)(\omega) = u(f(\omega)).$$

Then, function U is onto. If $U(f) = U(g)$, then $f \sim g$ by Axiom 4, which implies that $J(f) = J(g)$. Furthermore, $(\forall \alpha \in [0, 1])(\forall f, g \in L_0) \quad U(\alpha f + (1 - \alpha)g) = \alpha U(f) + (1 - \alpha)U(g)$. Now, we define an operator $I : B_0(K) \rightarrow \mathbb{R}$ by $I(a) = J(f)$ such that $U(f) = a$ for all $a \in B_0(K)$. Then, I is well-defined, and I is the Choquet integral with respect to capacity $v(T) = I(1_T)$.

Now, we prove the following property:

- (4) I is $\mathcal{E}$ -cominimum additive.

To prove this, it suffices to show the condition in Lemma 1. Suppose that f, g, h are pairwise $\mathcal{E}$ -cominimum, and let $a = U(f), b = U(g), c = U(h)$. Then, a, b, c are pairwise $\mathcal{E}$ -cominimum functions on $B_0(K)$. Now, suppose that $I(a) \geq I(b)$. Then, $f \succeq g$ since $J(f) \geq J(g)$. Thus, $J(\alpha f + (1 - \alpha)h) \geq J(\alpha g + (1 - \alpha)h)$ by Axiom 6. Note that $U(\alpha f + (1 - \alpha)h) = \alpha a + (1 - \alpha)c$ and $U(\alpha g + (1 - \alpha)h) = \alpha b + (1 - \alpha)c$ by the affinity of

u . Therefore, $I(\alpha a + (1 - \alpha)c) \geq I(\alpha b + (1 - \alpha)c)$. Hence, it holds that $I(a) \geq I(b)$ implies $I(\alpha a + (1 - \alpha)c) \geq I(\alpha b + (1 - \alpha)c)$. By Lemma 1, it is shown that I is $\mathcal{E}$ -cominimum additive. Therefore, it follows from Lemma 2 that I can be represented by the expression (3) in Lemma 2. By replacing a with $u(f)$ and $I(a)$ with $J(f)$ the desired representation is obtained. $\square$

5. Uncertainty Aversion and Totally Monotone Games

In this section, based on the following axiom (Axiom 7), that is, *uncertainty aversion*, proposed by Gilboa and Schmeidler (1989), we provide an axiomatization theorem of CEU in which all coefficients $\varepsilon_1, \dots, \varepsilon_n$ take non-negative values. This enables us to represent the decision maker's preferences by the *core* of a capacity that is mentioned below in detail. For that purpose, in addition to Axiom 7, the following restriction on a collection $\mathcal{E}$ is needed.

Definition 5. A collection $\mathcal{E} = \{E_1, E_2, \dots, E_n\} \subseteq \mathcal{F}$ with $|E_i| \geq 2$ for $i = 1, \dots, n$ is said to be *simple-complete* if $\mathcal{E}$ satisfies the following two conditions: (i) $\mathcal{E} \cup \mathcal{F}_1$ is complete, (ii) for every $E_i \in \mathcal{E}$, there exist two distinct elements $\omega_1, \omega_2 \in E_i$ such that there is no $E_j \in \mathcal{E}$ with $\{\omega_1, \omega_2\} \subseteq E_j \subsetneq E_i$.

To provide an axiomatic foundation for a rational decision maker whose beliefs are captured by a totally monotone game, we have to impose the simple-completeness on the collection $\mathcal{E}$ that is a stronger condition than completeness. However, as is clear from the following examples, the notion of being simple-complete is not so restrictive.

Example 1. Let $\mathcal{E} = \{\{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$. Then, this collection is simple-complete.

Example 2. Any partition of Ω is simple-complete, and so is $\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \dots, \{n-1, n\}\}$. More generally, for $\mathcal{E} = \{E_1, E_2, \dots, E_n\}$ with $|E_i| \geq 2$ for $i = 1, \dots, n$, if every $E, F \in \mathcal{E}$ with $E \neq F$ satisfy both $E \not\subseteq F$ and $F \not\subseteq E$, then $\mathcal{E}$ is simple-complete.

The following axiom is one of the key axioms to derive our representation theorem. This axiom was first proposed by Gilboa and Schmeidler (1989).

Axiom 7 (Uncertainty Aversion). For all $f, g \in L_0$ and all $\alpha \in (0, 1)$, $f \sim g$ implies $\alpha f + (1 - \alpha)g \succeq f$.

Gilboa and Schmeidler (1989) show that a binary relation $\succeq$ on L_0 satisfies A1, A3, A4, A5, and A7, and *Certainty-Independence*¹⁴ if and only if there exist a closed

¹⁴Certainty-Independence states that for all $f, g \in L_0$, all $h \in L_c$, and all $\alpha \in [0, 1]$, $f \succ g \Leftrightarrow \alpha f + (1 - \alpha)h \succ \alpha g + (1 - \alpha)h$.

and convex set of probability measures on (Ω, Σ) , $\mathcal{P} \subseteq \mathcal{M}$, and a non-constant function $u : X \rightarrow \mathbb{R}$ such that for all $f, g \in L_0$, $f \succeq g$ if and only if $\min_{p \in \mathcal{P}} \{\int_{\Omega} u(f(\omega)) dp(\omega)\} \geq \min_{p \in \mathcal{P}} \{\int_{\Omega} u(g(\omega)) dp(\omega)\}$, where $\mathcal{M}$ is the set of all probability measures on (Ω, Σ) . Furthermore, $\mathcal{P}$ is unique and u is unique up to positive linear transformations. This representation is called MEU.

The convexity of a game v serves as the bridge between CEU and MEU. Schmeidler (1986) shows that a game v is convex if and only if (i) $\text{core}(v) \neq \emptyset$ and (ii) for all bounded, real-valued, Σ -measurable functions on Ω , $\int_{\Omega} f(\omega) dv(\omega) = \min \{\int_{\Omega} f(\omega) dp(\omega) \mid p \in \text{core}(v)\}$. Using the above, Schmeidler (1989) shows that if a game v is convex, then the CEU with respect to v is equal to the MEU with the $\text{core}(v)$ as the decision maker's set of priors. Note that the core of v is defined by

$$\text{core}(v) \equiv \{p \mid p(\Omega) = v(\Omega), p \text{ is a probability measure on } \Sigma \text{ and } p(A) \geq v(A) \text{ for all } A \in \Sigma\}.$$

Schmeidler (1989) also shows that A7 is equivalent to the convexity of a capacity.

Then, our result in this section is in order.

