

Time reversal symmetries and properties of ground states in nonrelativistic QED

By

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Abstract

Time reversal symmetries of the Pauli-Fierz Hamiltonian are investigated in both Fock and Schrödinger representations. As an application, we investigate some properties of ground states expectations.

§ 1. Introduction

Let us consider one electron coupled with the quantized radiation field. This system is described by the so-called Pauli-Fierz Hamiltonian. By now some spectral properties of this model were successfully investigated by several authors [1, 2, 3, 4, 5, 6, 8, 10, 15, 16]. In particular the existence of a ground state is rather well understood. If we ignore the spin, the ground state is unique [12, 21]. A typical way to see the uniqueness is to prove that the heat kernel of the Hamiltonian improves the positivity in the Schrödinger representation [12]. Then the uniqueness is a direct consequence of the Perron-Frobenius theorem. On the other hand if the spin is included, it was shown that the ground state is always degenerate by the Kramers degeneracy theorem coming from the time reversal symmetry [14, 17, 18].

Usually we analyze the Pauli-Fierz Hamiltonian in the standard Fock representation where fields operators are described by the annihilation- and creation operators. However as mentioned above there is another important representation space for the

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Pauli-Fierz Hamiltonian, often called the Schrödinger representation where the fields operators are expressed as real valued multiplication operators in some L^2 -space. Each representation has its own advantage. For instance, since the quantized vector potential is a real multiplication operator in the Schrödinger representation, we can construct a path integral formula by modifying arguments about the standard Pauli operator. The point is that some properties could be easily proven in the Schrödinger representation even though these are hard to see in the Fock representation, and vice versa. Therefore choice of a suitable representation depends on each problem.

In previous works [17, 18], the time reversal symmetry was discussed in the Fock representation. Then it is rather natural to investigate the time reversal symmetry in the Schrödinger representation. Comparing results obtained in both representations, we could discover new aspects of the Pauli-Fierz Hamiltonian. This is a motivation of this little note. Indeed we will find a time reversal symmetry which is different from the one in the Fock representation. Then applying differences between two positively, we will investigate some properties of the ground states expectations.

§ 2. Pauli-Fierz Hamiltonian with spin 1/2

The Pauli-Fierz Hamiltonian is given by

$$(2.1) \quad H = \frac{1}{2}(-i\nabla_x + eA(x))^2 + \frac{e}{2}\sigma \cdot B(x) + V(x) + H_f$$

acting in $L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathfrak{F}$, where $\mathfrak{F}$ is the photon Fock space

$$(2.2) \quad \mathfrak{F} = \sum_{n \geq 0}^{\oplus} L^2(\mathbb{R}^3 \times \{1, 2\})^{\otimes_s n},$$

where $\mathfrak{h}^{\otimes_s n}$ means the n -fold symmetric tensor product of $\mathfrak{h}$ with the convention $\mathfrak{h}^{\otimes_s 0} = \mathbb{C}$. The quantized vector potential $A(x) = (A_1(x), A_2(x), A_3(x))$ is given by

$$(2.3) \quad A(x) = \sum_{\lambda=1,2} (2\pi)^{-3/2} \int_{|k| \leq \Lambda} dk \frac{\varepsilon(k, \lambda)}{\sqrt{2|k|}} \{e^{ik \cdot x} a(k, \lambda) + e^{-ik \cdot x} a(k, \lambda)^*\},$$

where $\varepsilon(k, \lambda)$ is a polarization vector which is real valued and measurable, Λ is the ultraviolet cutoff. Here $a(k, \lambda), a(k, \lambda)^*$ are the annihilation and creation operators which satisfy the standard commutation relations

$$(2.4) \quad [a(k, \lambda), a(q, \mu)^*] = \delta_{\lambda\mu} \delta(k - q), \quad [a(k, \lambda), a(q, \mu)] = 0 = [a(k, \lambda)^*, a(q, \mu)^*].$$

$B(x)$ is the quantized magnetic field defined by

$$(2.5) \quad \begin{aligned} B(x) &= \text{rot} A(x) \\ &= i \sum_{\lambda=1,2} (2\pi)^{-3/2} \int_{|k| \leq \Lambda} dk \frac{k \times \varepsilon(k, \lambda)}{\sqrt{2|k|}} \{e^{ik \cdot x} a(k, \lambda) - e^{-ik \cdot x} a(k, \lambda)^*\}. \end{aligned}$$

H_f is the field energy given by

$$(2.6) \quad H_f = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} dk |k| a(k, \lambda)^* a(k, \lambda).$$

In this note, we assume the following:

(V) V is infestimally small with respect to $-\Delta_x$. Moreover $V(-x) = V(x)$ for a.e. $x \in \mathbb{R}^3$.

