

BOUNDEDNESS OF THE SUCCESSIVE MINIMA ON ARITHMETIC VARIETIES

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Dedicated to the memory of Professor Masaki Maruyama.

Abstract

In this paper, we study the asymptotic behavior of the successive minima associated with high powers of a Hermitian invertible sheaf on an arithmetic variety. As a consequence, we prove that the arithmetic $\hat{\chi}$ -volume function, which is introduced by Yuan, is homogeneous, bi-rationally invariant, and continuous on the arithmetic Picard group. We also obtain the arithmetic Hilbert-Samuel formula for vertically nef Hermitian invertible sheaves.

Introduction

Let X be a projective arithmetic variety of dimension d and $\widehat{\text{Pic}}(X)$ the arithmetic Picard group of X . For a C^∞ -Hermitian invertible sheaf $\bar{L} \in \widehat{\text{Pic}}(X)$, we have two arithmetic analogues of the classical h^0 of invertible sheaves, which are

$$\hat{h}^0(H^0(X, L), \|\cdot\|_{\text{sup}}^{\bar{L}}) = \log \#\{s \in H^0(X, L) \mid \|s\|_{\text{sup}}^{\bar{L}} \leq 1\}$$

and

$$\hat{\chi}(H^0(X, L), \|\cdot\|_{\text{sup}}^{\bar{L}}) = \log \text{vol}_L\{s \in H^0(X, L)_{\mathbb{R}} \mid \|s\|_{\text{sup}}^{\bar{L}} \leq 1\},$$

where vol_L is the Haar measure on $H^0(X, L)_{\mathbb{R}}$ normalized as

$$\text{vol}_L(H^0(X, L)_{\mathbb{R}}/H^0(X, L)) = 1$$

(see Conventions and terminology, (7) and §1.1). In [15], Moriwaki defines the arithmetic volume of $\bar{L}$ as

$$\widehat{\text{vol}}(\bar{L}) = \limsup_{m \rightarrow \infty} \frac{\hat{h}^0(H^0(X, mL), \|\cdot\|_{\text{sup}}^{m\bar{L}})}{m^d/d!}$$

(cf. Definition 3.2.1) and studies its basic properties. He shows that the function $\widehat{\text{vol}} : \widehat{\text{Pic}}(X) \rightarrow \mathbb{R}$ is homogeneous and extends uniquely to a homogeneous

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function $\widehat{\text{vol}} : \widehat{\text{Pic}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \mathbb{R}$. The main results of Moriawaki [15], [16] are the continuity and the birational invariance of the arithmetic volume function $\widehat{\text{vol}} : \widehat{\text{Pic}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \mathbb{R}$ (cf. [16, Theorem 3.2] and [15, Theorem 4.3]). As is indicated in [15], Moriawaki's results have many applications such as the arithmetic Hilbert-Samuel formula, the generalized Hodge index theorem, the arithmetic Bogomolov-Gieseker inequality, etc.

In this paper, we define the arithmetic $\hat{\chi}$ -volume of $\bar{L}$ as

$$\widehat{\text{vol}}^{\hat{\chi}}(\bar{L}) = \limsup_{m \rightarrow \infty} \frac{\hat{\chi}(H^0(X, mL), \|\cdot\|_{\text{sup}}^{m\bar{L}})}{m^d/d!}$$

(cf. [24] and Definition 3.2.1) and study its properties. Actually, we shall see that almost all of the results in [15] and [16] also hold for our $\widehat{\text{vol}}^{\hat{\chi}}$. We prove that the arithmetic $\hat{\chi}$ -volume function takes values in $\mathbb{R}$, is homogeneous on $\widehat{\text{Pic}}(X)$, and extends uniquely to a homogenous function

$$\widehat{\text{vol}}^{\hat{\chi}} : \widehat{\text{Pic}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \mathbb{R}$$

(cf. Corollary 3.3.3). The purpose of this paper is to prove the continuity and the birational invariance of the arithmetic $\hat{\chi}$ -volume function on $\widehat{\text{Pic}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Theorem A (cf. Theorem 3.4.1). *Let $V \subseteq \widehat{\text{Pic}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ be a finite dimensional $\mathbb{Q}$ -vector subspace of $\widehat{\text{Pic}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\|\cdot\|_V$ a norm on $V \otimes_{\mathbb{Q}} \mathbb{R}$. Then, there is a positive constant $C_{\hat{\chi}}$ depending only on $(V, \|\cdot\|_V)$ such that the inequality*

$$|\widehat{\text{vol}}^{\hat{\chi}}(\xi^{\phi}) - \widehat{\text{vol}}^{\hat{\chi}}(\eta^{\psi})| \leq C_{\hat{\chi}}(\max\{\|\xi\|_V, \|\eta\|_V\})^{d-1} \cdot (\|\xi - \eta\|_V + \|\phi - \psi\|_{\text{sup}})$$

holds for any $\xi, \eta \in V$ and for any $\phi, \psi \in C^{\infty}(X)$, where $\xi^{\phi} = \xi + (\mathcal{O}_X, e^{-\phi} \|\cdot\|^{\text{can}})$ and $\eta^{\psi} = \eta + (\mathcal{O}_X, e^{-\psi} \|\cdot\|^{\text{can}})$.

Theorem B (cf. Theorem 3.1.1). *Let $\pi : X' \rightarrow X$ be a birational morphism of projective arithmetic varieties. Then, for any C^{∞} -Hermitian $\mathbb{Q}$ -invertible sheaf $\bar{L}$ on X ,*

$$\widehat{\text{vol}}^{\hat{\chi}}(\pi^* \bar{L}) = \widehat{\text{vol}}^{\hat{\chi}}(\bar{L}).$$

Roughly speaking, our function $\widehat{\text{vol}}^{\hat{\chi}}$ is “geometric” and related to *vertically nef* Hermitian invertible sheaves whereas the function $\widehat{\text{vol}}$ is “arithmetic” and related to *nef* ones (cf. Conventions and terminology, (8)). By the continuity of $\widehat{\text{vol}}^{\hat{\chi}}$, we can show the arithmetic Hilbert-Samuel formula for *vertically nef* Hermitian invertible sheaves. We note that Moriawaki [15] proves the formula only for *nef* Hermitian invertible sheaves.

Theorem C (cf. Theorem 3.5.1). *Let $\bar{L} = (L, h^{\bar{L}})$ be a C^{∞} -Hermitian $\mathbb{Q}$ -invertible sheaf on X . If L is relatively nef and $c_1(\bar{L})$ is semipositive on*

X_∞ , then

$$\widehat{\text{vol}}^{\hat{\chi}}(\bar{L}) = \widehat{\text{deg}}(\hat{c}_1(\bar{L})^d).$$

Our strategy to prove Theorems A and B is to study the boundedness of successive minima associated with high powers of a Hermitian invertible sheaf. Let $\bar{L}$ be a C^∞ -Hermitian invertible sheaf on X . If $H^0(X, L) \neq 0$, the last successive minimum of $(H^0(X, L), \|\cdot\|_{\text{sup}}^{\bar{L}})$ are defined as

$$\lambda_{\max}(H^0(X, L), \|\cdot\|_{\text{sup}}^{\bar{L}}) = \min \left\{ \max_{1 \leq i \leq l} \{\|e_i\|_{\text{sup}}\} \mid \begin{array}{l} e_1, \dots, e_l \in H^0(X, L) \text{ is a} \\ \mathbb{Q}\text{-basis for } H^0(X, L) \otimes_{\mathbb{Z}} \mathbb{Q} \end{array} \right\}$$

(see §1.1 for details). Our key technical result is the following theorem bounding the asymptotic behavior of the successive minima. Theorems A and B follows from Theorem D by the work of Yuan [22] and Moriwaki [15], [16].

Theorem D (cf. Theorem 1.2.5). *Suppose that X is normal. Let $r > 0$ be a positive integer and let $\bar{L}_1, \dots, \bar{L}_r$ be C^∞ -Hermitian invertible sheaves on X . Then, there are two positive constants $a_0, \sigma > 0$ and a non-zero polynomial with non-negative real coefficients, $P \in \mathbb{R}[T]$, which depend only on $\bar{L}_1, \dots, \bar{L}_r$ and X such that the inequality*

$$\begin{aligned} \lambda_{\max}(H^0(X, a_1 L_1 + \dots + a_r L_r), \|\cdot\|_{\text{sup}}^{a_1 \bar{L}_1 + \dots + a_r \bar{L}_r}) \\ \leq P(|a_1| + \dots + |a_r|) \cdot \sigma^{|a_1| + \dots + |a_r|} \end{aligned}$$

holds for any integers $a_1, \dots, a_r \in \mathbb{Z}$ with $|a_1| + \dots + |a_r| \geq a_0$.

We prove Theorem D in §1.2 and §1.4 by induction on dimension. Although our proofs are much involved, all of the tools we use are due to Yuan and Moriawaki and have already appeared in [22], [15], and [16].

This paper is organized as follows: in Section 1, we give a proof of the main technical estimate on asymptotic behavior of the successive minima (Theorem 1.2.1). In Section 2, we prove a multi-indexed version of the fundamental estimate, which is essential in the proof of the continuity of $\widehat{\text{vol}}^{\hat{\chi}}$ (Theorem 2.1.1). In Section 3, we introduce several types of arithmetic volume functions and study their properties: i.e. the birational invariance, the homogeneity, and the continuity (Theorems 3.1.1, 3.3.2, and 3.4.1). We also give some applications of our results.

Conventions and terminology. According to Moriawaki [15], [16], we fix the conventions and terminology used in this paper.