Theorem 3. *Let $\mathcal{E} = \{E_1, E_2, \dots, E_n\} \subseteq \mathcal{F}$ with $|E_i| \geq 2$ for $i = 1, \dots, n$ be a collection of events, and assume that $\mathcal{E}$ is simple-complete. A binary relation $\succeq$ defined on L_0 satisfies A1, A3, A4, A5, A6, and A7 if and only if there exist a unique capacity v on (Ω, Σ) , a unique finitely additive measure μ on (Ω, Σ) , an affine function $u : X \rightarrow \mathbb{R}$, and a set of coefficients $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$, such that for all f and g in L_0 ,*

$$J(f) = \int_{\Omega} u(f(\omega)) d\mu(\omega) + \sum_{i=1}^n \varepsilon_i \min_{\omega \in E_i} u(f(\omega)) \quad (4)$$

is the CEU with respect to capacity v satisfying $f \succeq g \Leftrightarrow J(f) \geq J(g)$ where coefficients $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ are all non-negative.

Note that all coefficients $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ take non-negative values in Theorem 3, which can be accomplished by assuming Axiom 7 together with $\mathcal{E}$ being simple-complete. This representation theorem provides an axiomatization of a totally monotone game v that can be written as $v = \sum_T \beta_T u_T$ where all β_T are non-negative. Collection $\{\beta_T\}_{T \in \mathcal{F}}$ is called the *Möbius inversion* of v . For the definitions of totally monotone games and Möbius inversions, see Appendix A. It should be emphasized that not all totally monotone games are axiomatized by Theorem 3 but only a subset of the set of all totally monotone games is. This restrictive result comes from the fact that $\mathcal{E}$ must be simple-complete.

The relation between capacity v 's convexity and collection $\mathcal{E}$'s simple-completeness should be mentioned. First, the convexity of v (equivalently, Axiom 7) and the simple-completeness of $\mathcal{E}$ imply that capacity v is a totally monotone game written by $v =$

$\sum_T \beta_T u_T$ where all coefficients β_T are non-negative, which also implies that all coefficients ε_i for all $i = 1, \dots, n$ are non-negative. If the non-negativity of the Möbius inversions, β_T , holds true, then the additivity of the core follows from Strassen (1964), as we mention below. This property does not follow only from the convexity of capacities. Some additional condition—the simple-completeness of $\mathcal{E}$ —should be imposed. If the simple-completeness of $\mathcal{E}$ together with the above axioms are assumed, then the decision maker's beliefs are captured by a totally monotone game $v = \sum_T \beta_T u_T$ where all coefficients β_T are non-negative, which implies that $\text{core}(v) = \sum_T \beta_T \text{core}(u_T)$, as shown by Strassen (1964). This additivity of the core enables us not only to represent the decision maker's preferences by the min-operator but also to represent the decision maker's beliefs by the core explicitly. For that purpose, Axioms 1, 3, 4, 5, 6, and 7 together with $\mathcal{E}$ being simple-complete should be imposed.

Proof. (if part) It suffices to prove A7. Assume that $f \sim g$. Then,

$$\int_{\Omega} u(f) d\mu(\omega) + \sum_{i=1}^n \varepsilon_i \min_{\omega \in E_i} u(f(\omega)) = \int_{\Omega} u(g) d\mu(\omega) + \sum_{i=1}^n \varepsilon_i \min_{\omega \in E_i} u(g(\omega)).$$

It follows that for any $\alpha \in (0, 1)$,

$$\begin{aligned} & \int_{\Omega} u(\alpha f + (1 - \alpha)g) d\mu(\omega) + \sum_{i=1}^n \varepsilon_i \min_{\omega \in E_i} u(\alpha f + (1 - \alpha)g) \\ & \geq \alpha \int_{\Omega} u(f) d\mu + (1 - \alpha) \int_{\Omega} u(g) d\mu(\omega) + \alpha \sum_{i=1}^n \varepsilon_i \min_{\omega \in E_i} u(f) + (1 - \alpha) \sum_{i=1}^n \varepsilon_i \min_{\omega \in E_i} u(g) \\ & = \int_{\Omega} u(f) d\mu(\omega) + \sum_{i=1}^n \varepsilon_i \min_{\omega \in E_i} u(f(\omega)). \end{aligned}$$

The inequity holds since $\varepsilon_i \geq 0$ for all $i = 1, \dots, n$, and u is affine. Thus, it holds that $\alpha f + (1 - \alpha)g \succeq f$.

(only if part) See the Appendix. □

Finally, as a corollary of Theorem 3, we provide a generalization of Eichberger and Kelsey (1999) that axiomatize the *E-capacity expected utility*. The E-capacity expected utility states that if a set of axioms are satisfied, then the decision maker's beliefs are captured by an E-capacity v and her preferences are represented by the Choquet integral with respect to capacity v .¹⁵ The ε -contamination is included as a special case as the

¹⁵Let $\{E_1, \dots, E_n\}$ be a partition of Ω . Let $\mathcal{E} = \{E_1, \dots, E_n\}$. A capacity $v \in \mathbb{R}^{\mathcal{F}}$ is called an E-capacity if there exist a finitely additive probability measure π , a real number $\varepsilon \in [0, 1]$, and a finitely additive probability measure ρ on $\mathcal{E}$ such that $v = (1 - \varepsilon)\pi + \varepsilon \sum_{i=1}^n \rho(E_i)u_{E_i}$, where u_{E_i} denotes the unanimity game on E_i for each i .

E-capacity expected utility.¹⁶ For a state space Ω and a collection of subsets of Ω , $\mathcal{E}$, while Eichberger and Kelsey (1999) assume that Ω is finite and $\mathcal{E}$ is a partition of Ω , this paper assumes that Ω is infinite and $\mathcal{E}$ is simple-complete. As pointed out by Example 2, any partition of Ω is simple-complete. Therefore, our framework in this paper is more general than that of Eichberger and Kelsey (1999).

Let $\Delta(\Omega)$ be the set of all finitely additive probability measures and let Π_E be the set of finitely additive probability measures that assign probability measure one to an event $E \in \mathcal{F}$: $\Pi_E \equiv \{p \in \Delta(\Omega) \mid p(E) = 1\}$.

Corollary 1. *Let $\mathcal{E} = \{E_1, E_2, \dots, E_n\} \subseteq \mathcal{F}$ with $|E_i| \geq 2$ for $i = 1, \dots, n$ be a collection of events, and assume that $\mathcal{E}$ is simple-complete. A binary relation $\succeq$ defined on L_0 satisfies A1, A3, A4, A5, A6, and A7 if and only if there exist a unique finitely additive probability measure π , numbers $\rho_1, \rho_2, \dots, \rho_n \in [0, 1]$ with $\sum_{i=1}^n \rho_i \leq 1$, and an affine function u such that*

$$f \succeq g \Leftrightarrow H(f) \geq H(g),$$

where $H(f) = \min\{\int u(f)dq \mid q = (1 - \sum_{i=1}^n \rho_i)\pi + \sum_{i=1}^n \rho_i p_i, p_i \in \Pi_{E_i}\}$.

