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On Siegel modular forms of degree 2 with square-free level

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Introduction

For representations of GL(2) over a p-adic field F there is a well-known theory of local newforms due to CASSELMAN, see [Cas]. This local theory together with the global strong multiplicity one theorem for cuspidal automorphic representations of GL(2) is reflected in the classical Atkin–Lehner theory for elliptic modular forms.

In contrast to this situation, there is currently no satisfactory theory of local newforms for the group $\mathrm{GSp}(2,F)$. As a consequence, there is no analogue of Atkin–Lehner theory for Siegel modular forms of degree 2. In this paper we shall present such a theory for the "square-free" case. In the local context this means that the representations in question are assumed to have nontrivial Iwahori–invariant vectors. In the global context it means that we are considering congruence subgroups of square-free level.

We shall begin by reviewing some well known facts from the classical theory of elliptic modular forms. Then we shall give a definition of a space $S_k(\Gamma_0(N)^{(2)})^{\mathrm{new}}$ of newforms in degree 2, where N is a square-free positive integer. Table 1 on page 8 lies at the heart of our theory. It contains the dimensions of the spaces of fixed vectors under each parahoric subgroup in every irreducible Iwahori–spherical representation of GSp(2) over a p-adic field F.

Section 4 deals with a global tool, namely a suitable L–function theory for certain cuspidal automorphic representations of PGSp(2). Since none of the existing results on the spin L–function seems to fully serve our needs, we have to make certain assumptions at this point. Having done so, we shall present our main result in the final section 5. It essentially says that given a cusp form $f\in S_k(\Gamma_0(N))^{\mathrm{new}}$, assumed to be an eigenform for almost all unramified Hecke algebras and also for certain Hecke operators at places $p|N$, we can attach a *global L–packet* π_f of automorphic representations of $\mathrm{PGSp}(2,\mathbb{A}_{\mathbb{Q}})$ to f. This allows us to associate with f a global (spin) L–function with a nice functional equation. We shall describe the local factors at the bad places explicitly in terms of certain Hecke eigenvalues.

1 Review of classical theory

We recall some well-known facts for classical holomorphic modular forms. Let $f \in S_k(\Gamma_0(N))$ be an elliptic cuspform, and let $G = \mathrm{GL}(2)$, considered as an algebraic $\mathbb{Q}$–group. It follows from strong approximation for SL(2) that there is a unique associated adelic function $\Phi_f : G(\mathbb{A}) \to \mathbb{C}$ with the following properties:

i) $\Phi_f(\rho g z) = \Phi_f(g)$ for all $g \in G(\mathbb{A})$, $\rho \in G(\mathbb{Q})$ and $z \in Z(\mathbb{A})$. Here Z is the center of GL(2).

ii) $\Phi_f(gh) - \Phi_f(g)$ for all $g \in G(\mathbb{A})$ and $h \in \prod_{p<\infty} K_p(N)$. Here $K_p(N) - \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}_p) : \ c \in N\mathbb{Z}_p\right\}$ is the local analogue of $\Gamma_0(N)$.

iii) $\Phi_f(g) = (f|_k g)(i) := \det(g)^{k/2} j(g,i)^{-k} f(g\langle i\rangle)$ for all $g \in \mathrm{GL}(2,\mathbb{R})^+$ (the identity component of $\mathrm{GL}(2,\mathbb{R})$).

Since f is a cusp form, Φ_f is an element of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/Z(\mathbb{A}))$. Let π_f be the unitary $\mathrm{PGL}(2,\mathbb{A})$–subrepresentation of this L^2–space generated by Φ_f.

1.1 Theorem. *With the above notations, the representation π_f is irreducible if and only if f is an eigenform for the Hecke operators $T(p)$ for almost all primes p. If this is the case, then f is automatically an eigenform for $T(p)$ for <u>all</u> $p \nmid N$.*

Idea of Proof: We decompose the representation π_f into irreducibles, $\pi_f = \bigoplus_i \pi_i$. Each π_i can be written as a restricted tensor product of local representations,

$$\pi_i \simeq \bigotimes_{p\leq\infty} \pi_{i,p}, \qquad \pi_{i,p} \text{ a representation of } \mathrm{PGL}(2,\mathbb{Q}_p).$$