(1) Let r be a positive integer. For $\mathbf{a} = (a(1), \dots, a(r)) \in \mathbb{R}^r$, we set

$$\|\mathbf{a}\|_1 := |a(1)| + \dots + |a(r)|,$$

the L^1 -norm of $\mathbf{a} \in \mathbb{R}^r$. Let R be a ring and M a left R -module. For $\mathbf{L} = (L(1), \dots, L(r)) \in M^{\oplus r}$ and for $\mathbf{a} = (a(1), \dots, a(r)) \in R^{\oplus r}$, we write

$$\mathbf{a} \cdot \mathbf{L} := a(1)L(1) + \cdots + a(r)L(r)$$

for simplicity.

(2) Let $\mathbb{K}$ be either $\mathbb{Q}$ or $\mathbb{R}$, V a $\mathbb{K}$ -vector space, and d a positive integer. A function $f : V \rightarrow \mathbb{R}$ on V is said to be *positively homogeneous of degree d* if the equality

$$f(pv) = p^d f(v)$$

holds for any non-negative $p \in \mathbb{K}_{\geq 0}$ and for any $v \in V$.

We endow the $\mathbb{K}$ -vector space V the inductive limit topology defined by $V = \varinjlim W$, where W runs over all finite dimensional $\mathbb{K}$ -vector subspaces of V and each W has the relative topology induced from $W \otimes_{\mathbb{K}} \mathbb{R}$. A function $f : V \rightarrow \mathbb{R}$ on V is said to be *continuous in the weak sense* if f is continuous with respect to the above topology on V : in other words, f is continuous in the weak sense if, for any finite dimensional $\mathbb{K}$ -vector subspace $W \subseteq V$, the restriction $f|_W : W \rightarrow \mathbb{R}$ extends uniquely to a continuous function $(f|_W)^\sim : W \otimes_{\mathbb{K}} \mathbb{R} \rightarrow \mathbb{R}$ (cf. [16, Lemma 4.2]).

(3) Let X be a scheme (resp. a complex analytic space). Let $L_1, \dots, L_r$ be invertible sheaves on X and $m_1, \dots, m_r$ integers. In this paper, the tensor product $L_1^{\otimes m_1} \otimes \cdots \otimes L_r^{\otimes m_r}$ of invertible sheaves is denoted by

$$m_1 L_1 + \cdots + m_r L_r$$

in the additive way.

Let $Y \rightarrow X$ be a morphism of schemes (resp. complex analytic spaces) and $\mathcal{F}$ an $\mathcal{O}_X$ -module. The image of the restriction map

$$\mathrm{Im}(H^0(X, \mathcal{F}) \rightarrow H^0(Y, \mathcal{F}|_Y))$$

is denoted by $H^0(X|Y, \mathcal{F})$.

(4) An *arithmetic variety* X over $\mathrm{Spec}(\mathbb{Z})$ is an integral scheme quasi-projective and flat over $\mathrm{Spec}(\mathbb{Z})$. For a field $\mathbb{K}$ over $\mathbb{Q}$, the fibred product $X \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{K})$ is denoted by $X_{\mathbb{K}}$. Let $\mathrm{Rat}(X)$ be the rational function field of X and K_X the algebraic closure of $\mathbb{Q}$ in $\mathrm{Rat}(X)$. X is said to be *generically smooth* if $X_{\mathbb{Q}}$ is smooth over $\mathrm{Spec}(\mathbb{Q})$. By Hironaka's resolution of singularities [11], there is a projective birational morphism $\mu : \tilde{X} \rightarrow X$ of arithmetic varieties such that $\tilde{X}$ is generically smooth. We call the morphism μ a *generic resolution of singularities* of X .

(5) Let X be a reduced complex analytic space. A C^0 -function $\phi : X \rightarrow \mathbb{C}$ on X is said to be a C^∞ -function on X if, for any complex manifold X'

and for any analytic map $\pi : X' \rightarrow X$, the pull-back $\pi^*(\phi) = \phi \circ \pi$ is a C^∞ -function on X' . In the same way, a C^0 -Hermitian invertible sheaf $\bar{L}$ on X is said to be C^∞ -Hermitian if, for any complex manifold X' and for any analytic map $\pi : X' \rightarrow X$, the pull-back $\pi^*\bar{L}$ is a C^∞ -Hermitian invertible sheaf on X' . Furthermore, we say that $c_1(\bar{L})$ is *semipositive* on X if, for any complex manifold X' and for any analytic map $\pi : X' \rightarrow X$, the first Chern form $c_1(\pi^*\bar{L})$ is semipositive pointwise on X' .

Assume that X is compact. Let L and A be two invertible sheaves on X and $s \in H^0(X, A)$ a global section that is non-zero on each irreducible component of X . Let $\|\cdot\|^L$ be a norm on $H^0(X, L)$. The subspace-norm induced from $(H^0(X, L), \|\cdot\|^L)$ via the injective homomorphism

$$H^0(X, L - A) \xrightarrow{\otimes s} H^0(X, L)$$

is denoted by $\|\cdot\|_{\text{sub}(s)}^L$.

(6) Let X be an arithmetic variety. The complex analytic space associated with $X_{\mathbb{C}}$ is denoted by X_∞ . The set of all isometry classes of C^∞ - (resp. C^0)-Hermitian invertible sheaves on X is denoted by $\widehat{\text{Pic}}(X)$ (resp. $\widehat{\text{Pic}}(X; C^0)$). Let $\mathbb{K}$ be either $\mathbb{Q}$ or $\mathbb{R}$. We set $\widehat{\text{Pic}}(X)_{\mathbb{K}} := \widehat{\text{Pic}}(X) \otimes_{\mathbb{Z}} \mathbb{K}$ and $\widehat{\text{Pic}}(X; C^0)_{\mathbb{K}} := \widehat{\text{Pic}}(X; C^0) \otimes_{\mathbb{Z}} \mathbb{K}$. Note that there is a canonical injection $\widehat{\text{Pic}}(X)_{\mathbb{K}} \rightarrow \widehat{\text{Pic}}(X; C^0)_{\mathbb{K}}$.

Let $\mathcal{O}_X^{\text{can}}$ be the structure sheaf on X with the canonical metric h^{can} . Let $\bar{L}$ be a C^0 -Hermitian $\mathbb{Q}$ -invertible sheaf on X and $\phi : X_\infty \rightarrow \mathbb{R}$ a C^0 -function on X . We define a C^0 -Hermitian $\mathbb{Q}$ -invertible sheaf $\bar{L}^\phi$ on X as

$$\bar{L}^\phi := \bar{L} + (\mathcal{O}_X, e^{-2\phi} h^{\text{can}}).$$

We can regard a real number λ as a constant function on X_∞ and define the C^0 -Hermitian $\mathbb{Q}$ -invertible sheaf $\bar{L}^\lambda$ on X in the same manner.

(7) Let X be a projective arithmetic variety and $\bar{L} = (L, h^{\bar{L}})$ a C^0 -Hermitian invertible sheaf on X . Since $H^0(X, L)$ is torsion-free, we can identify an element in $H^0(X, L)$ with its image in $H^0(X, L)_{\mathbb{Q}} := H^0(X, L) \otimes_{\mathbb{Z}} \mathbb{Q}$. Moreover, we have the natural inclusions $H^0(X, L)_{\mathbb{Q}} \subseteq H^0(X, L)_{\mathbb{R}} \subseteq H^0(X, L)_{\mathbb{C}} = H^0(X_\infty, L_\infty)$ by the GAGA theorem. The supremum-norm of a section $s \in H^0(X, L)_{\mathbb{R}}$ is defined as

$$\|s\|_{\text{sup}}^{\bar{L}} := \sup\{|s|_{\bar{L}}(x) \mid x \in X_\infty\},$$

where $|\cdot|_{\bar{L}}$ is the norm of $\bar{L}$. If X is generically smooth, then we fix a volume form Φ_X on X_∞ . The L^2 -metric on $H^0(X, L)_{\mathbb{C}}$ with respect to Φ_X is defined

as

$$h_{L^2}^{\bar{L}}(s, t) := \int_{X_\infty} h_x^{\bar{L}}(s(x), t(x)) \Phi_X$$

for $s, t \in H^0(X, L)_\mathbb{C}$.

(8) Let X be a projective arithmetic variety and $\bar{L}$ a C^∞ -Hermitian invertible sheaf on X . A section $s \in H^0(X, L)$ is said to be *small* if $\|s\|_{\sup}^{\bar{L}} \leq 1$. Moreover, $s \in H^0(X, L)$ is *strictly small* if $\|s\|_{\sup}^{\bar{L}} < 1$. In this paper, we consider the ampleness of Hermitian invertible sheaves only on *generically smooth* arithmetic varieties (see [15]).

- *ample*: Suppose that X is generically smooth. $\bar{L}$ is said to be *ample* if L is ample on X , the first Chern form $c_1(\bar{L})$ is positive pointwise on X_∞ , and, for sufficiently large $m \geq 1$, $H^0(X, mL)$ is generated by the strictly small sections in $H^0(X, mL)$.
- *nef*: $\bar{L}$ is *vertically nef* if L is relatively nef and $c_1(\bar{L})$ is semipositive on X_∞ (Conventions and terminology, (5)). $\bar{L}$ is *horizontally nef* if $\widehat{\deg}(\hat{c}_1(\bar{L}|_\Gamma)) \geq 0$ for any closed horizontal arithmetic curve Γ on X . If $\bar{L}$ is both vertically and horizontally nef, then $\bar{L}$ is said to be *nef*.
- *effective*: $\bar{L}$ is said to be *effective* if $\bar{L}$ has a non-zero small section. We write $\bar{L} \geq 0$ if $\bar{L}$ is effective.