Then, by [11, 13], H is self-adjoint on $\text{dom}(-\Delta_x) \cap \text{dom}(H_f)$, bounded from below.

§ 3. Time reversal symmetry in the Fock representation

On $L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathfrak{F} = \oplus^2 L^2(\mathbb{R}^3; \mathfrak{F})$, we take the following involution:

$$(3.1) \quad J_F \varphi = j\varphi_{\uparrow} \oplus j\varphi_{\downarrow},$$

$$(3.2) \quad j\varphi_{\sigma} = \sum_{n \geq 0}^{\oplus} \overline{\varphi^{(n)}_{\sigma}}(-x; k_1, \lambda_1, \dots, k_n, \lambda_n), \quad x \in \mathbb{R}^3, \quad (k_i, \lambda_i) \in \mathbb{R}^3 \times \{1, 2\}$$

for $\varphi = \varphi_{\uparrow} \oplus \varphi_{\downarrow} \in \oplus^2 L^2(\mathbb{R}^3; \mathfrak{F})$ with $\varphi_{\sigma} = \sum_{n \geq 0}^{\oplus} \varphi_{\sigma}^{(n)}(x; k_1, \lambda_1, \dots, k_n, \lambda_n) \in L^2(\mathbb{R}^3; \mathfrak{F})$, $\sigma = \uparrow, \downarrow$. In this note, we often denote the linear operator $X \oplus X$ acting in $\mathfrak{h} \oplus \mathfrak{h}$ simply by X .

Since the annihilation operator $a(k, \lambda)$ acts by

$$(3.3) \quad a(k, \lambda)\varphi_{\sigma} = \sum_{n \geq 0}^{\oplus} \sqrt{n+1} \varphi_{\sigma}^{(n+1)}(x; k, \lambda, k_1, \lambda_1, \dots, k_n, \lambda_n)$$

for $\varphi_{\sigma} = \sum_{n \geq 0}^{\oplus} \varphi_{\sigma}^{(n)} \in L^2(\mathbb{R}^3; \mathfrak{F})$ with $\sigma = \uparrow, \downarrow$, one has

$$(3.4) \quad J_F a(k, \lambda) = a(k, \lambda) J_F, \quad J_F a(k, \lambda)^* = a(k, \lambda)^* J_F.$$

Namely the annihilation and creation operators are reality preserving w.r.t. J_F . As a consequence, we obtain

$$(3.5) \quad J_F(-i\nabla_x) = (-i\nabla_x) J_F,$$

$$(3.6) \quad J_F A(x) = A(x) J_F,$$

$$(3.7) \quad J_F B(x) = -B(x) J_F,$$

$$(3.8) \quad J_F H_f = H_f J_F,$$

$$(3.9) \quad J_F V(x) = V(-x) J_F.$$

Now let introduce a time reversal operator in the Fock representation by

$$(3.10) \quad \vartheta_F = \sigma_2 J_F.$$

Clearly $\vartheta_F^2 = -\mathbb{1}$.

By the above relations, one arrives at the following:

Proposition 3.1. *[16] H has a time reversal symmetry in the Fock representation:*

$$(3.11) \quad \vartheta_F H = H \vartheta_F.$$

§ 4. Time reversal symmetry in the Schrödinger representation

There is a natural way to regard the fields operators $A(x)$ and $B(x)$ as real valued multiplication operators on L^2 -space. Such a representation is called the *Schrödinger representation*. To explain this, let us introduce

$$(4.1) \quad A(\mathbf{f}) = (A_1(f_1), A_2(f_2), A_3(f_3)),$$

$$(4.2) \quad A_j(f_j) = \sum_{\lambda=1,2} \int_{|k| \leq \Lambda} dk \frac{\varepsilon_j(k, \lambda)}{\sqrt{2|k|}} \{ \hat{f}_j(-k) a(k, \lambda) + \hat{f}_j(k) a(k, \lambda)^* \}$$