This corollary implies that our axiomatization theorem within CEU can be related to MEU through the core of a capacity. In addition to the existence of the core, our corollary clarifies how the core can be represented. That is, for capacity v derived in Theorem 3, the core of v can be written as follows:

$$\text{core}(v) = \left\{ q \in \Delta(\Omega) \mid q = (1 - \sum_{i=1}^n \rho_i)\pi + \sum_{i=1}^n \rho_i p_i, p_i \in \Pi_{E_i} \right\}.$$

This task is accomplished by assuming Axiom 7 together with $\mathcal{E}$ being simple-complete. As in Schmeidler (1989), Axiom 7 only guarantees the non-emptiness of the core. We prove this corollary directly, but it can also be shown by adopting Strassen (1964).

Proof. It suffices to show that $J(f) \geq J(g) \Leftrightarrow H(f) \geq H(g)$. By Theorem 3, there exist a unique finitely additive measure μ on (Ω, Σ) , an affine function u , and a set of non-negative coefficients $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$, such that

$$\begin{aligned} f \succeq g &\Leftrightarrow J(f) \geq J(g) \\ &\Leftrightarrow \int_{\Omega} u(f(\omega))d\mu(\omega) + \sum_{i=1}^n \varepsilon_i \min_{\omega \in E_i} u(f(\omega)) \geq \int_{\Omega} u(g(\omega))d\mu(\omega) + \sum_{i=1}^n \varepsilon_i \min_{\omega \in E_i} u(g(\omega)). \end{aligned}$$

¹⁶Let $\mathcal{M}(\Omega)$ be the set of all probability measures and let $\varepsilon \in [0, 1]$. Then, the set of probability measures defined by $\{(1 - \varepsilon)p + \varepsilon q \mid q \in \mathcal{M}(\Omega)\}$ is called the ε -contamination of p , where p is the true probability measure. Our paper derives a set of coefficients $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n$ endogenously.

Let us define $\zeta \equiv \mu(\Omega) + \sum_{i=1}^n \varepsilon_i$, and define $\pi \equiv \mu/\mu(\Omega)$. Then, π is a finitely additive probability measure on (Ω, Σ) . Thus,

$$\begin{aligned} J(f) \geq J(g) &\Leftrightarrow (\mu(\Omega)/\zeta) \int_{\Omega} u(f(\omega)) d\pi + \sum_{i=1}^n (\varepsilon_i/\zeta) \min_{\omega \in E_i} u(f(\omega)) \\ &\geq (\mu(\Omega)/\zeta) \int_{\Omega} u(g(\omega)) d\pi + \sum_{i=1}^n (\varepsilon_i/\zeta) \min_{\omega \in E_i} u(g(\omega)). \end{aligned}$$

Since $\zeta = \mu(\Omega) + \sum_{i=1}^n \varepsilon_i$, $1 - \sum_{i=1}^n \rho_i = \mu(\Omega)/\zeta$, where $\rho_i \equiv \varepsilon_i/\zeta$. Therefore,

$$\begin{aligned} J(f) \geq J(g) &\Leftrightarrow (1 - \sum_{i=1}^n \rho_i) \int_{\Omega} u(f(\omega)) d\pi + \sum_{i=1}^n \rho_i \min_{\omega \in E_i} u(f(\omega)) \\ &\geq (1 - \sum_{i=1}^n \rho_i) \int_{\Omega} u(g(\omega)) d\pi + \sum_{i=1}^n \rho_i \min_{\omega \in E_i} u(g(\omega)). \end{aligned}$$

Now, we have

$$\begin{aligned} &(1 - \sum_{i=1}^n \rho_i) \int_{\Omega} u(f(\omega)) d\pi + \sum_{i=1}^n \rho_i \min_{\omega \in E_i} u(f(\omega)) \\ &= \min \left\{ \int u(f) dq \mid q = (1 - \sum_{i=1}^n \rho_i) \pi + \sum_{i=1}^n \rho_i p_i, p_i \in \Pi_{E_i} \right\}. \end{aligned}$$

Indeed, take $p'_i = \delta_{x_i} \in \Pi_{E_i}$ for any $x_i \in \arg \min_{\omega \in E_i} u(f(\omega))$ ($i = 1, 2, \dots, n$), and put $q' = (1 - \sum_{i=1}^n \rho_i) \pi + \sum_{i=1}^n \rho_i p'_i$. Then,

$$\int u(f) dq' = (1 - \sum_{i=1}^n \rho_i) \int_{\Omega} u(f(\omega)) d\pi + \sum_{i=1}^n \rho_i \min_{\omega \in E_i} u(f(\omega)).$$

Moreover, for all $q = (1 - \sum_{i=1}^n \rho_i) \pi + \sum_{i=1}^n \rho_i p_i$ with $p_i \in \Pi_{E_i}$, it holds that

$$\begin{aligned} &\int u(f) dq \\ &= (1 - \sum_{i=1}^n \rho_i) \int u(f) d\pi + \sum_{i=1}^n \rho_i \int u(f) dp_i \\ &\geq (1 - \sum_{i=1}^n \rho_i) \int_{\Omega} u(f(\omega)) d\pi + \sum_{i=1}^n \rho_i \min_{\omega \in E_i} u(f(\omega)), \end{aligned}$$

which completes the proof. $\square$

6. Conclusion

The aim of this paper is to axiomatize the decision maker's rational behaviors within the framework of the Choquet Expected Utility (henceforth, CEU) theory by proposing the $\mathcal{E}$ -cominimum independence axiom that is stronger than the comonotonic independence axiom but weaker than the independence axiom. In order to achieve this aim, we generalize Kajii, Kojima, and Ui (2007) that characterize cominimum additive operators in terms of Choquet integrals into an infinite state space. Furthermore, we show that our axiomatization theorem within CEU can be related to MEU through the core of a capacity. In addition to the existence of the core, our corollary clarifies how the core can be represented. This task is accomplished by assuming Axiom 7 (Uncertainty Aversion) together with $\mathcal{E}$ being simple-complete. Our axiomatization theorem can also be interpreted as an axiomatization theorem of a totally monotone game, which enables us to derive Gilboa (1989) and Eichberger and Kelsey (1999) as a corollary.

However, some tasks are yet to be accomplished. It should be emphasized that not all totally monotone games are axiomatized by Theorem 3 but only a subset of the set of all totally monotone games is. This restrictive result comes from the fact that $\mathcal{E}$ must be simple-complete. An axiomatization of all totally monotone games is a topic for future research.

Appendices

Appendix A

As defined in Section 2, Ω is a nonempty finite or infinite set, Σ is a nonempty algebra of subsets of Ω , $\mathbb{R}^\Omega = \{x \mid x : \Omega \rightarrow \mathbb{R}\}$ denotes the set of all real valued functions on Ω , and $\mathcal{F}$ is the collection of all non-empty subsets of Ω .