Let $\bar{L}$ be a C^∞ -Hermitian $\mathbb{Q}$ -invertible sheaf on X . $\bar{L}$ is said to be *ample* (resp. *vertically nef*, *horizontally nef*, *nef*) if there is a positive integer $m \geq 1$ such that $m\bar{L}$ is an ample (resp. a vertically nef, a horizontally nef, a nef) C^∞ -Hermitian invertible sheaf on X . By the arithmetic Nakai-Mořezon criterion of Zhang, if X is generically smooth and $\bar{L}$ is nef, then, for any ample $\bar{A}$, $\bar{L} + \bar{A}$ is ample (cf. [26, Corollary (4.8)]).

1. Boundedness of the successive minima

1.1. Quick review of first properties. Let M be a finitely generated $\mathbb{Z}$ -module, M_{tor} the torsion part of M , and $M_{\text{free}} = M/M_{\text{tor}}$ the free part of M . Let $\|\cdot\|$ be a norm on $M_\mathbb{R} := M \otimes_\mathbb{Z} \mathbb{R}$ and vol a Haar measure on $M_\mathbb{R}$. We call the pair $(M, \|\cdot\|)$ a *normed $\mathbb{Z}$ -module*. Let

$$\Delta(M, \|\cdot\|) := \{v \in M_\mathbb{R} \mid \|v\| \leq 1\}.$$

It is well known that any compact centrally o -symmetric convex body $\Delta \subseteq M_\mathbb{R}$ is obtained thereby: i.e. Δ is the closed unit ball with respect to some norm on $M_\mathbb{R}$ (cf. [10, Ch. 1, §1, Theorem 4]). We abbreviate $\Delta(M, \|\cdot\|)$ as Δ if no confusion can arise. The *dual* normed $\mathbb{Z}$ -module $(M^\vee, \|\cdot\|^\vee)$ of $(M, \|\cdot\|)$ is

the dual $\mathbb{Z}$ -module $M^\vee = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ of M with the *dual norm* $\|\cdot\|^\vee$ defined as

$$\|f\|^\vee := \sup_{\substack{v \in M_{\mathbb{R}} \\ f(v)=1}} \|v\|^{-1}$$

for $f \in M^\vee$. For a normed $\mathbb{Z}$ -module $(M, \|\cdot\|)$, $M_{\text{free}} \cap \Delta$ is a non-empty finite set and we can define the invariants $\hat{h}^0$ and $\hat{h}^1$ of $(M, \|\cdot\|)$ as

$$\hat{h}^0(M, \|\cdot\|) := \log \#(M_{\text{free}} \cap \Delta) + \log \#M_{\text{tor}}$$

and

$$\hat{h}^1(M, \|\cdot\|) := \hat{h}^0(M^\vee, \|\cdot\|^\vee) = \log \#(M^\vee \cap \Delta^\vee),$$

respectively, where $\Delta^\vee$ is the polar set of Δ in $M_{\mathbb{R}}^\vee$. Moreover, if $M \neq M_{\text{tor}}$, we define the arithmetic Euler characteristic of $(M, \|\cdot\|)$ as

$$\hat{\chi}(M, \|\cdot\|) := \log \left(\frac{\text{vol}(\Delta)}{\text{vol}(M_{\mathbb{R}}/M_{\text{free}})} \right) + \log \#M_{\text{tor}},$$

which does not depend on the choice of the Haar measure vol . If $M = M_{\text{tor}}$, $\hat{\chi}(M, \|\cdot\|)$ is defined to be $\log \#M_{\text{tor}}$. Let h be a Hermitian metric of real type on $M_{\mathbb{C}} = M \otimes_{\mathbb{Z}} \mathbb{C}$ and $\|\cdot\|_h$ the norm on $M_{\mathbb{R}}$ associated with the metric h . We call the pair (M, h) a *Hermitian $\mathbb{Z}$ -module*. For a Hermitian $\mathbb{Z}$ -module (M, h) of rank l , we can take $e_1, \dots, e_l \in M$ having the property that $M/\mathbb{Z}e_1 + \dots + \mathbb{Z}e_l$ is torsion and define

$$\widehat{\deg}(M, h) := \log \#(M/\mathbb{Z}e_1 + \dots + \mathbb{Z}e_l) - \frac{1}{2} \log \det(h(e_i \otimes 1, e_j \otimes 1))_{1 \leq i, j \leq l},$$

which does not depend on the choice of $e_1, \dots, e_l$. Let vol_h be the Haar measure on $M_{\mathbb{R}}$ associated with h . Then, for any normed $\mathbb{Z}$ -module $(M, \|\cdot\|)$,

$$\hat{\chi}(M, \|\cdot\|) = \widehat{\deg}(M, h) + \log \text{vol}_h(\Delta).$$

If $M \neq M_{\text{tor}}$, we define the last and the first successive minima of $(M, \|\cdot\|)$ as

$$\lambda_{\max}(M, \|\cdot\|) := \inf\{\lambda > 0 \mid \dim_{\mathbb{Q}} \langle M_{\text{free}} \cap \lambda \Delta \rangle_{\mathbb{Q}} = \text{rk } M\}$$

and

$$\lambda_{\min}(M, \|\cdot\|) := \inf\{\lambda > 0 \mid \dim_{\mathbb{Q}} \langle M_{\text{free}} \cap \lambda \Delta \rangle_{\mathbb{Q}} = 1\}$$

respectively, where $\langle M_{\text{free}} \cap \lambda \Delta \rangle_{\mathbb{Q}}$ is the $\mathbb{Q}$ -vector subspace of $M_{\mathbb{Q}}$ spanned by the elements of $M_{\text{free}} \cap \lambda \Delta$. If $M = M_{\text{tor}}$, we set $\lambda_{\max}(M, \|\cdot\|) = \lambda_{\min}(M, \|\cdot\|) = 1$. It is clear that

$$\lambda_{\max}(M, \|\cdot\|) = \lambda_{\max}(M_{\text{free}}, \|\cdot\|) \quad \text{and} \quad \lambda_{\min}(M, \|\cdot\|) = \lambda_{\min}(M_{\text{free}}, \|\cdot\|).$$

Our starting point is the arithmetic Riemann-Roch formula on $\text{Spec}(\mathbb{Z})$ given by Gillet and Soulé (see [8, Theorem 1] or [15, Proposition 2.1 (1)]).

Lemma 1.1.1. *Let $(M, \|\cdot\|)$ be a normed $\mathbb{Z}$ -module. Then*

$$\begin{aligned} -\operatorname{rk} M \log 6 &\leq \hat{h}^0(M, \|\cdot\|) - \hat{h}^1(M, \|\cdot\|) - \hat{\chi}(M, \|\cdot\|) \\ &\leq 2 \log(\operatorname{rk} M!) + \operatorname{rk} M \log(3/2). \end{aligned}$$

The next lemma is a variant of the Minkowski convex-body-theorem [10, Ch. 2, §9, Theorems 1 and 2]. (1) is a direct consequence of [10, Ch. 2, §9, Theorems 1 and 2] and (2) and (3) are due to Yuan (see the proof of [22, Proposition 2.6(5) and (6)]).

Lemma 1.1.2. *Let $(M, \|\cdot\|)$ (resp. (M, h)) be a normed (resp. a Hermitian) $\mathbb{Z}$ -module.*

(1) *We have*

$$\begin{aligned} -\operatorname{rk} M \log(\lambda_{\max}(M, \|\cdot\|)/2) - \log(\operatorname{rk} M!) \\ \leq \hat{\chi}(M, \|\cdot\|) - \log \#M_{\text{tor}} \leq -\operatorname{rk} M \log(\lambda_{\min}(M, \|\cdot\|)/2). \end{aligned}$$

(2) *We have*

$$\widehat{\deg}(M, h) \geq -\operatorname{rk} M \log \lambda_{\max}(M, \|\cdot\|_h).$$

(3) *Let $M' \xrightarrow{\text{sub}} M$ be an injective homomorphism of $\mathbb{Z}$ -modules and h_{sub} the subspace-metric on $M'_{\mathbb{C}}$ induced from the metric h on $M_{\mathbb{C}}$. Then*

$$\widehat{\deg}(M, h) - \widehat{\deg}(M', h_{\text{sub}}) \geq -(\operatorname{rk} M - \operatorname{rk} M') \log \lambda_{\max}(M, \|\cdot\|_h).$$

Lemma 1.1.3. *Let $(M, \|\cdot\|)$ be a normed $\mathbb{Z}$ -module and $0 \rightarrow M' \rightarrow M \xrightarrow{f} M'' \rightarrow 0$ an exact sequence of $\mathbb{Z}$ -modules. Then,*

$$\lambda_{\max}(M, \|\cdot\|) \leq \operatorname{rk} M' \cdot \lambda_{\max}(M', \|\cdot\|_{\text{sub}}) + \lambda_{\max}(M'', \|\cdot\|_{\text{quot}})$$

and

$$\lambda_{\min}(M, \|\cdot\|) \geq \min \left\{ \frac{\lambda_{\min}(M', \|\cdot\|_{\text{sub}})}{\#M''_{\text{tor}}}, \lambda_{\min}(M'', \|\cdot\|_{\text{quot}}) \right\}.$$

Proof. The first inequality is well known (cf. [26, Lemma (4.4)]). We note that, if $M' = M'_{\text{tor}}$, then $\lambda_{\max}(M, \|\cdot\|) = \lambda_{\max}(M'', \|\cdot\|_{\text{quot}})$ and, if $M' \neq M'_{\text{tor}}$ and $M'' = M''_{\text{tor}}$, then $\lambda_{\max}(M, \|\cdot\|) \leq \lambda_{\max}(M', \|\cdot\|_{\text{sub}})$. Now, we prove the second inequality. If $M = M_{\text{tor}}$, then the inequality is trivial. Suppose that $M \neq M_{\text{tor}}$ and let $e \in M$ be a torsion-free element of M with $\|e\| = \lambda_{\min}(M, \|\cdot\|)$. If $f(e)$ is torsion-free, then $\|e\| \geq \|f(e)\|_{\text{quot}} \geq \lambda_{\min}(M'', \|\cdot\|_{\text{quot}})$. If $f(e)$ is torsion, then $\#M''_{\text{tor}} \cdot f(e) = 0$ and $\#M''_{\text{tor}} \cdot e \in M'$. Thus

$$\#M''_{\text{tor}} \cdot \|e\| \geq \lambda_{\min}(M', \|\cdot\|_{\text{sub}}).$$

Hence we obtain the assertion. $\square$

Remark 1.1.4. We do not have any natural definitions of $\lambda_{\max}(M, \|\cdot\|)$ and $\lambda_{\min}(M, \|\cdot\|)$ for torsion $\mathbb{Z}$ -modules. In order to ensure Lemma 1.1.3, we may define $\lambda_{\max}(M, \|\cdot\|)$ and $\lambda_{\min}(M, \|\cdot\|)$ as arbitrarily fixed positive real numbers.