for $\mathbf{f} = (f_1, f_2, f_3)$ with $|k|^{-1/2} \hat{f}_j \in L^2(\mathbb{R}^3)$ and $\hat{f}_j$ is real valued, where $\hat{f}_j$ means the Fourier transformation of f_j . Then one sees

$$(4.3) \quad \langle \Omega, e^{i \sum_{j=1}^3 A_j(f_j)} \Omega \rangle = e^{-\frac{1}{2} q(\mathbf{f}, \mathbf{f})},$$

$$(4.4) \quad q(\mathbf{f}, \mathbf{g}) = \int_{\mathbb{R}^3} \frac{dk}{|k|} \bar{\mathbf{f}}(k) (\mathbb{1} - |\hat{k}\rangle \langle \hat{k}|) \hat{\mathbf{g}}(k),$$

where $\hat{k} = k/|k|$ and $\Omega = 1 \oplus 0 \oplus 0 \oplus \dots$ is the Fock vacuum. The left hand side of (4.3) is the characteristic functional of a Gaussian measure, $d\mu(A)$ with mean 0 and covariance q . Hence, by Minlos' theorem, Fock space $\mathfrak{F}$ can be identified with $L^2(Q, d\mu)$, where Q is the dual space of $\oplus^3 \mathcal{S}_{\text{real}}(\mathbb{R}^3)$, the set of real valued Schwarz test functions. By the construction, $A(x)$ and $B(x)$ are multiplication operators on $L^2(Q, d\mu)$. More precise explanation can be found in [7], see also [12, 21].

Let J_S be a natural involution on $L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathfrak{F} = \oplus^2 L^2(\mathbb{R}^3 \times Q)$ defined by $J_S \varphi = \bar{\varphi}_\uparrow \oplus \bar{\varphi}_\downarrow$ for each $\varphi = \varphi_\uparrow \oplus \varphi_\downarrow \in \oplus^2 L^2(\mathbb{R}^3 \times Q)$. Then since $A(x)$, $B(x)$ and $V(x)$ are real valued multiplication operators, one sees

$$(4.5) \quad J_S A(x) = A(x) J_S,$$

$$(4.6) \quad J_S B(x) = B(x) J_S,$$

$$(4.7) \quad J_S V(x) = V(x) J_S,$$

$$(4.8) \quad J_S (-i \nabla_x) = -(-i \nabla_x) J_S.$$

Moreover since $e^{-\beta H_f}$ preserves the positivity [20], one has

$$(4.9) \quad J_S H_f = H_f J_S.$$

Let $\xi = e^{i\pi N_f/2}$ with $N_f = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} dk a(k, \lambda)^* a(k, \lambda)$, the number operator. Remark the following important relation

$$(4.10) \quad J_S \xi = e^{i\pi N_f} \xi J_S.$$

For a linear operator Z , let us denote $\tilde{Z} = \xi Z \xi^{-1}$. Then, by (4.10), one obtains

$$(4.11) \quad J_S \tilde{A}(x) = -\tilde{A}(x) J_S,$$

$$(4.12) \quad J_S \tilde{B}(x) = -\tilde{B}(x) J_S.$$

On the other hand, since $\tilde{V}(x) = V(x)$, $\widetilde{(-i\nabla_x)} = -i\nabla_x$ and $\tilde{H}_f = H_f$, we have

$$(4.13) \quad J_S \tilde{V}(x) = \tilde{V}(x) J_S, \quad J_S \widetilde{(-i\nabla_x)} = -\widetilde{(-i\nabla_x)} J_S, \quad J_S \tilde{H}_f = \tilde{H}_f J_S.$$

Now we define a time reversal operator in the Schrödinger representation by

$$(4.14) \quad \vartheta_S = \sigma_2 J_S.$$

Note that $\vartheta_S^2 = -\mathbb{1}$. Summarizing the above discussion, we arrive at the following.

Proposition 4.1. *$\tilde{H}$ has a time reversal symmetry in the Schrödinger representation, namely,*

$$(4.15) \quad \vartheta_S \tilde{H} = \tilde{H} \vartheta_S.$$

§ 5. Ground states properties

Throughout this section we assume the following.