A set function $v : 2^\Omega \rightarrow \mathbb{R}$ with $v(\emptyset) = 0$ is *k-monotone* for $k \geq 2$ if $v(\cup_{i=1}^k A_i) \geq \sum_{\{I: \emptyset \neq I \subset \{1, \dots, k\}\}} (-1)^{|I|+1} v(\cap_{i \in I} A_i)$ for all $A_1, \dots, A_k \in 2^\Omega$, it is a *capacity* if $v(E) \leq v(F)$ for all $E \subseteq F$ and $v(\Omega) = 1$, and it is *totally monotone* if it is non-negative and *k-monotone* for all $k \geq 2$. A totally monotone game v with $v(\Omega) = 1$ is called a *belief function*.

Let Ω be finite. For $T \in \mathcal{F}$, let $u_T \in \mathbb{R}^\mathcal{F}$ be the *unanimity game* on T that is defined by the following: $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise. The following lemma has been proved by Shapley (1953).

Lemma 3 (Shapley (1953)). *Let Ω be finite. Collection $\{u_T\}_{T \in \mathcal{F}}$ is a linear base for $\mathbb{R}^\mathcal{F}$. The unique collection of coefficients $\{\beta_T\}_{T \in \mathcal{F}}$ satisfying the form*

$$v = \sum_{T \in \mathcal{F}} \beta_T u_T, \quad (5)$$

or equivalently $v(E) = \sum_{T \subseteq E} \beta_T$ for all $E \in \mathcal{F}$, is provided by $\beta_T = \sum_{E \subseteq T} (-1)^{|T|-|E|} v(E)$.

Collection $\{\beta_T\}_{T \in \mathcal{F}}$ is called the *Möbius inversion* of v . A totally monotone game v can be characterized by coefficients β_T for all $T \in \mathcal{F}$. The following lemma is shown by Shafer (1976).

Lemma 4 (Shafer (1976)). *For any $v \in \mathbb{R}^\mathcal{F}$, v is totally monotone if and only if in unique representation (5), β_T is non-negative for all $T \in \mathcal{F}$.*

Gilboa and Schmeidler (1994) prove the following lemma with respect to the additivity of Choquet integrals through Möbius inversions.

Lemma 5. *Let $x \in \mathbb{R}^\Omega$ and let $v = \sum_{T \in \mathcal{F}} \beta_T u_T \in \mathbb{R}^\mathcal{F}$. Then,*

$$\int x dv = \sum_{T \in \mathcal{F}} \beta_T \int x du_T = \sum_{T \in \mathcal{F}} \beta_T \min_T x.$$

The following lemma is a special case of Proposition 3 in Chateauneuf and Jaffray (1987).¹⁷

¹⁷We are grateful to an anonymous referee who pointed out that Lemma 6 is very similar to a result in Chateauneuf and Jaffray (1987).

Lemma 6. Let $v = \sum_{T \in \mathcal{F}} \beta_T u_T$. Then, for all $X, Y \in \mathcal{F}$,

$$v(X \cup Y) + v(X \cap Y) - v(X) - v(Y) = \sum_{T \in \mathcal{F}_{X,Y}} \beta_T,$$

where $\mathcal{F}_{X,Y} \equiv \{T \mid T \cap (X \cap Y^c) \neq \emptyset, T \cap (X^c \cap Y) \neq \emptyset, \text{ and } T \subseteq X \cup Y\}$.

Kajii, Kojima, and Ui (2007) characterize the Möbius inversion of a game v by v 's $\mathcal{E}$ -cominimum additivity.

Theorem 4 (Kajii, Kojima, and Ui (2007)). Let Ω be finite. Let $v = \sum_{T \in \mathcal{F}} \beta_T u_T \in \mathbb{R}^{\mathcal{F}}$ be a game. v is $\mathcal{E}$ -cominimum additive if and only if $\beta_T = 0$ for any $T \notin \Upsilon(\mathcal{E})$. If $\mathcal{E}$ is complete, that is, $\mathcal{E} = \Upsilon(\mathcal{E})$, v is $\mathcal{E}$ -cominimum additive if and only if $\beta_T = 0$ for any $T \notin \mathcal{E}$.

Appendix B: Proofs

Proof of Lemma 1

Proof. Suppose that for all pairwise $\mathcal{E}$ -cominimum functions $a, b, c \in \Phi_{\Sigma}$ and for all $\alpha \in (0, 1)$, $I(a) \geq I(b)$ implies $I(\alpha a + (1 - \alpha)c) \geq I(\alpha b + (1 - \alpha)c)$. First, let us prove the following claim: if $x, y \in \Phi_{\Sigma}$ are $\mathcal{E}$ -cominimum and $\alpha \in (0, 1)$, then $I(\alpha x + (1 - \alpha)y) = \alpha I(x) + (1 - \alpha)I(y)$. Indeed, for any $\varepsilon > 0$, $(I(x) + \varepsilon)1_{\Omega}$ satisfies $I((I(x) + \varepsilon)1_{\Omega}) > I(x)$ and $(I(y) + \varepsilon)1_{\Omega}$ satisfies $I((I(y) + \varepsilon)1_{\Omega}) > I(y)$ by the assumption that $I(\lambda 1_{\Omega}) = \lambda$. Hence, $\alpha I(x) + (1 - \alpha)I(y) + \varepsilon = I(\alpha(I(x) + \varepsilon)1_{\Omega} + (1 - \alpha)(I(y) + \varepsilon)1_{\Omega}) > I(\alpha x + (1 - \alpha)(I(y) + \varepsilon)1_{\Omega}) > I(\alpha x + (1 - \alpha)y)$. The first inequality holds since $(I(x) + \varepsilon)1_{\Omega}$, x , and $(I(y) + \varepsilon)1_{\Omega}$ are pairwise $\mathcal{E}$ -cominimum and the second inequality holds since $(I(y) + \varepsilon)1_{\Omega}$, y , and x are pairwise $\mathcal{E}$ -cominimum. Since ε is any positive number, we obtain that $\alpha I(x) + (1 - \alpha)I(y) \geq I(\alpha x + (1 - \alpha)y)$. Furthermore, the converse inequality can be shown by using a similar argument as that for $\varepsilon < 0$. Therefore, it is proved that $I(\alpha x + (1 - \alpha)y) = \alpha I(x) + (1 - \alpha)I(y)$. Then, our claim is proved. Next, let us use this claim twice. First, let $\alpha = 1/2$, $x = 2a$, and $y = 0$. Then, $I(a) = (1/2)I(2a)$ for all $a \in \Phi_{\Sigma}$. Similarly, let $\alpha = 1/2$, $y = 2b$, and $x = 0$. Then, $I(b) = (1/2)I(2b)$ for all $b \in \Phi_{\Sigma}$. Second, let $\alpha = 1/2$, $x = 2a$, and $y = 2b$. Then, for all $a, b \in \Phi_{\Sigma}$, $I(a + b) = (1/2)I(2a) + (1/2)I(2b) = I(a) + I(b)$. Now we show Lemma 1. $\square$

Proof of Lemma 2

Proof. Since the proof for (ii) $\Rightarrow$ (i) has been already provided in Section 3, we prove the converse.