Assuming that f is an eigenform, one can show easily that for almost all p we have $\pi_{i,p} \simeq \pi_{j,p}$. But *Strong Multiplicity One* for GL(2) says that two cuspidal automorphic representations coincide (as spaces of automorphic forms) if their local components are isomorphic at almost every place. It follows that π_f must be irreducible. ■

Thus to each eigenform f we can attach an *automorphic representation* $\pi_f = \otimes\pi_p$. A natural problem is to identify the local representations π_p given only the classical function f. This is easy at the archimedean place: π_∞ is the discrete series representation of $\mathrm{PGL}(2,\mathbb{R})$ with a lowest weight vector of weight k. It is also easy for finite primes p not dividing N. At such places π_p is an unramified principal series representation, i.e., π_p is an infinite-dimensional representation containing a non-zero $\mathrm{GL}(2,\mathbb{Z}_p)$–fixed vector. These representations are characterized by their Satake parameter $\alpha \in \mathbb{C}^*$, and the relationship between α and the Hecke-eigenvalue λ_p is $\lambda_p = p^{(k-1)/2}(\alpha + \alpha^{-1})$.

In general it is not easy to identify the local components π_p at places $p|N$. But if N is square-free, we have the following result.

1.2 Theorem. *Assume that N is a square-free positive integer, and let $f \in S_k(\Gamma_0(N))$ be an eigenform. Further assume that f is a* <u>*newform*</u>. *Then the local component π_p of the associated automorphic representation π_f at a place $p|N$ is given as follows:*

$$\pi_p = \begin{cases} \mathrm{St}_{\mathrm{GL}(2)} & \text{if } a_1 f = -f, \\ \xi\, \mathrm{St}_{\mathrm{GL}(2)} & \text{if } a_1 f = f. \end{cases}$$

Here $\mathrm{St}_{\mathrm{GL}(2)}$ is the Steinberg representation of $\mathrm{GL}(2,\mathbb{Q}_p)$, and ξ is the unique non-trivial unramified quadratic character of $\mathbb{Q}_p^*$. The operator a_1 is the Atkin–Lehner involution at p.

Idea of Proof: It follows from the fact that f is a modular form for $\Gamma_0(N)$ that π_p contains non-trivial vectors invariant under the Iwahori subgroup

$$I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}_p) : \; c \in p\mathbb{Z}_p \right\}.$$

The following is a complete list of all such Iwahori-spherical representations together with the dimensions of their spaces of fixed vectors under I and under $K = \mathrm{GL}(2,\mathbb{Z}_p)$.

representation	K	I
$\pi(\chi,\chi^{-1})$, χ unramified, $\chi^2 \neq \vert\,\vert^{\pm 1}$	1	2
$\mathrm{St}_{\mathrm{GL}(2)}$ or $\xi\,\mathrm{St}_{\mathrm{GL}(2)}$	0	1

(1)

We recall the definition of newforms, for notational simplicity assuming that $N = p$. We have two operators

$$T_0, T_1 : \; S_k(\mathrm{SL}(2,\mathbb{Z})) \longrightarrow S_k(\Gamma_0(p)), \tag{2}$$

where T_0 is simply the inclusion and T_1 is given by $(T_1 f)(\tau) = f(p\tau)$. Then the space of *oldforms* is defined as

$$S_k(\Gamma_0(p))^{\text{old}} = \text{im}(T_0) + \text{im}(T_1), \tag{3}$$

and the space of *newforms* $S_k(\Gamma_0(p))^{\text{new}}$ is by definition the orthogonal complement of $S_k(\Gamma_0(p))^{\text{old}}$ with respect to the Petersson inner product. Now it is easily checked that *locally*, in an unramified principal series representation $\pi(\chi, \chi^{-1})$ realized on a space V, we have

$$V^I = T_0 V^K + T_1 V^K. \tag{4}$$

Hence the fact that f is a newform means precisely that π_p cannot be an unramified principal series representation $\pi(\chi, \chi^{-1})$. Therefore $\pi_p = \text{St}_{\text{GL}(2)}$ or $\pi_p = \xi\,\text{St}_{\text{GL}(2)}$, and easy computations show the connection with the Atkin–Lehner eigenvalue (cf. [Sch], section 3). ∎