- (G) The Schrödinger operator $h_{\text{at}} = -\frac{1}{2}\Delta_x + V(x)$ has a unique strictly positive ground state ϕ_{at} with corresponding eigenvalue $E_{\text{at}} < 0$.

Then H has a ground state [3, 4, 10, 15]. Let Ψ be a ground state of H . Then, by Proposition 3.1, $\vartheta_F \Psi$ is a ground state of H as well. Moreover we have

$$\langle \Psi, \vartheta_F \Psi \rangle = \langle \vartheta_F \Psi, \vartheta_F \vartheta_F \Psi \rangle^* = -\langle \vartheta_F \Psi, \Psi \rangle^* = -\langle \Psi, \vartheta_F \Psi \rangle$$

which means $\Psi \perp \vartheta_F \Psi$. Let P_g be an orthogonal projection onto a closed subspace spanned by the ground states of H . Then above argument tells us

$$(5.1) \quad \dim \text{ran} P_g \geq 2.$$

Namely H has degenerate ground states. Similar arguments are still valid when we consider $\tilde{H}$ instead of H in the Schrödinger representation and apply Proposition 4.1. (In general we can show each eigenvalue is degenerate by the parallel argument. The degeneracy coming from the time reversal symmetry is called the *Kramers' degeneracy* [17, 18].)

In this section, we consider one of the following.

$$(\mathbf{H} \uparrow) \quad P_g s_\uparrow \otimes \phi_{\text{at}} \otimes \Omega \neq 0,$$

$$(\mathbf{H} \downarrow) \quad P_g s_\downarrow \otimes \phi_{\text{at}} \otimes \Omega \neq 0,$$

$$\text{where } s_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } s_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Remark that if e is sufficiently small, then we can actually check $(\mathbf{H} \uparrow)$ and $(\mathbf{H} \downarrow)$ by the pull-through arguments [4, 14].

Proposition 5.1. $(\mathbf{H} \uparrow)$ implies $(\mathbf{H} \downarrow)$. Conversely $(\mathbf{H} \downarrow)$ implies $(\mathbf{H} \uparrow)$.

Proof. By Proposition 3.1, one sees $\vartheta_F P_g = P_g \vartheta_F$. Assume $(\mathbf{H} \uparrow)$. Then

$$(5.2) \quad 0 \neq \vartheta_F P_g s_\uparrow \otimes \phi_{\text{at}} \otimes \Omega = P_g \vartheta_F s_\uparrow \otimes \phi_{\text{at}} \otimes \Omega = i P_g s_\downarrow \otimes \phi_{\text{at}} \otimes \Omega$$

which implies $(\mathbf{H} \downarrow)$. □

Henceforth we always assume $(\mathbf{H} \uparrow)$ or $(\mathbf{H} \downarrow)$. Define

$$(5.3) \quad \psi_\uparrow = s\text{-}\lim_{\beta \rightarrow \infty} \frac{e^{-\beta H} s_\uparrow \otimes \phi_{\text{at}} \otimes \Omega}{\|e^{-\beta H} s_\uparrow \otimes \phi_{\text{at}} \otimes \Omega\|},$$

$$(5.4) \quad \psi_\downarrow = s\text{-}\lim_{\beta \rightarrow \infty} \frac{e^{-\beta H} s_\downarrow \otimes \phi_{\text{at}} \otimes \Omega}{\|e^{-\beta H} s_\downarrow \otimes \phi_{\text{at}} \otimes \Omega\|}.$$

Then both $\psi_\uparrow$ and $\psi_\downarrow$ are normalized ground states of H .

Proposition 5.2. *One has the following.*

- (i) $\vartheta_F \psi_\uparrow = i \psi_\downarrow$, $\vartheta_F \psi_\downarrow = -i \psi_\uparrow$.
- (ii) Let $\tilde{\psi}_\uparrow = \xi \psi_\uparrow$ and $\tilde{\psi}_\downarrow = \xi \psi_\downarrow$. Then $\vartheta_S \tilde{\psi}_\uparrow = i \tilde{\psi}_\downarrow$, $\vartheta_S \tilde{\psi}_\downarrow = -i \tilde{\psi}_\uparrow$.
- (iii) $\langle \psi_\uparrow, \psi_\downarrow \rangle = 0$.