(i) $\Rightarrow$ (ii). Let Ω be an arbitrary infinite set and $\mathcal{E} = \{E_1, E_2, \dots, E_n\} \subseteq \mathcal{F}$ with $|E_i| \geq 2$

for $i = 1, \dots, n$ be a collection of events. Denote by Π the set of all finite partitions of Ω and by $\Pi_{\mathcal{E}}$ the set of all finite partitions of Ω , which separate each E_i for $i = 1, \dots, n$ to at least two nonempty subsets. That is, if $\mathcal{P} = \{P_1, P_2, \dots, P_k\} \in \Pi_{\mathcal{E}}$ is a partition of Ω , then for every E_i for $i = 1, \dots, n$, there are at least two nonempty sets in $P_1 \cap E_i, P_2 \cap E_i, \dots, P_k \cap E_i$. Fix a partition $\mathcal{P} = \{P_1, P_2, \dots, P_k\} \in \Pi_{\mathcal{E}}$. Let $\sigma(\mathcal{P}, \mathcal{E}) \subseteq \Sigma$ be the algebra generated by $\mathcal{P}$ and $\mathcal{E}$: the smallest algebra containing $P_1, P_2, \dots, P_k, E_1, E_2, \dots, E_n$. Denote by $\Phi_{\sigma(\mathcal{P}, \mathcal{E})}$ the set of all $\sigma(\mathcal{P}, \mathcal{E})$ -measurable functions in Φ_{Σ} . Note that $\bigcup_{\mathcal{P} \in \Pi_{\mathcal{E}}} \Phi_{\sigma(\mathcal{P}, \mathcal{E})} = \bigcup_{\mathcal{P} \in \Pi} \Phi_{\sigma(\mathcal{P}, \mathcal{E})} = \Phi_{\Sigma}$. Let $\Omega_{\mathcal{P}, \mathcal{E}}$ be the collection of minimal elements of $\sigma(\mathcal{P}, \mathcal{E})$, which constitutes a well defined partition of Ω . Note that $\Omega_{\mathcal{P}, \mathcal{E}}$ is a collection of the subsets of Ω , not a set of states. We explain this set $\Omega_{\mathcal{P}, \mathcal{E}}$ in detail. Let $E_1^i, E_2^i, \dots, E_{m_i}^i$ be nonempty sets in $\sigma(\mathcal{P}, \mathcal{E})$ that are subsets of E_i , where $m_i \geq 2$. That is, $\{E_1^i, E_2^i, \dots, E_{m_i}^i\}$ constitutes a partition of E_i for $i = 1, \dots, n$. Moreover, denote the nonempty sets in $\Omega_{\mathcal{P}, \mathcal{E}}$, which are in $(\bigcup_{1 \leq i \leq n} E_i)^c$, by $Q_1, Q_2, \dots, Q_m$. Thus, the collection $\{E_1^1, E_2^1, \dots, E_{m_1}^1, \dots, E_1^n, E_2^n, \dots, E_{m_n}^n, Q_1, Q_2, \dots, Q_m\}$ constitutes $\Omega_{\mathcal{P}, \mathcal{E}}$. Choose one element e_j^i from each E_j^i for $i = 1, \dots, n$ and $j = 1, \dots, m_i$ arbitrarily, and choose q_i from each Q_i for $i = 1, \dots, m$ arbitrarily. Denote finite set $\{e_j^i | 1 \leq i \leq n, 1 \leq j \leq m_i\} \cup \{q_i | 1 \leq i \leq m\}$ by $\Omega_{\mathcal{P}, \mathcal{E}}^*$, finite set $\{e_j^i | 1 \leq j \leq m_i\}$ by E_i^* for each $i = 1, \dots, n$, and finite set $\{q_i | 1 \leq i \leq m\}$ by Q^* . Thus, $\Omega_{\mathcal{P}, \mathcal{E}}^* = E_1^* \cup \dots \cup E_n^* \cup Q^*$. Note that by construction, $\Omega_{\mathcal{P}, \mathcal{E}}^*$ is a subset of Ω . Denote by $\mathcal{E}^*$ collection $\{E_1^*, E_2^*, \dots, E_n^*\}$ and by $\mathcal{F}_1^*$ the collection of singleton subsets in $2^{\Omega_{\mathcal{P}, \mathcal{E}}^*}$. Moreover, denote by $\Phi_{\Omega_{\mathcal{P}, \mathcal{E}}^*}^*$ the set of real valued functions on finite set $\Omega_{\mathcal{P}, \mathcal{E}}^*$.

Now, we define capacity $v_{\mathcal{P}, \mathcal{E}}^*$ on power set $2^{\Omega_{\mathcal{P}, \mathcal{E}}^*}$ as follows: for every $X \in 2^{\Omega_{\mathcal{P}, \mathcal{E}}^*}$, $v_{\mathcal{P}, \mathcal{E}}^*(X) = v((\bigcup_{e_j^i \in X} E_j^i) \cup (\bigcup_{q_i \in X} Q_i))$. For example, $v_{\mathcal{P}, \mathcal{E}}^*(\emptyset) = 0$, $v_{\mathcal{P}, \mathcal{E}}^*(\{e_2^1, q_3\}) = v(E_2^1 \cup Q_3)$, $v_{\mathcal{P}, \mathcal{E}}^*(E_i^*) = v_{\mathcal{P}, \mathcal{E}}^*(\{e_1^i, \dots, e_{m_i}^i\}) = v(\bigcup_{1 \leq j \leq m_i} E_j^i) = v(E_i)$.

Next, we define a function a^* on $\Omega_{\mathcal{P}, \mathcal{E}}^*$ that corresponds to a function $a \in \Phi_{\sigma(\mathcal{P}, \mathcal{E})}$. For every $\sigma(\mathcal{P}, \mathcal{E})$ -measurable function $a \in \Phi_{\sigma(\mathcal{P}, \mathcal{E})}$, define the function $a^* \in \Phi_{\Omega_{\mathcal{P}, \mathcal{E}}^*}^*$ as follows: for any $\omega \in \Omega_{\mathcal{P}, \mathcal{E}}^*$, $a^*(\omega) = a(\omega)$ naturally; that is, $a^*(e_j^i) = a(e_j^i)$ and $a^*(q_j) = a(q_j)$. The value $a^*(\omega)$ is the same on each E_j^i and on each Q_j regardless of the choice of the representation element, since a is constant on each E_j^i and on each Q_j by the assumption of a being $\sigma(\mathcal{P}, \mathcal{E})$ -measurable.