To end this subsection, we note, without proof, a trivial fact on induced norms.

Lemma 1.1.5. *Let V , W , V' , and W' be four $\mathbb{R}$ -vector spaces and $\|\cdot\|^{W'}$ a norm on W' . We consider a commutative diagram:*

$$\begin{array}{ccc} V & \xrightarrow{\text{sub}} & W \\ \uparrow \text{quot} & & \uparrow \text{quot} \\ V' & \xrightarrow{\text{sub}} & W' \end{array}$$

where the horizontal two arrows are injective and the vertical two are surjective. We endow V' (resp. W) with the subspace-norm $\|\cdot\|_{\text{sub}}^{W'}$ (resp. the quotient-norm $\|\cdot\|_{\text{quot}}^{W'}$) induced from $(W', \|\cdot\|^{W'})$. Then we have two norms on V : i.e. the quotient-norm $\|\cdot\|_{\text{sub,quot}}^{W'}$ induced from $(V', \|\cdot\|_{\text{sub}}^{W'})$ and the subspace-norm $\|\cdot\|_{\text{quot,sub}}^{W'}$ induced from $(W, \|\cdot\|_{\text{quot}}^{W'})$. These two norms are related as

$$\|v\|_{\text{quot,sub}}^{W'} \leq \|v\|_{\text{sub,quot}}^{W'}$$

for any $v \in V$. The equality holds if $\text{Ker}(V' \rightarrow V) = \text{Ker}(W' \rightarrow W)$.

1.2. Bounds for the successive minima. Let X be a projective arithmetic variety of dimension d . In this subsection, X is always assumed to be normal. Let L be an invertible sheaf on X and $\|\cdot\|^{(m)}$ a norm on $H^0(X, mL)_{\mathbb{R}}$ for $m \geq 0$ (needless to say, if $m = 0$, mL stands for $\mathcal{O}_X$). We consider the normed graded $\mathbb{Z}$ -module

$$\bar{R}(X, L) := \bigoplus_{m \geq 0} (H^0(X, mL), \|\cdot\|^{(m)}).$$

We say that $\bar{R}(X, L)$ is of *bounded type* if $\bar{R}(X, L)$ has the following property:

- (B) For any positive integer $m_0 \geq 1$ and for any non-zero section $s \in H^0(X, m_0 L)$, there are two positive constants $\sigma_0(s) > 0$ and $\tau_0(s) > 0$ depending on s such that the inequalities

$$\tau_0(s)^{m+k} \cdot \|u\|^{(m)} \leq \|s^{\otimes k} \otimes u\|^{(m+m_0 k)} \leq \sigma_0(s)^{m+k} \cdot \|u\|^{(m)}$$

hold for any non-negative integers $m, k \geq 0$ and for any section $u \in H^0(X, mL)$.

The following theorem is the core of this paper and shall be proved in §1.4.

Theorem 1.2.1. *Suppose that X is generically smooth and normal. If $\bar{R}(X, L)$ is of bounded type, then there are two positive constants $m_0, \sigma > 0$ two non-zero polynomials with non-negative real coefficients, $P, Q \in \mathbb{R}[T]$, of degree at most $d(d-1)/2$ and of degree at most d respectively, all of which depend only on $\bar{R}(X, L)$ and X , such that the inequalities*

$$\begin{aligned} \exp(-Q(m)) &\leq \lambda_{\min}(H^0(X, mL), \|\cdot\|^{(m)}) \\ &\leq \lambda_{\max}(H^0(X, mL), \|\cdot\|^{(m)}) \leq P(m) \cdot \sigma^m \end{aligned}$$

hold for any integer m with $m \geq m_0$.

Remark 1.2.2. (1) Without supposing that X is generically smooth and normal, it is easy to see that, if the section ring $R(X, L)_{\mathbb{Q}}$ is finitely generated over $\mathbb{Q}$, then the last logarithmic successive minima are bounded from above by a linear function of m : i.e.

$$\log \lambda_{\max}(H^0(X, mL), \|\cdot\|^{(m)}) \leq O(m).$$

(2) In [4, Proposition 2.6], Chambert-Loir proves that, without supposing that X is generically smooth and normal, the first logarithmic successive minima are bounded from below by a linear function of m : i.e.

$$\log \lambda_{\min}(H^0(X, mL), \|\cdot\|_{\sup}^{m\bar{L}}) \geq O(m)$$

for any C^0 -Hermitian invertible sheaf $\bar{L}$ on X . Thus the lower bound in the above theorem is not optimal.

Let $r \geq 1$ be a positive integer and $\bar{\mathbf{L}} = (\bar{\mathbf{L}}(1), \dots, \bar{\mathbf{L}}(r))$ a family of C^∞ -Hermitian invertible sheaves on X . For $\mathbf{a} = (\mathbf{a}(1), \dots, \mathbf{a}(r)) \in \mathbb{Z}^r$, we write

$$\mathbf{a} \cdot \bar{\mathbf{L}} := \mathbf{a}(1)\bar{\mathbf{L}}(1) + \dots + \mathbf{a}(r)\bar{\mathbf{L}}(r)$$

for simplicity (Conventions and terminology, (1)). We denote the family of invertible sheaves on X obtained by forgetting the Hermitian metrics of $\bar{\mathbf{L}}$ by $\mathbf{L}$: i.e. $\mathbf{L} = (\mathbf{L}(1), \dots, \mathbf{L}(r))$.

Theorem 1.2.3. *Suppose that X is generically smooth and normal. We fix a volume form Φ_X on X_∞ and consider the L^2 -norms with respect to Φ_X . Then, there are two positive constants $a_0, \sigma > 0$ and a non-zero polynomial with non-negative real coefficients, $P \in \mathbb{R}[T]$, all of which depend only on $\bar{\mathbf{L}}$ and (X, Φ_X) such that the inequality*

$$\lambda_{\max}(H^0(X, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{\mathbf{L}}}) \leq P(\|\mathbf{a}\|_1) \cdot \sigma^{\|\mathbf{a}\|_1}$$

holds for any integers $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$ with $\|\mathbf{a}\|_1 \geq a_0$.

Proof of Theorem 1.2.1 $\Rightarrow$ Theorem 1.2.3. Let $\bar{E} = (E, h^{\bar{E}}) = \bar{\mathbf{L}}(1) \oplus^\perp \dots \oplus^\perp \bar{\mathbf{L}}(r)$ be the C^∞ -Hermitian locally free sheaf on X , $\pi : \mathbb{P}(E) := \text{Proj}_X(\text{Sym}(E)) \rightarrow X$ the projective bundle on X associated with E , and

$L = \mathcal{O}_{\mathbb{P}(E)}(1)$ the fundamental sheaf on $\mathbb{P}(E)$. Let $m\bar{E} := (mE, h^{mE})$ be the m -th tensor power of $\bar{E}$ and $\mathrm{Sym}^m E$ the m -th symmetric power of E . For a sequence of integers $\mathbf{a} = (\mathbf{a}(1), \dots, \mathbf{a}(r)) \in \mathbb{Z}_{\geq 0}^r$ with $\|\mathbf{a}\|_1 = m$, the Hermitian invertible sheaf $\mathbf{a} \cdot \bar{L}$ is isometrically embedded in the Hermitian locally free sheaf $m\bar{E}$. The natural projection $mE \rightarrow \mathrm{Sym}^m E$ induces an isomorphism

$$mE \supseteq \bigoplus_{\substack{\mathbf{a} \in \mathbb{Z}_{\geq 0}^r \\ \|\mathbf{a}\|_1 = m}} \mathbf{a} \cdot L \xrightarrow{\sim} \mathrm{Sym}^m E.$$

Now we recall that the metrized tensor product is associative and commutative. For any positive integers $m_1, \dots, m_k$, there is a natural isometry

$$\begin{aligned} \left(\sum_i m_i \right) \bar{E} &\supseteq \left(\bigoplus_{\|\mathbf{a}_1\|_1 = m_1}^\perp \mathbf{a}_1 \cdot \bar{L} \right) \otimes \cdots \otimes \left(\bigoplus_{\|\mathbf{a}_k\|_1 = m_k}^\perp \mathbf{a}_k \cdot \bar{L} \right) \\ &\xrightarrow{\rho(m_1, \dots, m_k)} \bigoplus_{\|\mathbf{a}\|_1 = \sum_i m_i}^\perp \mathbf{a} \cdot \bar{L} \subseteq \left(\sum_i m_i \right) \bar{E}, \end{aligned}$$

which is locally a rearrangement of a basis for $(\sum m_i)E$. In the rest of the proof, we denote such isometries simply by ρ .