Proof. (i) and (ii) immediately follow from (5.3), (5.4) and Propositions 3.1, 4.1. To see (iii) we observe, by (i), that

$$(5.5) \quad \langle \psi_\uparrow, \psi_\downarrow \rangle = \langle \vartheta_F \psi_\downarrow, \vartheta_F \psi_\uparrow \rangle = -\langle \psi_\uparrow, \psi_\downarrow \rangle$$

which implies (iii). □

Remark that if we choose e sufficiently small, then $\dim \text{ran } P_g = 2$ by [14]. Hence, by Proposition 5.2 (iii), we have

$$P_g = |\psi_\uparrow\rangle\langle\psi_\uparrow| + |\psi_\downarrow\rangle\langle\psi_\downarrow|$$

provided e sufficiently small, where $|\psi_\sigma\rangle\langle\psi_\sigma|$ stands for the orthogonal projection onto the one dimensional subspace spanned by ψ_σ .

Theorem 5.3. *Let $F \in L^\infty(\mathbb{R}^3)$. Assume $F^*(-x) = F(x)$. Then one has the following.*

(i) *For all $i_1, \dots, i_n \in \{1, 2, 3\}$ and $n \in \mathbb{N}$,*

$$(5.6) \quad \begin{aligned} & \langle \psi_\uparrow, F(x)A_{i_1}(x)A_{i_2}(x) \cdots A_{i_n}(x)\psi_\uparrow \rangle \\ &= \langle \psi_\downarrow, F(x)A_{i_1}(x)A_{i_2}(x) \cdots A_{i_n}(x)\psi_\downarrow \rangle. \end{aligned}$$

(ii) *For all $i_1, \dots, i_n \in \{1, 2, 3\}$ and $n \in \mathbb{N}$,*

$$(5.7) \quad \langle \psi_\uparrow, F(x)A_{i_1}(x)A_{i_2}(x) \cdots A_{i_n}(x)\psi_\downarrow \rangle = 0.$$

(iii) *For all $i_1, \dots, i_n \in \{1, 2, 3\}$ and $n \in \mathbb{N}$,*

$$(5.8) \quad \begin{aligned} & \langle \psi_\uparrow, F(x)A_{i_1}(x)A_{i_2}(x) \cdots A_{i_{2n+1}}(x)\psi_\uparrow \rangle \\ &= \langle \psi_\downarrow, F(x)A_{i_1}(x)A_{i_2}(x) \cdots A_{i_{2n+1}}(x)\psi_\downarrow \rangle \\ &= 0. \end{aligned}$$

Remark. In [19], similar properties played some important roles, where the spinless Pauli-Fierz Hamiltonian was studied.

Proof. It suffices to show the assertions for each real $F \in L^\infty(\mathbb{R}^3)$ with $F(-x) = F(x)$.

(i) One has

$$\begin{aligned} \langle \psi_\uparrow, FA_{i_1} \cdots A_{i_n} \psi_\uparrow \rangle &= \langle \vartheta_F \psi_\uparrow, \vartheta_F FA_{i_1} \cdots A_{i_n} \psi_\uparrow \rangle^* \\ &= \langle \vartheta_F \psi_\uparrow, FA_{i_1} \cdots A_{i_n} \vartheta_F \psi_\uparrow \rangle^* \\ &= \langle \psi_\downarrow, FA_{i_1} \cdots A_{i_n} \psi_\downarrow \rangle \end{aligned}$$

by Proposition 5.2.

(ii) One has

$$\begin{aligned} \langle \psi_\uparrow, FA_{i_1} \cdots A_{i_n} \psi_\downarrow \rangle &= \langle \vartheta_F \psi_\uparrow, \vartheta_F FA_{i_1} \cdots A_{i_n} \psi_\downarrow \rangle^* \\ &= \langle \vartheta_F \psi_\uparrow, FA_{i_1} \cdots A_{i_n} \vartheta_F \psi_\downarrow \rangle^* \\ &= -\langle \psi_\uparrow, FA_{i_1} \cdots A_{i_n} \psi_\downarrow \rangle \end{aligned}$$

by Proposition 5.2.