Define by $I^*(a^*)$ the Choquet integral of a^* with respect to capacity $v_{\mathcal{P}, \mathcal{E}}^*$; that is, $I^*(a^*) \equiv \int_{\Omega_{\mathcal{P}, \mathcal{E}}^*} a^* dv_{\mathcal{P}, \mathcal{E}}^*$. We show that $I^*(a^*) = I(a)$ for all $a \in \Phi_{\sigma(\mathcal{P}, \mathcal{E})}$. Suppose $a^*(\omega_1) \geq a^*(\omega_2) \geq \dots \geq a^*(\omega_t)$. Since a is $\sigma(\mathcal{P}, \mathcal{E})$ -measurable, it follows that

$$\begin{aligned} & I^*(a^*) \\ &= (a^*(\omega_1) - a^*(\omega_2))v_{\mathcal{P}, \mathcal{E}}^*(\{\omega_1\}) + (a^*(\omega_2) - a^*(\omega_3))v_{\mathcal{P}, \mathcal{E}}^*(\{\omega_1, \omega_2\}) \\ & \quad + (a^*(\omega_3) - a^*(\omega_4))v_{\mathcal{P}, \mathcal{E}}^*(\{\omega_1, \omega_2, \omega_3\}) + \dots \end{aligned}$$

$$\begin{aligned}
&= (a(\omega_1) - a(\omega_2))v(\cup_{\omega_1 \in X_1 \in \Omega_{\mathcal{P},\mathcal{E}}} X_1) + (a(\omega_2) - a(\omega_3))v(\cup_{\omega_j \in X_j \in \Omega_{\mathcal{P},\mathcal{E}}, (j=1,2)} X_j) \\
&\quad + (a(\omega_3) - a(\omega_4))v(\cup_{\omega_j \in X_j \in \Omega_{\mathcal{P},\mathcal{E}}, (j=1,2,3)} X_j) + \cdots = I(a).
\end{aligned}$$

Here, we claim that for every $a, b \in \Phi_{\sigma(\mathcal{P},\mathcal{E})}$, the condition that a and b are $\mathcal{E}$ -cominimum on Ω is equivalent to the condition that a^* and b^* are $\mathcal{E}^*$ -cominimum on $\Omega_{\mathcal{P},\mathcal{E}}^*$. Indeed, suppose that a^* and b^* are $\mathcal{E}^*$ -cominimum. Then, there exists an $\omega_i \in \operatorname{argmin}_{\omega \in E_i^*} a^*(\omega) \cap \operatorname{argmin}_{\omega \in E_i^*} b^*(\omega)$ for every $i = 1, \dots, n$. Such an ω_i satisfies $\omega_i \in \operatorname{argmin}_{\omega \in E_i} a(\omega) \cap \operatorname{argmin}_{\omega \in E_i} b(\omega)$ since $a \in \Phi_{\sigma(\mathcal{P},\mathcal{E})}$ is a $\sigma(\mathcal{P},\mathcal{E})$ -measurable function. Thus, a and b are $\mathcal{E}$ -cominimum. The converse also holds similarly. Hence, our claim is shown.

Now, let $a, b \in \Phi_{\sigma(\mathcal{P},\mathcal{E})}$ and suppose that a^* and b^* are $\mathcal{E}^*$ -cominimum. Then, a and b are $\mathcal{E}$ -cominimum, and thus, $I(a+b) = I(a) + I(b)$ since I is $\mathcal{E}$ -cominimum additive. By $I^*(a^*) = I(a)$ for all $a \in \Phi_{\sigma(\mathcal{P},\mathcal{E})}$, it holds that $I^*(a^* + b^*) = I^*(a^*) + I^*(b^*)$ since $(a+b)^* = a^* + b^*$. Thus, it holds that I^* is $\mathcal{E}^*$ -cominimum additive. Note that if $\mathcal{F}_1 \cup \mathcal{E}$ is complete, then $\mathcal{F}_1^* \cup \mathcal{E}^*$ is clearly complete. Since $\Omega_{\mathcal{P},\mathcal{E}}^*$ is a finite set, we can apply Theorem 4. Thus, there exist coefficients $\{\beta_{\{\omega\}}^{\mathcal{P},\mathcal{E}}\}_{\omega \in \Omega_{\mathcal{P},\mathcal{E}}^*}, \{\beta_{E_i^*}^{\mathcal{P},\mathcal{E}}\}_{1 \leq i \leq n}$ such that

$$v_{\mathcal{P},\mathcal{E}}^* = \sum_{\omega \in \Omega_{\mathcal{P},\mathcal{E}}^*} \beta_{\{\omega\}}^{\mathcal{P},\mathcal{E}} u_{\{\omega\}} + \sum_{1 \leq i \leq n} \beta_{E_i^*}^{\mathcal{P},\mathcal{E}} u_{E_i^*}. \quad (6)$$

Note that for all $\omega \in \Omega_{\mathcal{P},\mathcal{E}}^*$, $\beta_{\{\omega\}}^{\mathcal{P},\mathcal{E}} = v_{\mathcal{P},\mathcal{E}}^*(\{\omega\}) \geq 0$ since $v_{\mathcal{P},\mathcal{E}}^*$ is a capacity. Take any $a \in \Phi_{\sigma(\mathcal{P},\mathcal{E})}$. Then, for $a^* \in \Phi_{\Omega_{\mathcal{P},\mathcal{E}}^*}^*$, by Lemma 5, it holds that

$$I^*(a^*) = \int_{\Omega_{\mathcal{P},\mathcal{E}}^*} a^* dv_{\mathcal{P},\mathcal{E}}^* = \sum_{\omega \in \Omega_{\mathcal{P},\mathcal{E}}^*} \beta_{\{\omega\}}^{\mathcal{P},\mathcal{E}} a^*(\omega) + \sum_{1 \leq i \leq n} \beta_{E_i^*}^{\mathcal{P},\mathcal{E}} \min_{\omega \in E_i^*} a^*(\omega). \quad (7)$$