Knowing the local components π_p allows to correctly attach local factors to the modular form f. For example, if f is a newform as in Theorem 1.2, one would define for $p|N$

$$L_p(s, f) = L_p(s, \pi_p) = \begin{cases} (1 - p^{-1/2-s})^{-1} & \text{if } a_1 f = -f, \\ (1 + p^{-1/2-s})^{-1} & \text{if } a_1 f = f. \end{cases}$$

$$\varepsilon_p(s, f) = \varepsilon_p(s, \pi_p) = \begin{cases} -p^{1/2-s} & \text{if } a_1 f = -f, \\ p^{1/2-s} & \text{if } a_1 f = f. \end{cases}$$

With these definitions, and unramified and archimedean factors as usual, the functional equation $L(s, f) = \varepsilon(s, f) L(1 - s, f)$ holds for $L(s, f) = \prod_p L_p(s, f)$ and $\varepsilon(s, f) = \prod_p \varepsilon_p(s, f)$.

2 Newforms in degree 2

It is our goal to develop a similar theory as outlined in the previous section for the space of Siegel cusp forms $S_k(\Gamma_0(N)^{(2)})$ of degree 2 and square-free level N. Here we are facing several difficulties.

- Strong multiplicity one fails for the underlying group GSp(2), and even weak multiplicity one is presently not known. Thus it is not clear how to attach an automorphic representation of GSp(2, $\mathbb{A}$) to a classical cusp form f.

- The local representation theory of $\mathrm{GSp}(2,\mathbb{Q}_p)$ is much more complicated than that of $\mathrm{GL}(2,\mathbb{Q}_p)$. In particular, there are 13 different types of infinite-dimensional representations containing non-trivial vectors fixed under the local Siegel congruence subgroup, while in the $\mathrm{GL}(2)$ case we had only 2 (see table (1)).
- There is currently no generally accepted notion of newforms for Siegel modular forms of degree 2.

The last two problems are of course related. Let P_1 be the Siegel congruence subgroup of level p, i.e.,

$$P_1 = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}(2,\mathbb{Z}_p) : C \equiv 0 \bmod p \right\}. \tag{5}$$

Every classical definition of newforms with respect to P_1 must in particular be designed to exclude K–spherical representations, where $K = \mathrm{GSp}(2,\mathbb{Z}_p)$. Since an unramified principal series representation of $\mathrm{GSp}(2,\mathbb{Q}_p)$ contains a four–dimensional space of P_1–invariant vectors (see Table 1 below), we expect *four* operators

$$T_0, T_1, T_2, T_3 : S_k(\mathrm{Sp}(2,\mathbb{Z})) \longrightarrow S_k(\Gamma_0(p))$$

whose images would span the space of oldforms. (From now on, when we write $\Gamma_0(N)$, we mean groups of 4×4–matrices.) For this purpose we are now going to introduce four endomorphisms $T_0(p), \ldots, T_3(p)$ of the space $S_k(\Gamma_0(N))$, where N is square-free and $p|N$.

- $T_0(p)$ is simply the identity map.
- $T_1(p)$ is the Atkin–Lehner involution at p, defined as follows. Choose integers α, β such that $p\alpha - \frac{N}{p}\beta = 1$. Then the matrix

$$\eta_p = \begin{pmatrix} p\alpha & & 1 & \\ & p\alpha & & 1 \\ N\beta & & p & \\ & N\beta & & p \end{pmatrix}$$

is in $\mathrm{GSp}(2,\mathbb{R})^+$ with multiplier p. It normalizes $\Gamma_0(N)$, hence the map $f \mapsto f\big|_k \eta_p$ defines an endomorphism of $S_k(\Gamma_0(N))$. Since $\eta_p^2 \in p\Gamma_0(N)$, this endomorphism is an involution (we always normalize the slash operator as

$$(f\big|_k g)(Z) = \mu(g)^k j(g,Z)^{-k} f(g\langle Z\rangle) \qquad (\mu \text{ is the multiplier}),$$

which makes the center of $\mathrm{GSp}(2,\mathbb{R})^+$ act trivially). This is the *Atkin–Lehner involution* at p. It is independent of the choice of α and β.