Since we have a canonical isomorphism $\mathrm{Sym}^m E = \pi_* m L$,

$$H^0(\mathbb{P}(E), mL) = H^0(X, \mathrm{Sym}^m E) \simeq \bigoplus_{\|\mathbf{a}\|_1 = m} H^0(X, \mathbf{a} \cdot L).$$

By use of this isomorphism, we define a Hermitian $\mathbb{Z}$ -module $(H^0(\mathbb{P}(E), mL), h^{(m)})$ as

$$(H^0(\mathbb{P}(E), mL), h^{(m)}) := \bigoplus_{\|\mathbf{a}\|_1 = m}^\perp (H^0(X, \mathbf{a} \cdot L), h_{L^2}^{\mathbf{a} \cdot \bar{L}}).$$

Claim 1.2.4. We denote the norm on $H^0(\mathbb{P}(E), mL)_{\mathbb{R}}$ associated with the metric $h^{(m)}$ by $\|\cdot\|^{(m)}$. The graded normed $\mathbb{Z}$ -module

$$\bar{R}(\mathbb{P}(E), L) := \bigoplus_{m \geq 0}^\perp (H^0(\mathbb{P}(E), mL), \|\cdot\|^{(m)})$$

is of bounded type.

Proof. This claim follows from [15, Lemma 1.1.4]. Let $s \in H^0(\mathbb{P}(E), m_0 L)$ be a non-zero global section of $m_0 L$. We can regard the section s as a section in $H^0(X, m_0 E)$ via the injective homomorphism

$$H^0(\mathbb{P}(E), m_0 L) = H^0(X, \mathrm{Sym}^{m_0} E) \simeq \bigoplus_{\|\mathbf{a}\|_1 = m_0} H^0(X, \mathbf{a} \cdot L) \subseteq H^0(X, m_0 E).$$

Let U be a sufficiently small open subset of X_∞ such that U is not empty on each connected component of X_∞ and that the inequality $\inf_{x \in U} \{|s|_{m_0 \bar{E}}(x)\} > 0$ holds. We put

$$\tau_1(s) := \inf_{x \in U} \{|s|_{m_0 \bar{E}}(x)\} > 0,$$

the positive constant depending on s . We may assume without loss of generality that the volume form Φ_X is given by the volume form $\omega^{\wedge(d-1)}/(d-1)!$ associated with a Kähler form ω on X_∞ since there is a positive constant $D \geq 1$ such that

$$D^{-1} \cdot \frac{\omega^{\wedge(d-1)}}{(d-1)!} \leq \Phi_X \leq D \cdot \frac{\omega^{\wedge(d-1)}}{(d-1)!}$$

at each point on X_∞ . Applying [15, Lemma 1.1.4] to this situation, we can find a positive constant τ_2 depending on U such that

$$\int_{X_\infty} h^{\mathbf{a} \cdot \bar{L}}(u_{\mathbf{a}}, u_{\mathbf{a}}) \Phi_X \leq \tau_2^{2\|\mathbf{a}\|_1} \cdot \int_U h^{\mathbf{a} \cdot \bar{L}}(u_{\mathbf{a}}, u_{\mathbf{a}}) \Phi_X$$

for any $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$ and for any section $u_{\mathbf{a}} \in H^0(X, \mathbf{a} \cdot \mathbf{L})$. Hence we have the inequality

$$\int_{X_\infty} h^{m\bar{E}}(u, u) \Phi_X \leq \tau_2^{2m} \cdot \int_U h^{m\bar{E}}(u, u) \Phi_X$$

for any positive integer $m > 0$ and for any $u \in \bigoplus_{\|\mathbf{a}\|_1=m} H^0(X, \mathbf{a} \cdot \mathbf{L})$. Therefore,

$$\begin{aligned} & \int_{X_\infty} h^{(m+m_0k)\bar{E}}(\rho(s^{\otimes k} \otimes u), \rho(s^{\otimes k} \otimes u)) \Phi_X \\ & \geq \int_U h^{(m+m_0k)\bar{E}}(\rho(s^{\otimes k} \otimes u), \rho(s^{\otimes k} \otimes u)) \Phi_X \\ & = \int_U \left(h^{m_0\bar{E}}(s, s) \right)^k \cdot h^{m\bar{E}}(u, u) \Phi_X \\ & \geq \tau_1(s)^{2k} \cdot \int_U h^{m\bar{E}}(u, u) \Phi_X \\ & \geq \tau_1(s)^{2k} \cdot \tau_2^{-2m} \int_{X_\infty} h^{m\bar{E}}(u, u) \Phi_X, \end{aligned}$$

so that

$$\|s^{\otimes k} \otimes u\|^{(m+m_0k)} \geq (\min\{\tau_1(s), \tau_2^{-1}\})^{m+k} \cdot \|u\|^{(m)}$$

for any two integers $m, k \geq 0$ and for any section $u \in H^0(\mathbb{P}(E), mL)$. Next, we put

$$\sigma_1(s) := \sup_{x \in X_\infty} \{|s|_{m_0 \bar{E}}(x)\}.$$

Then, it is clear that

$$\|s^{\otimes k} \otimes u\|^{(m+m_0k)} \leq \sigma_1(s)^k \cdot \|u\|^{(m)} \leq (\max\{1, \sigma_1(s)\})^{m+k} \cdot \|u\|^{(m)}$$

for any integers $m, k \geq 0$ and for any section $u \in H^0(\mathbb{P}(E), mL)$. $\square$

We recall that $\mathbb{P}(E)$ is normal. By Theorem 1.2.1, there are positive constants m_0 , σ , and a non-zero polynomial, $P \in \mathbb{R}[T]$, of degree at most $(d+r-1)(d+r-2)/2$, which depend only on $\bar{R}(\mathbb{P}(E), L)$, such that the inequality

$$\lambda_{\max}(H^0(\mathbb{P}(E), mL), \|\cdot\|^{(m)}) \leq P(m) \cdot \sigma^m$$

holds for any integer $m \geq m_0$. On the other hand,

$$\lambda_{\max}(H^0(\mathbb{P}(E), mL), \|\cdot\|^{(m)}) = \max_{\substack{\mathbf{a} \in \mathbb{Z}_{\geq 0}^r \\ \|\mathbf{a}\|_1 = m}} \left\{ \lambda_{\max}(H^0(X, \mathbf{a} \cdot \bar{L}), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{L}}) \right\},$$

by definition of the norms $\|\cdot\|^{(m)}$. Hence, we obtain Theorem 1.2.3. $\square$

Let r be a positive integer. Let X be a projective arithmetic variety and $\bar{L} = (\bar{L}(1), \dots, \bar{L}(r))$ a family of C^∞ -Hermitian invertible sheaves on X .

Theorem 1.2.5. *Suppose that X is normal, but not necessarily generically smooth. Then, there are two positive constants a_0 , $\sigma > 0$ and a non-zero polynomial with non-negative real coefficients, $P \in \mathbb{R}[T]$, all of which depend only on $\bar{L}$ and X , such that the inequalities*

$$\lambda_{\max}(H^0(X, \mathbf{a} \cdot \bar{L}), \|\cdot\|_{\sup}^{\mathbf{a} \cdot \bar{L}}) \leq P(\|\mathbf{a}\|_1) \cdot \sigma^{\|\mathbf{a}\|_1}$$

holds for any integers $\mathbf{a} \in \mathbb{Z}^r$ with $\|\mathbf{a}\|_1 \geq a_0$.

Proof of Theorem 1.2.3 $\Rightarrow$ Theorem 1.2.5. Let $\mu: \tilde{X} \rightarrow X$ be the normalization of a generic resolution of singularities. Since X is normal, we have $\mu_* \mathcal{O}_{\tilde{X}} = \mathcal{O}_X$ and a natural isometry

$$(H^0(X, \mathbf{a} \cdot \bar{L}), \|\cdot\|_{\sup}^{\mathbf{a} \cdot \bar{L}}) \xrightarrow{\sim} (H^0(\tilde{X}, \mathbf{a} \cdot \mu^* \bar{L}), \|\cdot\|_{\sup}^{\mathbf{a} \cdot \mu^* \bar{L}}).$$

Thus we may assume without loss of generality that X is generically smooth and normal. Next, by [15, Corollary 1.1.2], there is a positive constant D such that

$$\|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{L}} \leq \|\cdot\|_{\sup}^{\mathbf{a} \cdot \bar{L}} \leq D(1 + \|\mathbf{a}\|_1)^{d-1} \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{L}}$$

for any $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$. Thus we may replace L^2 with $\sup$ in Theorem 1.2.3.

For any $\epsilon \in \{\pm 1\}^r$, we put $\epsilon \bar{L} = (\epsilon(1) \bar{L}(1), \dots, \epsilon(r) \bar{L}(r))$. By Theorem 1.2.3, there are positive constants $a_0(\epsilon)$, $\sigma(\epsilon)$, and non-zero polynomials $P(\epsilon) \in \mathbb{R}[T]$, depending only on ϵ , $\bar{L}$ and X , such that

$$\lambda_{\max}(H^0(X, \mathbf{a} \cdot \epsilon \bar{L}), \|\cdot\|_{\sup}^{\mathbf{a} \cdot \epsilon \bar{L}}) \leq P(\epsilon)(\|\mathbf{a}\|_1) \cdot \sigma(\epsilon)^{\|\mathbf{a}\|_1}$$

holds for any non-negative $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$ with $\|\mathbf{a}\|_1 \geq a_0(\epsilon)$. We put

$$a_0 = \max\{a_0(\epsilon) \mid \epsilon \in \{\pm 1\}^r\}, \quad \sigma = \max\{\sigma(\epsilon) \mid \epsilon \in \{\pm 1\}^r\},$$

and we can find non-zero polynomial with non-negative real coefficients, $P \in \mathbb{R}[T]$, such that

$$P(t) \geq \max\{P(\epsilon)(t) \mid \epsilon \in \{\pm 1\}^r\},$$

for any non-negative real number $t \in \mathbb{R}_{\geq 0}$. Hence we conclude. $\square$

Remark 1.2.6. Theorems 1.2.3 and 1.2.5 hold also for C^0 -Hermitian invertible sheaves $\bar{L}$ by standard arguments as in §3.