Note that $\min_{\omega \in E_i^*} a^*(\omega) = \min_{\omega \in E_i} a(\omega)$ for all $a \in \Phi_{\sigma(\mathcal{P},\mathcal{E})}$. Thus, we can use notation $\{\beta_{E_i^*}^{\mathcal{P},\mathcal{E}}\}_{1 \leq i \leq n}$ instead of $\{\beta_{E_i^*}^{\mathcal{P},\mathcal{E}}\}_{1 \leq i \leq n}$. Moreover, it holds that $a^*(\omega) = a(\omega)$ for all $\omega \in \Omega_{\mathcal{P},\mathcal{E}}^*$. Thus, by $I^*(a^*) = I(a)$, (6) and (7) can be rewritten as follows:

$$v_{\mathcal{P},\mathcal{E}}^* = \sum_{\omega \in \Omega_{\mathcal{P},\mathcal{E}}^*} \beta_{\{\omega\}}^{\mathcal{P},\mathcal{E}} u_{\{\omega\}} + \sum_{1 \leq i \leq n} \beta_{E_i^*}^{\mathcal{P},\mathcal{E}} u_{E_i^*}; \quad (8)$$

$$I(a) = \int_{\Omega} a dv = \sum_{\omega \in \Omega_{\mathcal{P},\mathcal{E}}^*} \beta_{\{\omega\}}^{\mathcal{P},\mathcal{E}} a(\omega) + \sum_{1 \leq i \leq n} \beta_{E_i^*}^{\mathcal{P},\mathcal{E}} \min_{\omega \in E_i} a(\omega). \quad (9)$$

Now, define a finitely additive measure $\mu_{\mathcal{P},\mathcal{E}}$ on $\sigma(\mathcal{P},\mathcal{E})$ as follows: $\mu_{\mathcal{P},\mathcal{E}}(E_j^i) = \beta_{\{e_j^i\}}^{\mathcal{P},\mathcal{E}}$ for every $E_j^i \in \Omega_{\mathcal{P},\mathcal{E}}$ for $i = 1, \dots, n$ and $j = 1, \dots, m_i$; and $\mu_{\mathcal{P},\mathcal{E}}(Q_j) = \beta_{\{q_j\}}^{\mathcal{P},\mathcal{E}}$ for every $Q_j \in \Omega_{\mathcal{P},\mathcal{E}}$ for $j = 1, \dots, m$. Then, (9) can be rewritten as follows:

$$I(a) = \int_{\Omega} a(\omega) d\mu_{\mathcal{P},\mathcal{E}}(\omega) + \sum_{1 \leq i \leq n} \beta_{E_i^*}^{\mathcal{P},\mathcal{E}} \min_{\omega \in E_i} a(\omega). \quad (10)$$

Next, we shall show that finitely additive measure $\mu_{\mathcal{P},\mathcal{E}}$ and coefficients $\{\beta_{E_i}^{\mathcal{P},\mathcal{E}}\}_{1 \leq i \leq n}$ in expression (10) does not depend on the choice of partition $\mathcal{P} \in \Pi_{\mathcal{E}}$. In the proof, the uniqueness of the Möbius inversion plays a crucial role. Let us take another partition $\mathcal{P}' \in \Pi_{\mathcal{E}}$ such that $\sigma(\mathcal{P},\mathcal{E}) \subseteq \sigma(\mathcal{P}',\mathcal{E})$, and repeat the above procedure. Let us define another finite set $\Omega_{\mathcal{P}',\mathcal{E}}^*$. We can choose the elements in $\Omega_{\mathcal{P}',\mathcal{E}}^*$ such that $\Omega_{\mathcal{P},\mathcal{E}}^* \subseteq \Omega_{\mathcal{P}',\mathcal{E}}^*$. For such a set $\Omega_{\mathcal{P}',\mathcal{E}}^*$, let us denote $E_i'^* = \{\omega \in \Omega_{\mathcal{P}',\mathcal{E}}^* | \omega \in E_i\}$, which is the set of representation elements in E_i . Then, it holds that $E_i^* \subseteq E_i'^*$ for all $i = 1, \dots, n$. By defining capacity $v_{\mathcal{P}',\mathcal{E}}^*$ on finite power set $2^{\Omega_{\mathcal{P}',\mathcal{E}}^*}$, we can show that there exist coefficients $\{\beta_{\{\omega\}}^{\mathcal{P}',\mathcal{E}}\}_{\omega \in \Omega_{\mathcal{P}',\mathcal{E}}^*}, \{\beta_{E_i}^{\mathcal{P}',\mathcal{E}}\}_{1 \leq i \leq n}$ such that

$$v_{\mathcal{P}',\mathcal{E}}^* = \sum_{\omega \in \Omega_{\mathcal{P}',\mathcal{E}}^*} \beta_{\{\omega\}}^{\mathcal{P}',\mathcal{E}} u_{\{\omega\}} + \sum_{1 \leq i \leq n} \beta_{E_i}^{\mathcal{P}',\mathcal{E}} u_{E_i'^*}, \quad (11)$$

and that there exists a finitely additive measure $\mu_{\mathcal{P}',\mathcal{E}}(\omega)$ on $\sigma(\mathcal{P}',\mathcal{E})$ such that for all $a \in \Phi_{\sigma(\mathcal{P}',\mathcal{E})}$,

$$I(a) = \int_{\Omega} a(\omega) d\mu_{\mathcal{P}',\mathcal{E}}(\omega) + \sum_{1 \leq i \leq n} \beta_{E_i}^{\mathcal{P}',\mathcal{E}} \min_{\omega \in E_i} a(\omega). \quad (12)$$

To complete our proof, we have to show that $\beta_{E_i}^{\mathcal{P}',\mathcal{E}} = \beta_{E_i}^{\mathcal{P},\mathcal{E}}$ for all $i = 1, \dots, n$ and that $\mu_{\mathcal{P}',\mathcal{E}} = \mu_{\mathcal{P},\mathcal{E}}$ on $\sigma(\mathcal{P},\mathcal{E})$. To do so, we need to define another capacity V^* on $2^{\Omega_{\mathcal{P},\mathcal{E}}^*}$ as follows. First, for every element $\omega \in \Omega_{\mathcal{P},\mathcal{E}}^*$, define $\Gamma(\omega) = \{X \in \Omega_{\mathcal{P}',\mathcal{E}} | X \subseteq Y \text{ such that } \omega \in Y \in \Omega_{\mathcal{P},\mathcal{E}}\}$. In other words, $\Gamma(\omega)$ is the partition of Y with respect to algebra $\sigma(\mathcal{P}',\mathcal{E})$ where Y is the cell in partition $\Omega_{\mathcal{P},\mathcal{E}}$ such that $\omega \in Y$. Second, for every element $\omega \in \Omega_{\mathcal{P},\mathcal{E}}^*$, define $\gamma_{\{\omega\}} = \sum_{X \in \Gamma(\omega)} \mu_{\mathcal{P}',\mathcal{E}}(X)$. Third, define a capacity V^* on $2^{\Omega_{\mathcal{P},\mathcal{E}}^*}$ by

$$V^* = \sum_{\omega \in \Omega_{\mathcal{P},\mathcal{E}}^*} \gamma_{\{\omega\}} u_{\{\omega\}} + \sum_{1 \leq i \leq n} \beta_{E_i}^{\mathcal{P}',\mathcal{E}} u_{E_i^*}. \quad (13)$$