- We define $T_2(p)$ by

$$(T_2(p)f)(Z) = \sum_{g\in\Gamma_0(N)\backslash\Gamma_0(N)\begin{pmatrix}1&\\&p1\end{pmatrix}\Gamma_0(N)} (f|_k g)(Z)$$

$$= \sum_{x,\mu,\kappa\in\mathbb{Z}/p\mathbb{Z}} \left(f\Big|_k \begin{pmatrix}1&&&\\&1&&\\&&p&\\&&&p\end{pmatrix}\begin{pmatrix}1&&x&\mu\\&1&\mu&\kappa\\&&1&\\&&&1\end{pmatrix}\right)(Z). \qquad (6)$$

This is a well-known operator in the classical theory. In terms of Fourier expansions, if $f(Z) = \sum_{n,r,m} c(n,r,m)e^{2\pi i(n\tau+rz+m\tau')}$ with $Z = \begin{pmatrix}\tau & z\\ z & \tau'\end{pmatrix}$, then

$$(T_2(p)f)(Z) = \sum_{n,r,m} c(np,rp,mp)e^{2\pi i(n\tau+rz+m\tau')}. \qquad (7)$$

- Finally, we define $T_3(p) := T_1(p)\circ T_2(p)$.

Now we are ready to define newforms in degree 2.

2.1 Definition. *Let N be a square-free positive integer. In $S_k(\Gamma_0(N))$ we define the subspace of* **oldforms** *$S_k(\Gamma_0(N))^{\rm old}$ to be the sum of the spaces*

$$T_i(p)S_k(\Gamma_0(Np^{-1})), \qquad i=0,1,2,3, \quad p|N.$$

The subspace of **newforms** *$S_k(\Gamma_0(N))^{\rm new}$ is defined as the orthogonal complement of $S_k(\Gamma_0(N))^{\rm old}$ inside $S_k(\Gamma_0(N))$ with respect to the Petersson scalar product.*

Note that this definition is analogous to the definition of oldforms in the degree 1 case. The operator T_1 given in (2) has the same effect as the Atkin–Lehner involution on modular forms for $\mathrm{SL}(2,\mathbb{Z})$.

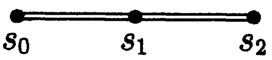
See [Ib] for more comments on the topic of old and new Siegel modular forms.

3 Local newforms

Let us realize $G = \mathrm{GSp}(2)$ using the symplectic form $\begin{pmatrix} & 1\\ -1 & \end{pmatrix}$. In this section we shall consider G as an algebraic group over a p-adic field F. Let $\mathfrak{o}$ be the ring of integers of F and $\mathfrak{p}$ its maximal ideal. Let $K = G(\mathfrak{o})$ be the standard special maximal compact subgroup of $G(F)$. As an Iwahori subgroup we choose

$$I = \left\{ g \in K : g \equiv \begin{pmatrix} * & & * & * \\ * & * & * & * \\ & & * & * \\ & & & * \end{pmatrix} \bmod \mathfrak{p} \right\}$$

The *parahoric subgroups* of $G(F)$ correspond to subsets of the simple Weyl group elements in the Dynkin diagram of the affine Weyl group C_2:



The Iwahori subgroup corresponds to the empty subset of $\{s_0, s_1, s_2\}$. The numbering is such that s_1 and s_2 generate the usual 8–element Weyl group of GSp(2). The corresponding parahoric subgroup is $P_{12} = K$. The Atkin-Lehner element

$$\eta = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \varpi & & \\ \varpi & & & \end{pmatrix} \in \mathrm{GSp}(2, F) \qquad (\varpi \text{ a uniformizer}) \tag{8}$$

induces an automorphism of the Dynkin diagram. The parahoric subgroup P_{01} corresponding to $\{s_0, s_1\}$ is therefore conjugate to K via η. We further have the Siegel congruence subgroup P_1 (see (5)), the Klingen congruence subgroup P_2, its conjugate $P_0 = \eta P_2 \eta^{-1}$, and the *paramodular group*

$$P_{02} = \left\{ g \in G(F) : g, g^{-1} \in \begin{pmatrix} \mathfrak{o} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-1} \\ \mathfrak{o} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{o} \end{pmatrix} \right\}.$$

K and P_{02} represent the two conjugacy classes of maximal compact subgroups of $\mathrm{GSp}(2, F)$. By a well-known result of BOREL (see [Bo]) the Iwahori–spherical irreducible representations are precisely the constituents of representations induced from an unramified character of the Borel subgroup. For GSp(2), such representations were first classified by RODIER, see [Rod], but in the following we shall use the notation of SALLY–TADIC [ST]. The following Table 1 gives a complete list of all the irreducible representations of $\mathrm{GSp}(2, F)$ with non-trivial I–invariant vectors. Behind each representation we have listed the dimension of the spaces of vectors fixed under each parahoric subgroup (modulo conjugacy). The last column gives the exponent of the conductor of the local parameter of each representation.