1.3. Preliminary results. Let Y be a smooth projective algebraic variety over the rationals $\mathbb{Q}$ and $K := H^0(Y, \mathcal{O}_Y)$. Let A be an invertible sheaf on Y . The *base locus* $\text{Bs}(A)$ of A is a subset of Y defined as

$$\text{Bs}(A) := \{y \in Y \mid s(y) = 0 \text{ for all sections } s \in H^0(Y, A)\}$$

(cf. [14, I, Definition 2.1.20]). We can endow the base locus $\text{Bs}(A)$ with the scheme structure defined by the ideal sheaf

$$\mathfrak{b}(A) := \text{Im}(H^0(Y, A) \otimes_{\mathbb{Q}} A^{\otimes -1} \rightarrow \mathcal{O}_Y).$$

We call the base locus $\text{Bs}(A)$ with the above scheme structure the *base scheme* of A . Note that there is a natural surjection $H^0(Y, A) \otimes_{\mathbb{Q}} \mathcal{O}_Y \rightarrow H^0(Y, A) \otimes_K \mathcal{O}_Y$. We say that A is *free* if $\text{Bs}(A) = \emptyset$. The following lemma is easy.

- Lemma 1.3.1.** (1) *For any invertible sheaves A, A' on Y , $\text{Bs}(A + A') \subseteq \text{Bs}(A) \cup \text{Bs}(A')$. In particular, if A' is free and $A - A'$ is effective, then $\text{Bs}(A) \subseteq \text{Supp}(A - A')$.*
- (2) *Let Z be a closed algebraic subvariety of Y . Then $\text{Bs}(A|_Z) \subseteq \text{Bs}(A) \cap Z$.*
- (3) *Let $\pi : Y' \rightarrow Y$ be a birational morphism of smooth projective algebraic varieties. Then $\mathfrak{b}(\pi^*A) = \pi^*\mathfrak{b}(A)$.*

An invertible sheaf A on Y is said to be *very big* if, for a very ample invertible sheaf B on Y , $A - B$ is effective.

- Lemma 1.3.2.** (1) *If A is big on Y , then mA is very big on Y for a sufficiently large integer $m \geq 1$.*
- (2) *If A is very big on Y , then mA is very big on Y for any positive integer $m \geq 1$.*
- (3) *Let A be a very big invertible sheaf on Y . There is a proper Zariski-closed subset $S \subseteq Y$ having the property that, for any closed algebraic subvariety $Z \subseteq Y$ not contained in S , the restriction $A|_Z$ is very big on Z .*
- (4) *Suppose that A is very big and free on Y . Let $f : Y \rightarrow \mathbb{P}_K(H^0(Y, A))$ be the K -morphism defined by A . Then $\dim \overline{f(Y)} = \dim Y$.*

Proof. (1) Let B be a very ample invertible sheaf on Y . By Kodaira's lemma [14, I, Proposition 2.2.6], there exists a positive integer m with the property that $mA - B$ is effective.

(2) is clear.

(3) We can put $S := \text{Supp}(A - B)$. If a closed algebraic subvariety Z is not contained in S , then $(A - B)|_Z$ is effective and $B|_Z$ is very ample.

(4) We consider a commutative diagram,

$$\begin{array}{ccc} H^0(Y, A) \otimes_K \mathcal{O}_Y & \longrightarrow & A \\ \uparrow & & \uparrow \\ H^0(Y, B) \otimes_K \mathcal{O}_Y & \longrightarrow & B, \end{array}$$

where B is very ample on Y and the vertical two arrows are given by a non-zero global section of $A - B$. The homomorphism $H^0(Y, B) \rightarrow H^0(Y, A)$ yields a rational map $\text{pr} : \mathbb{P}_K(H^0(Y, A)) \dashrightarrow \mathbb{P}_K(H^0(Y, B))$. Since B is very ample, $\dim \overline{\text{pr}(f(Y))} = \dim Y$. Hence we conclude. $\square$

Proposition 1.3.3 (Bertini). *Suppose that A is very big and free on Y and let $f : Y \rightarrow \mathbb{P}_K(H^0(Y, A))$ be the K -morphism defined by A . Let $y_1, \dots, y_m$ be closed points on Y . Then, there is a Zariski-open subset U of the complete linear system $|A| := \mathbb{P}_K(H^0(Y, A)^\vee)$ with the following two properties:*

- (i) *If $\dim Y \geq 2$, $f^{-1}(H)$ is geometrically irreducible and smooth for any $H \in U$. If $\dim Y = 1$, $f^{-1}(H)$ is geometrically reduced for any $H \in U$.*
- (ii) *For any $H \in U$ and for $i = 1, \dots, m$, $y_i \notin f^{-1}(H)$.*

Proof. Since Y is smooth over $\mathbb{Q}$, $Y \otimes_{\text{Spec}(K)} \text{Spec}(\mathbb{C})$ is irreducible and smooth for any geometric point $\text{Spec}(\mathbb{C}) \rightarrow \text{Spec}(K)$. By Lemma 1.3.2(4), $\dim \overline{f(Y)} = \dim Y$. Thus, there is a Zariski-open subset of $|A|$ with the property (i) by virtue of the Bertini theorem [12, Corollaire 6.11, 2) and 3)]. For each $y_i \in Y$ with $1 \leq i \leq m$, all of the hyperplanes $H \in |A|$ with $y_i \in f^{-1}(H)$ are contained in a hyperplane of $|A|$. Hence we conclude. $\square$

1.4. Proof of Theorem 1.2.1. X is assumed to be generically smooth and normal. Let L and A be two invertible sheaves on X . Suppose that $A_{\mathbb{Q}}$ is big and free on $X_{\mathbb{Q}}$ and one of the following two conditions is satisfied:

- $L_{\mathbb{Q}}$ is non-big on $X_{\mathbb{Q}}$ (Case I).
- For a certain positive integer $a_0 > 0$, $a_0 L_{\mathbb{Q}} - A_{\mathbb{Q}}$ is effective and non-big on $X_{\mathbb{Q}}$ (Case II).

(The crucial case is Case II. Case I is easy.) Since $A_{\mathbb{Q}}$ is big, there is a positive integer $b_0 > 0$ (resp. $c_0 > 0$) such that $b_0 A_{\mathbb{Q}}$ is very big (resp. $c_0 A_{\mathbb{Q}} - L_{\mathbb{Q}}$ is effective). Let $Y \subseteq X$ be a reduced closed subscheme of X of dimension $d' \geq 1$

with the following six properties:

- (Y1) Y is either a generically smooth closed arithmetic subvariety of X of dimension $d' \geq 2$ or a finite union of closed horizontal arithmetic curves on X .
- (Y2) There is a section $t_1 \in H^0(X, A)$ that is non-zero on each irreducible component of Y .
- (Y3) In Case II, there is a section $t_2 \in H^0(X, a_0L - A)$ that is non-zero on each irreducible component of Y .
- (Y4) If $d' \geq 2$, the restriction $b_0A_{\mathbb{Q}}|_{Y_{\mathbb{Q}}}$ is very big and free on $Y_{\mathbb{Q}}$.
- (Y5) If $d' \geq 2$, the restriction $(c_0A_{\mathbb{Q}} - L_{\mathbb{Q}})|_{Y_{\mathbb{Q}}}$ is effective on $Y_{\mathbb{Q}}$.
- (Y6) If $d' \geq 2$, the restriction map $H^0(X, kb_0A)_{\mathbb{Q}} \rightarrow H^0(Y, kb_0A|_Y)_{\mathbb{Q}}$ is surjective for any integer k with $1 \leq k < 2^{d'-1}$.

We put $A_{d'} = 2^{d'-1}b_0A$, $d' \geq 1$, for simplicity.

Proposition 1.4.1. *Suppose that X is generically smooth. Let $s \in H^0(X, A_{d'})$ be any non-zero section of $A_{d'}$. Then, there are two positive constants m_0 , $\sigma(s)$ and two non-zero polynomials with non-negative real coefficients, $P, Q \in \mathbb{R}[T]$, of degree at most $d'(d'-1)/2$ and of degree at most d' respectively, depending only on $\bar{R}(X, L)$, $(A_{d'}, s)$, t_1 , t_2 , X , and Y , such that the inequalities:*

$$\begin{aligned} \exp(-Q(m+k)) \\ &\leq \lambda_{\min}(H^0(X|Y, mL - kA_{d'}), \|\cdot\|_{\text{sub}(s^{\otimes k}), \text{quot}}^{(m)}) \\ &\leq \lambda_{\max}(H^0(X|Y, mL - kA_{d'}), \|\cdot\|_{\text{sub}(s^{\otimes k}), \text{quot}}^{(m)}) \\ &\leq P(m+k) \cdot \sigma(s)^{m+k} \end{aligned}$$

hold for any non-negative integers $m, k \geq 0$ with $m+k \geq m_0$ (for the definition of $\|\cdot\|_{\text{sub}(s^{\otimes k}), \text{quot}}^{(m)}$, see Conventions and terminology, (5) and Lemma 1.1.5).

We start the proof of Proposition 1.4.1 with the following claim.

Claim 1.4.2. If Proposition 1.4.1 holds for a fixed non-zero section $s_0 \in H^0(X, A_{d'})$, then the assertion holds also for any non-zero section $s \in H^0(X, A_{d'})$.