(iii) To show (iii), we move to the Schrödinger representation $L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathfrak{F} = \oplus^2 L^2(\mathbb{R}^3 \times Q)$. Recall $\vartheta_S \tilde{A}_i(x) = -\tilde{A}_i(x) \vartheta_S$ with $\tilde{A}_i(x) = \xi A_i(x) \xi^{-1}$. Then one sees

$$\begin{aligned}
\langle \psi_\uparrow, F A_{i_1} \cdots A_{i_{2n+1}} \psi_\uparrow \rangle &= \langle \tilde{\psi}_\uparrow, F \tilde{A}_{i_1} \cdots \tilde{A}_{i_n} \tilde{\psi}_\uparrow \rangle \\
&= \langle \vartheta_S \tilde{\psi}_\uparrow, \vartheta_S F \tilde{A}_{i_1} \cdots \tilde{A}_{i_n} \tilde{\psi}_\uparrow \rangle^* \\
&= -\langle \vartheta_S \tilde{\psi}_\uparrow, F \tilde{A}_{i_1} \cdots \tilde{A}_{i_n} \vartheta_S \tilde{\psi}_\uparrow \rangle^* \\
&= -\langle \tilde{\psi}_\downarrow, F \tilde{A}_{i_1} \cdots \tilde{A}_{i_n} \tilde{\psi}_\downarrow \rangle \quad (\text{By Proposition 5.2}) \\
&= -\langle \psi_\downarrow, F A_{i_1} \cdots A_{i_n} \psi_\downarrow \rangle \\
&= -\langle \psi_\uparrow, F A_{i_1} \cdots A_{i_n} \psi_\uparrow \rangle. \quad (\text{By (i)})
\end{aligned}$$

This proves (iii). □

Theorem 5.4. *One has the following.*

(i) For all $i_1, \dots, i_n \in \{1, 2, 3\}$ and $n \in \mathbb{N}$,

$$\begin{aligned}
&\langle \psi_\uparrow, A_{i_1}(x) e^{-\beta_1 H} A_{i_2}(x) e^{-\beta_2 H} \cdots A_{i_n}(x) e^{-\beta_n H} \psi_\uparrow \rangle \\
(5.9) \quad &= \langle \psi_\downarrow, e^{-\beta_n H} A_{i_n}(x) e^{-\beta_{n-1} H} A_{i_{n-1}}(x) \cdots e^{-\beta_1 H} A_{i_1}(x) \psi_\downarrow \rangle.
\end{aligned}$$

(ii) For all $i_1, \dots, i_n \in \{1, 2, 3\}$ and $n \in \mathbb{N}$,

$$\begin{aligned}
&\langle \psi_\uparrow, A_{i_1}(x) e^{-\beta_1 H} A_{i_2}(x) e^{-\beta_2 H} \cdots A_{i_{2n+1}}(x) e^{-\beta_{2n+1} H} \psi_\uparrow \rangle \\
&= \langle \psi_\downarrow, A_{i_1}(x) e^{-\beta_1 H} A_{i_2}(x) e^{-\beta_2 H} \cdots A_{i_{2n+1}}(x) e^{-\beta_{2n+1} H} \psi_\downarrow \rangle \\
(5.10) \quad &= 0.
\end{aligned}$$

Proof. (i) Observe that

$$\begin{aligned}
\langle \psi_\uparrow, A_{i_1} e^{-\beta_1 H} \cdots A_{i_n} e^{-\beta_n H} \psi_\uparrow \rangle &= \langle \vartheta_F \psi_\uparrow, \vartheta_F A_{i_1} e^{-\beta_1 H} \cdots A_{i_n} e^{-\beta_n H} \psi_\uparrow \rangle^* \\
&= \langle \vartheta_F \psi_\uparrow, A_{i_1} e^{-\beta_1 H} \cdots A_{i_n} e^{-\beta_n H} \vartheta_F \psi_\uparrow \rangle^* \\
&= \langle \psi_\downarrow, A_{i_1} e^{-\beta_1 H} \cdots A_{i_n} e^{-\beta_n H} \psi_\downarrow \rangle^* \\
&= \langle \psi_\downarrow, e^{-\beta_n H} A_{i_n} \cdots e^{-\beta_1 H} A_{i_1} \psi_\downarrow \rangle.
\end{aligned}$$