		representation	K	P_{02}	P_2	P_1	I	a
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	**1**	2 +−	4	4 ++ −−	8 ++++ −−−−	0
II	a	$\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	0	**1** −	2	**1** −	4 +−−−	1
	b	$\chi \mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$	**1**	1 +	2	3 ++−	4 +++−	0
III	a	$\chi \rtimes \sigma \mathrm{St}_{\mathrm{GSp}(1)}$	0	0	**1**	**2** +−	4 ++−−	2
	b	$\chi \rtimes \sigma \mathbf{1}_{\mathrm{GSp}(1)}$	**1**	2 +−	3	2 +−	4 ++−−	0
IV	a	$\sigma \mathrm{St}_{\mathrm{GSp}(2)}$	0	0	0	0	**1** −	3
	b	$L((\nu^2, \nu^{-1}\sigma \mathrm{St}_{\mathrm{GSp}(1)}))$	0	0	**1**	**2** +−	3 ++−	2
	c	$L((\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma))$	0	**1** −	2	**1** −	3 +−−	1
	d	$\sigma \mathbf{1}_{\mathrm{GSp}(2)}$	**1**	1 +	1	1 +	1 +	0
V	a	$\delta([\xi_0, \nu\xi_0], \nu^{-1/2}\sigma)$	0	0	**1**	0	2 +−	2
	b	$L((\nu^{1/2}\xi_0 \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma))$	0	**1** +	1	**1** +	2 ++	1
	c	$L((\nu^{1/2}\xi_0 \mathrm{St}_{\mathrm{GL}(2)}, \xi_0\nu^{-1/2}\sigma))$	0	**1** −	1	**1** −	2 −−	1
	d	$L((\nu\xi_0, \xi_0 \rtimes \nu^{-1/2}\sigma))$	**1**	0	1	2 +−	2 +−	0
VI	a	$\tau(S, \nu^{-1/2}\sigma)$	0	0	**1**	**1** −	3 +−−	2
	b	$\tau(T, \nu^{-1/2}\sigma)$	0	0	0	**1** +	1 +	2
	c	$L((\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma))$	0	**1** −	1	0	1 −	1
	d	$L((\nu, \mathbf{1}_{F^*} \rtimes \nu^{-1/2}\sigma))$	**1**	1 +	2	2 +−	3 ++−	0

Table 1: Dimensions of spaces of invariant vectors in Iwahori–spherical representations of $\mathrm{GSp}(2, F)$.

The signs under the entries for the "symmetric" subgroups P_{02}, P_1 and I indicate how these spaces of fixed vectors split into Atkin–Lehner eigenspaces, provided the central character of the representation is trivial. The signs listed in Table 1 are correct if one assumes that

- in Group II, where the central character is $\chi^2\sigma^2$, the character $\chi\sigma$ is trivial.
- in Groups IV, V and VI, where the central character is σ^2, the character σ itself is trivial.

If these assumptions are not met, then one has to interchange the plus and minus signs in Table 3 to get the correct dimensions.

The information in Table 1 is essentially obtained by computations in the standard models of these induced representations. Details will appear elsewhere.

Imitating the classical theory, one can define *oldforms* by introducing natural operators from fixed vectors for bigger to fixed vectors for smaller parahoric subgroups. Here "bigger" not always means inclusion, since we also consider K "bigger" than P_{02}. More precisely, we consider R' bigger than R, and shall write $R' \succ R$, if there is an arrow from R' to R in the following diagram.

$$\begin{tikzcd}
P_{01} \arrow[dd] \arrow[dr] \arrow[ddr] & & P_{12} \arrow[dl] \arrow[ddl] \arrow[dd] \\
& P_{02} \arrow[dl] \arrow[dr] & \\
P_0 \arrow[dr] & P_1 \arrow[d] & P_2 \arrow[dl] \\
& I &
\end{tikzcd} \tag{9}$$

Whenever $R' \succ R$, one can define natural operators from $V^{R'}$ to V^R, where V is any representation space. For example, our previously defined global operators $T_0(p)$ and $T_2(p)$ correspond to two natural maps $V^K \to V^{P_1}$. Our $T_1(p)$ and $T_3(p)$ correspond to two natural maps $V^{P_{01}} \to V^{P_1}$, composed with the Atkin–Lehner element $V^K \to V^{P_{01}}$.