Proof. If $L_{\mathbb{Q}}$ is non-big, then $H^0(X, mL - kA) = 0$ for any $k > 0$. Hence, the statement of Proposition 1.4.1 is clearly independent of the choice of s . Next, we consider the commutative diagram:

$$\begin{array}{ccc} H^0(X, mL) & \xrightarrow{\otimes (s_0 \otimes t'_2)^{\otimes k}} & H^0(X, (m + 2^{d'-1}b_0a_0k)L) \\ \otimes s^{\otimes k} \uparrow & & \uparrow \otimes (s \otimes t'_2)^{\otimes k} \\ H^0(X, mL - kA_{d'}) & \xrightarrow{\otimes s_0^{\otimes k}} & H^0(X, mL), \end{array}$$

where we put $t'_2 = t_2^{\otimes 2^{d'-1}b_0} \in H^0(X, 2^{d'-1}b_0a_0L - A_{d'})$. Since $\bar{R}(X, L)$ is of bounded type,

$$\begin{aligned} \tau_0(s_0 \otimes t'_2)^{m+k} \cdot \|u \otimes s^{\otimes k}\|^{(m)} \\ \leq \|u \otimes s_0^{\otimes k} \otimes s^{\otimes k} \otimes t'_2{}^{\otimes k}\|^{(m+2^{d'-1}b_0a_0k)} \\ \leq \sigma_0(s \otimes t'_2)^{m+k} \cdot \|u \otimes s_0^{\otimes k}\|^{(m)} \end{aligned}$$

and

$$\begin{aligned} \tau_0(s \otimes t'_2)^{m+k} \cdot \|u \otimes s_0^{\otimes k}\|^{(m)} \\ \leq \|u \otimes s_0^{\otimes k} \otimes s^{\otimes k} \otimes t'_2{}^{\otimes k}\|^{(m+2^{d'-1}b_0a_0k)} \\ \leq \sigma_0(s_0 \otimes t'_2)^{m+k} \cdot \|u \otimes s^{\otimes k}\|^{(m)} \end{aligned}$$

for any $u \in H^0(X, mL - kA_{d'})$. Thus

$$\begin{aligned} \left(\frac{\tau_0(s \otimes t'_2)}{\sigma_0(s_0 \otimes t'_2)} \right)^{m+k} \cdot \|u\|_{\text{sub}(s_0^{\otimes k}), \text{quot}}^{(m)} \\ \leq \|u\|_{\text{sub}(s^{\otimes k}), \text{quot}}^{(m)} \leq \left(\frac{\sigma_0(s \otimes t'_2)}{\tau_0(s_0 \otimes t'_2)} \right)^{m+k} \cdot \|u\|_{\text{sub}(s_0^{\otimes k}), \text{quot}}^{(m)} \end{aligned}$$

hold for any $u \in H^0(X|Y, mL - kA_{d'})$. □

Claim 1.4.3. If $d' \geq 2$, there is a non-zero section $s_0 \in H^0(X, A_{d'-1})$ with the following seven properties:

- (Y0') The section s_0 is non-zero on each irreducible component of Y . We denote the Weil divisor $\text{div}(s_0|_Y)$ on Y by Y^1 . The horizontal (resp. the vertical) part of Y^1 is denoted by Y_{hor}^1 (resp. Y_{ver}^1).
- (Y1') If $d' \geq 3$, Y_{hor}^1 is a generically smooth closed arithmetic subvariety of X . If $d' = 2$, Y_{hor}^1 is a finite union of reduced closed horizontal arithmetic curves on X .
- (Y2') The section $t_1 \in H^0(X, A)$ is non-zero on each horizontal irreducible component of Y^1 .
- (Y3') In Case II, the section $t_2 \in H^0(X, a_0L - A)$ is non-zero on each horizontal irreducible component of Y^1 .
- (Y4') If $d' \geq 3$, the restriction $b_0A_{\mathbb{Q}}|_{Y_{\mathbb{Q}}^1}$ is very big and free on $Y_{\mathbb{Q}}^1$.
- (Y5') If $d' \geq 3$, the restriction $(c_0A_{\mathbb{Q}} - L_{\mathbb{Q}})|_{Y_{\mathbb{Q}}^1}$ is effective on $Y_{\mathbb{Q}}^1$.
- (Y6') If $d' \geq 3$, the restriction map $H^0(X, kb_0A)_{\mathbb{Q}} \rightarrow H^0(Y^1, kb_0A|_{Y^1})_{\mathbb{Q}}$ is surjective for any integer k with $1 \leq k < 2^{d'-2}$.

Proof. Let $K_Y := H^0(Y, \mathcal{O}_Y)_{\mathbb{Q}}$. By the properties (Y4) and (Y6), $A_{d'-1, \mathbb{Q}}|_{Y_{\mathbb{Q}}}$ is very big and free on $Y_{\mathbb{Q}}$ and the homomorphism

$$H^0(X, A_{d'-1})_{\mathbb{Q}} \rightarrow H^0(Y, A_{d'-1}|_Y)_{\mathbb{Q}}$$

is surjective. The image of

$$\begin{array}{ccc} H^0(X, A_{d'-1}) & \rightarrow & |A_{d'-1, \mathbb{Q}}|_{Y_{\mathbb{Q}}} \\ s & \mapsto & \langle s|_{Y_{\mathbb{Q}}} \rangle_{K_Y} \end{array}$$

is not contained in a proper Zariski-closed subset of $|A_{d'-1, \mathbb{Q}}|_{Y_{\mathbb{Q}}}|$. We can take a non-zero section $s_0 \in H^0(X, A_{d'-1})$ with (Y0'), (Y1'), (Y2'), (Y3'), (Y4'), and (Y5') of the above properties by using Proposition 1.3.3 on $Y_{\mathbb{Q}}$.

Next, we consider the exact sequence

$$0 \rightarrow (k - 2^{d'-2})b_0A|_Y \xrightarrow{\otimes (s_0|_Y)} kb_0A|_Y \rightarrow kb_0A|_{Y^1} \rightarrow 0.$$

We recall that $Y_{\mathbb{Q}}$ is smooth if $d' \geq 2$. If $d' \geq 3$, we have

$$H^1(Y, (k - 2^{d'-2})b_0A|_Y)_{\mathbb{Q}} = 0$$

for any integer k with $1 \leq k < 2^{d'-2}$ by virtue of the Kawamata-Viehweg vanishing theorem [14, I, Theorem 4.3.1]. Thus the restriction

$$H^0(Y, kb_0A|_Y)_{\mathbb{Q}} \rightarrow H^0(Y^1, kb_0A|_{Y^1})_{\mathbb{Q}}$$

is surjective for $1 \leq k < 2^{d'-2}$. □

We prove Proposition 1.4.1 by induction on d' . The case where Y is a union of arithmetic curves on X is treated in Lemma 1.4.4 below. Thus we suppose $d' \geq 2$. If $k \geq \lceil c_0/2^{d'-2}b_0 \rceil m + 1$, then one knows that $H^0(X, mL - kA_{d'-1}) = 0$ since $(k-1)A_{d'-1} - mL$ is effective. We put $c'_0 = \lceil c_0/2^{d'-2}b_0 \rceil + 1$ and assume $k < c'_0m$. We consider the exact sequence

$$\begin{aligned} 0 \rightarrow m(L - c'_0A_{d'-1}) &\xrightarrow{\otimes s_0^{\otimes (c'_0m-k)}} mL - kA_{d'-1} \\ &\rightarrow (mL - kA_{d'-1})|_{(c'_0m-k)\operatorname{div}(s_0)} \rightarrow 0. \end{aligned}$$

Since $Y_{\mathbb{Q}}$ is reduced and $((c'_0 - 1)A_{d'-1, \mathbb{Q}} - L_{\mathbb{Q}})|_{Y_{\mathbb{Q}}}$ is effective by (Y4) and by (Y5), we obtain $H^0(Y, m(L - c'_0A_{d'-1})|_Y) = 0$ and an isomorphism

$$H^0(X|_Y, mL - kA_{d'-1}) \xrightarrow{\sim} H^0(X|(c'_0m - k)Y^1, mL - kA_{d'-1}).$$

By the induction hypothesis, Proposition 1.4.1 holds for Y^1_{hor} . Since

$$H^0(X|Y^1, mL - kA_{d'-1})_{\operatorname{free}} = H^0(X|Y^1_{\operatorname{hor}}, mL - kA_{d'-1}),$$

there are positive constants m_0 , $\sigma(s_0)$, and $\tau(s_0)$ and non-zero polynomials, $P', Q' \in \mathbb{R}[T]$, of degree at most $(d'-1)(d'-2)/2$ and of degree at most $d'-1$

respectively, all of which may depend on the choice of $s_0 \in H^0(X, A_{d'-1})$, such that the inequalities

$$\begin{aligned} \exp(-Q'(m+k)) &\leq \lambda_{\min}(H^0(X|Y^1, mL - kA_{d'-1}), \|\cdot\|_{\text{sub}(s_0^{\otimes k}), \text{quot}}^{(m)}) \\ &\leq \lambda_{\max}(H^0(X|Y^1, mL - kA_{d'-1}), \|\cdot\|_{\text{sub}(s_0^{\otimes k}), \text{quot}}^{(m)}) \\ &\leq P'(m+k) \cdot \sigma(s_0)^{m+k} \end{aligned}$$

hold for any non-negative integers $m, k \geq 0$ with $m+k \geq m_0$.