Note that coefficient $\beta_{E_i}^{\mathcal{P}',\mathcal{E}}$ has a prime symbol on $\mathcal{P}$ and that unanimity game $u_{E_i^*}$ has no prime symbol on E . Let us show that V^* in (13) coincides with $v_{\mathcal{P},\mathcal{E}}^*$ in (8). Indeed, pick any $T \in 2^{\Omega_{\mathcal{P},\mathcal{E}}^*}$ and define $\tilde{T} \in \sigma(\mathcal{P},\mathcal{E})$ by $\tilde{T} = \bigcup_{T \cap X \neq \emptyset, X \in \Omega_{\mathcal{P},\mathcal{E}}} X$. This set $\tilde{T}$ is the minimal set in $\sigma(\mathcal{P},\mathcal{E})$ that contains all the elements of T . Then,

$$\begin{aligned} V^*(T) &= \sum_{\omega \in T} \gamma_{\{\omega\}} + \sum_{E_i^* \subseteq T} \beta_{E_i}^{\mathcal{P}',\mathcal{E}} \\ &= \sum_{\omega \in T} \sum_{X \in \Gamma(\omega)} \mu_{\mathcal{P}',\mathcal{E}}(X) + \sum_{E_i^* \subseteq \tilde{T}} \beta_{E_i}^{\mathcal{P}',\mathcal{E}} \\ &= \sum_{X \subseteq \tilde{T}, X \in \Omega_{\mathcal{P}',\mathcal{E}}} \mu_{\mathcal{P}',\mathcal{E}}(X) + \sum_{E_i^* \subseteq \tilde{T}} \beta_{E_i}^{\mathcal{P}',\mathcal{E}} \end{aligned}$$

$$\begin{aligned}
&= \mu_{\mathcal{P}', \mathcal{E}}(\tilde{T}) + \sum_{E_i \subseteq \tilde{T}} \beta_{E_i}^{\mathcal{P}', \mathcal{E}} \\
&= I(1_{\tilde{T}}) = I^*(1_T^*) = v_{\mathcal{P}, \mathcal{E}}^*(T).
\end{aligned}$$

Hence, V^* coincides with $v_{\mathcal{P}, \mathcal{E}}^*$ for all $T \in 2^{\Omega_{\mathcal{P}, \mathcal{E}}^*}$. Therefore, by the uniqueness of the Möbius inversion, it must hold that $\beta_{E_i}^{\mathcal{P}', \mathcal{E}} = \beta_{E_i}^{\mathcal{P}, \mathcal{E}}$ for all $i = 1, \dots, n$ and that $\mu_{\mathcal{P}', \mathcal{E}} = \mu_{\mathcal{P}, \mathcal{E}}$ on $\sigma(\mathcal{P}, \mathcal{E})$. Then, when defining $\beta_{E_i}^{\mathcal{P}, \mathcal{E}} = \varepsilon_i$ and $\mu_{\mathcal{P}, \mathcal{E}} = \mu$, we obtain our expression (1). $\square$

Proof of (only if part) of Theorem 3

Proof. Since A1, A3, A4, A5, and A6 hold, by Theorem 2, there exist a unique finitely additive measure μ on (Ω, Σ) , an affine function u , and a set of coefficients $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$, such that $f \succeq g \Leftrightarrow J(f) \geq J(g)$ where $J(f) = \int_{\Omega} u(f(\omega)) d\mu(\omega) + \sum_{i=1}^n \varepsilon_i \min_{\omega \in E_i} u(f(\omega))$. Therefore, it suffices to show that all coefficients $\varepsilon_1, \dots, \varepsilon_n$ are non-negative. Let U be the function from L_0 to $B_0(K)$ and I be the operator on $B_0(K)$, both of which are defined in Property (3) of the proof of Theorem 2. Then, for every $a = U(f)$, $I(a)$ is Choquet integral $\int a dv$ with respect to capacity $v(T) = I(1_T)$. Moreover, by A7, it holds that $v(X \cup Y) + v(X \cap Y) \geq v(X) + v(Y)$ for all $X, Y \in 2^{\Omega}$.

Now, pick any $E_i \in \mathcal{E}$. By the assumption of $\mathcal{E}$ being simple-complete, we can find a two-element set $\{p, q\} \subseteq E_i$ such that there is no $E_j \in \mathcal{E}$ satisfying $\{p, q\} \subseteq E_j \subsetneq E_i$. Moreover, take a partition $\mathcal{P} \in \Pi_{\mathcal{E}}$ in the proof of Lemma 2 such that $p, q \in \Omega_{\mathcal{P}, \mathcal{E}}^*$. Recall that for every $X \in 2^{\Omega_{\mathcal{P}, \mathcal{E}}^*}$, $v_{\mathcal{P}, \mathcal{E}}^*(X) = v(\bigcup_{e_j^i \in X} E_j^i \cup \bigcup_{q_i \in X} Q_i)$. Thus, $v(X \cup Y) + v(X \cap Y) \geq v(X) + v(Y)$ for all $X, Y \in 2^{\Omega}$ implies that $v_{\mathcal{P}, \mathcal{E}}^*(X \cup Y) + v_{\mathcal{P}, \mathcal{E}}^*(X \cap Y) \geq v_{\mathcal{P}, \mathcal{E}}^*(X) + v_{\mathcal{P}, \mathcal{E}}^*(Y)$ for all $X, Y \in 2^{\Omega_{\mathcal{P}, \mathcal{E}}^*}$.

Here, it holds that there is no $E_j^* \in \mathcal{E}^*$ satisfying $\{p, q\} \subseteq E_j^* \subsetneq E_i^*$. Let $T_1 = E_i^* \setminus \{p\}$ and $T_2 = E_i^* \setminus \{q\}$. Thus, $T_1 \cup T_2 = E_i^*$. Note that for every $S \subseteq T_1 \cup T_2$, $S \not\subseteq T_1$ and $S \not\subseteq T_2$ are equivalent to $\{p, q\} \subseteq S$. It follows that

$$\begin{aligned}
0 &\leq v_{\mathcal{P}, \mathcal{E}}^*(T_1 \cup T_2) - v_{\mathcal{P}, \mathcal{E}}^*(T_1) - v_{\mathcal{P}, \mathcal{E}}^*(T_2) + v_{\mathcal{P}, \mathcal{E}}^*(T_1 \cap T_2) \\
&= \sum_{E_j^* \subseteq T_1 \cup T_2, E_j^* \not\subseteq T_1, E_j^* \not\subseteq T_2} \varepsilon_j = \sum_{\{p, q\} \subseteq E_j^* \subseteq E_i^*} \varepsilon_j = \varepsilon_i,
\end{aligned}$$

where the inequality holds by the convexity of $v_{\mathcal{P}, \mathcal{E}}^*$ and the first equality holds by Lemma 6. $\square$

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