This can be done for any parahoric subgroup, and it is natural to call any fixed vector that can be obtained from any bigger parahoric subgroup an *oldform*. Everything else would naturally be called a *newform*, but the meaning of "everything else" has to be made precise. Let it suffice to say that if the representation is unitary one can work with orthogonal complements as in the classical theory.

Once these notions of oldforms and newforms are defined, one can verify the decisive fact that *each space of fixed vectors listed in Table 1 consists either completely of oldforms or completely of newforms.* If this were not true, our notions of oldforms and newforms would make little sense. In Table 1 we have indicated the spaces of newforms by writing their dimensions in bold face. We see that they are not always one-dimensional.

4 L–functions

For the applications we have in mind we need the spin L–function of cuspidal automorphic representations of $\mathrm{GSp}(2,\mathbb{A})$ as a global tool. There are several results on this L–function, see [No], [PS] or [An]. Unfortunately none of these results fully serves our needs. What we need is the following.

4.1 L–Function Theory for $\mathrm{GSp}(2)$.

i) To every cuspidal automorphic representation π of $\mathrm{PGSp}(2,\mathbb{A})$ *is associated a global L–function $L(s,\pi)$ and a global ε–factor $\varepsilon(s,\pi)$, both defined as Euler products, such that $L(s,\pi)$ has meromorphic continuation to all of $\mathbb{C}$ and such that a functional equation*

$$L(s,\pi) = \varepsilon(s,\pi)L(1-s,\pi)$$

of the standard kind holds.

ii) For Iwahori–spherical representations, the local factors $L_v(s,\pi_v)$ and $\varepsilon_v(s,\pi_v,\psi_v)$ coincide with the spin local factors defined via the local Langlands correspondence as in [KL].

Of course such an L–function theory is predicted by general conjectures over any number field. For our classical applications we shall only need it over $\mathbb{Q}$. Furthermore, we can restrict to the archimedean component being a lowest weight representation with scalar minimal K–type (a discrete series representation if the weight is ≥ 3). All we need to know about ε–factors is in fact that they are of the form cp^{ms} with a constant $c \in \mathbb{C}^*$ and an integer m.

The local Langlands correspondence is not yet a theorem for $\mathrm{GSp}(2)$ (but see [Pr], [Rob]), but for Iwahori–spherical representations it is known by [KL].

In fact, the local parameters (four–dimensional representations of the Weil–Deligne group) of all the representations in Table 1 can easily be written down explicitly. Hence we know all their local factors. There is one case of L–indistinguishability in Table 1, namely, the representations VIa and VIb constitute an L–packet. The representation Va also lies in a two-element L–packet. Its partner is a θ_{10}–type supercuspidal representation.

4.2 Theorem. *We assume that an L–function theory as in 4.1 exists. Let $\pi_1 = \otimes\pi_{1,p}$ and $\pi_2 = \otimes\pi_{2,p}$ be two cuspidal automorphic representations of* $\mathrm{PGSp}(2, \mathbb{A}_{\mathbb{Q}})$. *Let S be a finite set of prime numbers such that the following holds:*

i) $\pi_{1,p} \simeq \pi_{2,p}$ *for each* $p \notin S$.

ii) For each $p \in S$, both $\pi_{1,p}$ and $\pi_{2,p}$ possess non-trivial Iwahori–invariant vectors.

Then, for each $p \in S$, the representations $\pi_{1,p}$ and $\pi_{2,p}$ are constituents of the same induced representation (from an unramified character of the Borel subgroup).

Idea of proof: We divide the two functional equations for $L(s, \pi_1)$ and $L(s, \pi_2)$ and obtain *finite* Euler products by hypothesis i). Since we are over $\mathbb{Q}$, and since the expressions p^{-s} for different p can be treated as independent variables, it follows that we get equalities

$$\frac{L_p(s, \pi_{1,p})}{L_p(s, \pi_{2,p})} = cp^{ms} \frac{L_p(1-s, \pi_{1,p})}{L_p(1-s, \pi_{2,p})}, \qquad c \in \mathbb{C}^*,\ m \in \mathbb{Z},$$

for each $p \in S$. But we have the complete list of all possible local Euler factors. One can check that such a relation is only possible if $\pi_{1,p}$ and $\pi_{2,p}$ are constituents of the same induced representation. ∎

Remark: In Table 1, for two representations to be constituents of the same induced representation means that they are in the same group I–VI.