(ii)

$$\begin{aligned}
\langle \psi_\uparrow, A_{i_1} e^{-\beta_1 H} \cdots A_{i_{2n+1}} e^{-\beta_{2n+1} H} \psi_\uparrow \rangle &= \langle \tilde{\psi}_\uparrow, \tilde{A}_{i_1} e^{-\beta_1 \tilde{H}} \cdots \tilde{A}_{i_{2n+1}} e^{-\beta_{2n+1} \tilde{H}} \tilde{\psi}_\uparrow \rangle \\
&= \langle \vartheta_S \tilde{\psi}_\uparrow, \vartheta_S \tilde{A}_{i_1} e^{-\beta_1 \tilde{H}} \cdots \tilde{A}_{i_{2n+1}} e^{-\beta_{2n+1} \tilde{H}} \tilde{\psi}_\uparrow \rangle^* \\
&= -\langle \vartheta_S \tilde{\psi}_\uparrow, \tilde{A}_{i_1} e^{-\beta_1 \tilde{H}} \cdots \tilde{A}_{i_{2n+1}} e^{-\beta_{2n+1} \tilde{H}} \vartheta_S \tilde{\psi}_\uparrow \rangle^* \\
&= -\langle \tilde{\psi}_\downarrow, \tilde{A}_{i_1} e^{-\beta_1 \tilde{H}} \cdots \tilde{A}_{i_{2n+1}} e^{-\beta_{2n+1} \tilde{H}} \tilde{\psi}_\downarrow \rangle^* \\
&= -\langle \psi_\downarrow, e^{-\beta_{2n+1} H} A_{i_{2n+1}} \cdots e^{-\beta_1 H} A_{i_1} \psi_\downarrow \rangle \\
&= -\langle \psi_\uparrow, A_{i_1} e^{-\beta_1 H} \cdots A_{i_{2n+1}} e^{-\beta_{2n+1} H} \psi_\uparrow \rangle.
\end{aligned}$$

In the last line, we used (i). □

Theorem 5.5. *Let F be a measurable function on $\mathbb{R}^3$ satisfying*

$$(5.11) \quad \sup_x |F(x)|e^{-\varepsilon|x|} < \infty$$

for sufficiently small $\varepsilon > 0$. Then one has the following.

$$(i) \quad \langle \psi_\uparrow, F(x)\psi_\uparrow \rangle = \langle \psi_\downarrow, F(x)\psi_\downarrow \rangle.$$

$$(ii) \quad \langle \psi_\uparrow, F(x)\psi_\downarrow \rangle = 0.$$

(iii) *If F is odd, i.e., $F(-x) = -F(x)$, then*

$$(5.12) \quad \langle \psi_\uparrow, F(x)\psi_\uparrow \rangle = 0 = \langle \psi_\downarrow, F(x)\psi_\downarrow \rangle.$$

Proof. First we remark that, the ground states $\psi_\uparrow$ and $\psi_\downarrow$ have the exponential decay property $\|e^{\varepsilon|x|}\psi_\sigma\| < \infty$ for sufficiently small $\varepsilon > 0$ [4, 9, 10]. Hence by the assumption (5.11), each ψ_σ belongs to the domain of the multiplication operator F . We also remark that, it suffices to show the assertions for real F .

(i)

$$\begin{aligned} \langle \psi_\uparrow, F\psi_\uparrow \rangle &= \langle \tilde{\psi}_\uparrow, F\tilde{\psi}_\uparrow \rangle = \langle \vartheta_S \tilde{\psi}_\uparrow, \vartheta_S F \tilde{\psi}_\uparrow \rangle^* = \langle \vartheta_S \tilde{\psi}_\uparrow, F \vartheta_S \tilde{\psi}_\uparrow \rangle^* = \langle \tilde{\psi}_\downarrow, F\tilde{\psi}_\downarrow \rangle^* \\ &= \langle \psi_\downarrow, F\psi_\downarrow \rangle. \end{aligned}$$

Similarly we can see (ii).

(iii) Note that if F is odd, then $\vartheta_F F = -F \vartheta_F$. Hence

$$\langle \psi_\uparrow, F\psi_\uparrow \rangle = \langle \vartheta_F \psi_\uparrow, \vartheta_F F \psi_\uparrow \rangle^* = -\langle \vartheta_F \psi_\uparrow, F \vartheta_F \psi_\uparrow \rangle^* = -\langle \psi_\downarrow, F\psi_\downarrow \rangle = -\langle \psi_\uparrow, F\psi_\uparrow \rangle.$$

This proves (iii). □

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