With some additional information on the representation this result sometimes allows to attach a unique *equivalence class* of automorphic representations to a classical cuspform f. For example, if N is square-free and $f \in S_k(\Gamma_0(N))^{\mathrm{new}}$ is an eigenform for almost all the unramified Hecke algebras and also an eigenvector for the Atkin–Lehner involutions for all $p|N$, then Theorem 4.2 together with the information in Table 1 show that the associated adelic function Φ_f generates a multiple of an automorphic representation π_f of $\mathrm{PGSp}(2, \mathbb{A})$.

5 The main result

Let N be a square-free positive integer. In the degree 1 case, given an eigenform $f \in S_k(\Gamma_0(N))^{\text{new}}$, knowing the Atkin–Lehner eigenvalues for $p|N$ was enough to identify the local representations and attach the correct local factors. In the degree 2 case, since there are more possibilities for the local representations, and since some of them have parameters, we need more information than just the Atkin–Lehner eigenvalues. For example, the representations IIa or IIIa, both of which have local newforms with respect to P_1, depend on characters χ and σ. Hence there are additional Satake parameters which enter into the L–factor. What we need are suitable Hecke operators on $S_k(\Gamma_0(N))^{\text{new}}$ to extract this information from the modular form f. It turns out that the previously defined operator $T_2(p)$ works well, but we need even more information. We are now going to define an additional endomorphism $T_4(p)$ of $S_k(\Gamma_0(N))^{\text{new}}$.

For notational simplicity assume $N = p$ is a prime and consider the following linear maps:

$$S_k(\Gamma_0(p))^{\text{new}} \underset{d_1}{\overset{d_{02}}{\rightleftarrows}} S_k(\Gamma^{\text{para}}(p))^{\text{new}} \tag{10}$$

Here d_1 and d_{02} are *trace operators* which always exist between spaces of modular forms for commensurable groups. Explicitly,

$$d_{02}f = \frac{1}{(\Gamma^{\text{para}}(p) : \Gamma_0(p) \cap \Gamma^{\text{para}}(p))} \sum_{\gamma \in (\Gamma_0(p) \cap \Gamma^{\text{para}}(p)) \backslash \Gamma^{\text{para}}(p)} f\big|_k \gamma.$$

It is obvious from Table 1 that these operators indeed map newforms to newforms. The additional endomorphism of $S_k(\Gamma_0(p))^{\text{new}}$ we require is

$$T_4(p) := (1+p)^2\, d_1 \circ d_{02}. \tag{11}$$

Similarly we can define endomorphisms $T_4(p)$ of $S_k(\Gamma_0(N))^{\text{new}}$ for each $p|N$. Looking at local representations, the following is almost trivial.

5.1 Proposition. *Let N be square-free. The space $S_k(\Gamma_0(N))^{\text{new}}$ has a basis consisting of common eigenfunctions for the operators $T_2(p)$ and $T_4(p)$, all $p|N$, and for the unramified Hecke algebras at all good places $p \nmid N$.*

We can now state our main result.

5.2 Theorem. *We assume that an L–function theory as in 4.1 exists. Let N be a square-free positive integer, and let $f \in S_k(\Gamma_0(N))^{\text{new}}$ be a newform in the sense of Definition 2.1. We assume that f is an eigenform for the unramified local Hecke algebras $\mathcal{H}_p$ for almost all primes p. We further assume that f is an eigenfunction for $T_2(p)$ and $T_4(p)$ for all $p|N$,*

$$T_2(p)f = \lambda_p f, \qquad T_4(p)f = \mu_p f \qquad \text{for } p|N. \tag{12}$$

Then:

i) f is an eigenfunction for the local Hecke algebras $\mathcal{H}_p$ for all primes $p \nmid N$.

ii) Only the combinations of λ_p and μ_p as given in the following table can occur. Here ε is ± 1.

λ	μ	rep.	$L_p(s,f)^{-1}$	$\varepsilon_p(s,f)$
$-\varepsilon p$	$\notin\{0,2p\}$	IIa	$(1+\varepsilon(p+1)(p-\mu)p^{-3/2-s}+p^{-2s})(1+\varepsilon p^{-1/2-s})$	$\varepsilon p^{1/2-s}$
$\neq\pm p$	0	IIIa	$(1-\lambda p^{-3/2-s})(1-\lambda^{-1}p^{1/2-s})$	p^{1-2s}
$-\varepsilon p$	$2p$	Vb,c	$(1-\varepsilon p^{1/2-s})(1-p^{-1/2-s})(1+p^{-1/2-s})$	$\varepsilon p^{1/2-s}$
$-\varepsilon p$	0	VIa,b	$(1+\varepsilon p^{-1/2-s})^2$	p^{1-2s}

(We omit some indices p.)

iii) We define archimedean local factors according to our L–function theory and unramified spin Euler factors for $p \nmid N$ as usual. For places $p|N$ we define L– and ε–factors according to the table in ii). Then the resulting L–function has meromorphic continuation to the whole complex plane and satisfies the functional equation

$$L(s,f) = \varepsilon(s,f)L(1-s,f), \tag{13}$$

where $L(s,f) = \prod_{p\leq\infty} L_p(s,f)$ and $\varepsilon(s,f) = \prod_{p|N\infty} \varepsilon_p(s,f)$.

Sketch of proof: Statement i) follows from Theorem 4.2. Statement ii) follows by explicitly computing the possible eigenvalues of $T_2(p)$ and $T_4(p)$ in local representations. In the present case we cannot conclude that in the global representation $\pi_f = \oplus\pi_i$ all the irreducible components π_i must be isomorphic, because the eigenvalues in (12) cannot tell apart local representations VIa and VIb. This is however the only ambiguity, so that we can at least associate a *global L–packet* with f. (As mentioned before, VIa and VIb constitute a local L–packet.) The table in ii) indicates the possible representations depending on the Hecke eigenvalues.

The L–factors given in the table are those coming from the local Langlands correspondence. By hypothesis they coincide with the factors in our L–function theory. Hence the L–function in (13) coincides with the L–function of any one of the automorphic representations in our global L–packet. By our L–function theory we get the functional equation. ∎

5.3 Corollary. *If a cusp form $f \in S_k(\mathrm{Sp}(2,\mathbb{Z}))$ is an eigenfunction for the unramified Hecke algebras $\mathcal{H}_p$ for almost all primes p, then it is an eigenfunction for those Hecke algebras for all p.*

Remarks:

i) The corollary does not claim that f generates an irreducible automorphic representation of $\mathrm{PGSp}(2,\mathbb{A})$, but a multiple of such a representation. Without knowing multiplicity one for PGSp(2) we cannot conclude that f is determined by all its Hecke eigenvalues.

ii) The local factors given in Theorem 5.2 are the Langlands L– and ε–factors for the *spin* (degree 4) L–function. The following table lists the Langlands factors for the *standard* (degree 5) L–function.

λ	μ	rep.	$L_p(s,f,\mathrm{st})^{-1}$	$\varepsilon_p(s,f,\mathrm{st})$
$-\varepsilon p$	$\notin\{0,2p\}$	IIa	$(1-(p+1)(p-\mu)p^{-2-s}+p^{-1-2s})(1-p^{-s})$	p^{1-2s}
$\neq\pm p$	0	IIIa	$(1-\lambda^2 p^{-2-s})(1-\lambda^{-2}p^{2-s})(1-p^{-1-s})$	p^{1-2s}
$-\varepsilon p$	$2p$	Vb,c	$(1+p^{-1-s})(1+p^{-s})(1-p^{-s})$	p^{1-2s}
$-\varepsilon p$	0	VIa,b	$(1-p^{-s})^2(1-p^{-1-s})$	p^{1-2s}

iii) There is a statement analogous to Theorem 5.2 for modular forms with respect to the paramodular group $\Gamma^{\mathrm{para}}(N)$. Instead of $T_4(p)$ as defined in (11) this result makes use of the "dual" endomorphism $T_5(p) := (1+p)^2\, d_{02} \circ d_1$ of $S_k(\Gamma^{\mathrm{para}}(N))^{\mathrm{new}}$.

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