For a positive integer n , we have the divisor $nY^1 := \text{div}(s_0^{\otimes n}|_Y)$ on Y . According to [15, Claim 3.5.8], the commutative diagram:

$$\begin{array}{ccccccc} & & -nA_{d'-1}|_{\text{div}(s_0)} & & & & \\ & & \uparrow & & & & \\ 0 & \longrightarrow & -nA_{d'-1} & \xrightarrow{\otimes s_0^{\otimes n}} & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{n \text{ div}(s_0)} \longrightarrow 0 \\ & & \uparrow \otimes s_0 & & \parallel & & \uparrow \\ 0 & \longrightarrow & -(n+1)A_{d'-1} & \xrightarrow{\otimes s_0^{\otimes (n+1)}} & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{(n+1) \text{ div}(s_0)} \longrightarrow 0 \end{array}$$

yields an injective homomorphism $\alpha_n : -nA_{d'-1}|_{\text{div}(s_0)} \rightarrow \mathcal{O}_{(n+1) \text{ div}(s_0)}$ and a commutative diagram of exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & -nA_{d'-1}|_{\text{div}(s_0)} & \xrightarrow{\alpha_n} & \mathcal{O}_{(n+1) \text{ div}(s_0)} & \longrightarrow & \mathcal{O}_{n \text{ div}(s_0)} \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \parallel \\ 0 & \longrightarrow & -nA_{d'-1} & \xrightarrow{\otimes s_0^{\otimes n}} & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{n \text{ div}(s_0)} \longrightarrow 0. \end{array}$$

By tensoring the diagram with $mL - kA_{d'-1}$, we have

$$\begin{array}{ccccccc} 0 & \longrightarrow & (mL - (k+n)A_{d'-1})|_{\text{div}(s_0)} & \xrightarrow{\alpha_n} & mL - kA_{d'-1}|_{(n+1) \text{ div}(s_0)} & \longrightarrow & mL - kA_{d'-1}|_{n \text{ div}(s_0)} \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \parallel \\ 0 & \longrightarrow & mL - (k+n)A_{d'-1} & \xrightarrow{\otimes s_0^{\otimes n}} & mL - kA_{d'-1} & \longrightarrow & mL - kA_{d'-1}|_{n \text{ div}(s_0)} \longrightarrow 0 \end{array}$$

and an exact sequence:

$$\begin{aligned} 0 &\rightarrow H^0(X|Y^1, mL - (k+n)A_{d'-1}) \\ &\xrightarrow{\alpha_n} H^0(X|(n+1)Y^1, mL - kA_{d'-1}) \\ &\rightarrow H^0(X|nY^1, mL - kA_{d'-1}) \rightarrow 0. \end{aligned}$$

Consider a commutative diagram:

$$\begin{array}{ccc}
 H^0(X|Y^1, mL - (k+n)A_{d'-1}) & \xrightarrow{\alpha_n} & H^0(X|(n+1)Y^1, mL - kA_{d'-1}) \\
 \uparrow \text{quot} & & \uparrow \text{quot} \\
 H^0(X, mL - (k+n)A_{d'-1}) & \xrightarrow{\otimes s_0^{\otimes n}} & H^0(X, mL - kA_{d'-1}).
 \end{array}$$

Since the vertical two arrows have the same kernels, we have

$$\begin{aligned}
 & (H^0(X|Y^1, mL - (k+n)A_{d'-1}), \|\cdot\|_{\text{sub}(s_0^{\otimes k}), \text{quot}, \text{sub}(\alpha_n)}^{(m)}) \\
 &= (H^0(X|Y^1, mL - (k+n)A_{d'-1}), \|\cdot\|_{\text{sub}(s_0^{\otimes(k+n)}), \text{quot}}^{(m)})
 \end{aligned}$$

by using Lemma 1.1.5. Therefore,

$$\begin{aligned}
 & \lambda_{\max}(H^0(X|(n+1)Y^1, mL - kA_{d'-1}), \|\cdot\|_{\text{sub}(s_0^{\otimes k}), \text{quot}}^{(m)}) \\
 & \leq \text{rk } H^0(X|Y^1, mL - (k+n)A_{d'-1}) \\
 & \quad \cdot \lambda_{\max}(H^0(X|Y^1, mL - (k+n)A_{d'-1}), \|\cdot\|_{\text{sub}(s_0^{\otimes(k+n)}), \text{quot}}^{(m)}) \\
 & \quad + \lambda_{\max}(H^0(X|nY^1, mL - kA_{d'-1}), \|\cdot\|_{\text{sub}(s_0^{\otimes k}), \text{quot}}^{(m)})
 \end{aligned}$$

and

$$\begin{aligned}
 & \lambda_{\min}(H^0(X|(n+1)Y^1, mL - kA_{d'-1}), \|\cdot\|_{\text{sub}(s_0^{\otimes k}), \text{quot}}^{(m)}) \\
 & \geq \min \left\{ \frac{\lambda_{\min}(H^0(X|Y^1, mL - (k+n)A_{d'-1}), \|\cdot\|_{\text{sub}(s_0^{\otimes(k+n)}), \text{quot}}^{(m)})}{\# H^0(X|Y^1, mL - (k+n)A_{d'-1})_{\text{tor}}}, \right. \\
 & \quad \left. \lambda_{\min}(H^0(X|nY^1, mL - kA_{d'-1}), \|\cdot\|_{\text{sub}(s_0^{\otimes k}), \text{quot}}^{(m)}) \right\}
 \end{aligned}$$

by Lemma 1.1.3. Hence there are non-zero polynomials with non-negative real coefficients, $P, Q \in \mathbb{R}[T]$, of degree at most $d'(d'-1)/2$ and of degree at

most d' respectively such that

$$\begin{aligned}
 & \lambda_{\max}(\mathbf{H}^0(X|Y, mL - kA_{d'-1}), \|\cdot\|_{\substack{\text{sub}(s_0^{\otimes k}), \text{quot}}}^{(m)}) \\
 &= \lambda_{\max}(\mathbf{H}^0(X|(c'_0 m - k)Y^1, mL - kA_{d'-1}), \|\cdot\|_{\substack{\text{sub}(s_0^{\otimes k}), \text{quot}}}^{(m)}) \\
 &\leq \sum_{n=0}^{c'_0 m - k - 1} \text{rk } \mathbf{H}^0(X|Y^1, mL - (k+n)A_{d'-1}) \\
 &\quad \cdot \lambda_{\max}(\mathbf{H}^0(X|Y^1, mL - (k+n)A_{d'-1}), \|\cdot\|_{\substack{\text{sub}(s_0^{\otimes(k+n)}), \text{quot}}}^{(m)}) \\
 &\leq \text{rk } \mathbf{H}^0(Y_{\text{hor}}^1, (mL - kA_{d'-1})|_{Y_{\text{hor}}^1}) \\
 &\quad \cdot \sum_{n=0}^{c'_0 m - k - 1} \lambda_{\max}(\mathbf{H}^0(X|Y^1, mL - (k+n)A_{d'-1}), \|\cdot\|_{\substack{\text{sub}(s_0^{\otimes(k+n)}), \text{quot}}}^{(m)}) \\
 &\leq \text{rk } \mathbf{H}^0(Y_{\text{hor}}^1, (mL - kA_{d'-1})|_{Y_{\text{hor}}^1}) \cdot \sum_{n=0}^{c'_0 m - k - 1} P'(m+k+n) \cdot \sigma(s_0)^{m+k+n} \\
 &\leq P(m+k) \cdot (\max\{1, \sigma(s_0)\})^{(c'_0+1)m}
 \end{aligned}$$

and

$$\begin{aligned}
 & \lambda_{\min}(\mathbf{H}^0(X|Y, mL - kA_{d'-1}), \|\cdot\|_{\substack{\text{sub}(s_0^{\otimes k}), \text{quot}}}^{(m)}) \\
 &= \lambda_{\min}(\mathbf{H}^0(X|(c'_0 m - k)Y^1, mL - kA_{d'-1}), \|\cdot\|_{\substack{\text{sub}(s_0^{\otimes k}), \text{quot}}}^{(m)}) \\
 &\geq \min_{0 \leq n \leq c'_0 m - k - 1} \left\{ \frac{\lambda_{\min}(\mathbf{H}^0(X|Y^1, mL - (k+n)A_{d'-1}), \|\cdot\|_{\substack{\text{sub}(s_0^{\otimes(k+n)}), \text{quot}}}^{(m)})}{\# \mathbf{H}^0(X|Y^1, mL - (k+n)A_{d'-1})_{\text{tor}}} \right\} \\
 &\geq \frac{1}{\# \mathbf{H}^0(Y_{\text{ver}}^1, (mL - kA_{d'-1})|_{Y_{\text{ver}}^1})} \\
 &\quad \cdot \min_{0 \leq n \leq c'_0 m - k - 1} \left\{ \lambda_{\min}(\mathbf{H}^0(X|Y^1, mL - (k+n)A_{d'-1}), \|\cdot\|_{\substack{\text{sub}(s_0^{\otimes(k+n)}), \text{quot}}}^{(m)}) \right\} \\
 &\geq \frac{1}{\# \mathbf{H}^0(Y_{\text{ver}}^1, (mL - kA_{d'-1})|_{Y_{\text{ver}}^1})} \cdot \min_{0 \leq n \leq c'_0 m - k - 1} \{ \exp(-Q'(m+k+n)) \} \\
 &\geq (-Q(m+k))
 \end{aligned}$$

hold for any non-negative integers $m, k \geq 0$ with $m+k \geq m_0$. Since $s_0^{\otimes 2}$ is a non-zero section of $2A_{d'-1} = A_{d'}$, Proposition 1.4.1 holds for Y by use of Claim 1.4.2.

Finally, we treat the case where Y is a union of arithmetic curves and complete the proof of Proposition 1.4.1.

Lemma 1.4.4. *Let Y be a finite union of closed horizontal arithmetic curves on X . Then Proposition 1.4.1 holds for Y .*

Proof. (Case I) Since $L_{\mathbb{Q}}$ is non-big on $X_{\mathbb{Q}}$, $H^0(X|Y, mL - kA) = 0$ for any $k > 0$. We may assume $k = 0$ and the rest is easy (see the arguments below).

(Case II) Since $a_0L_{\mathbb{Q}} - A_{\mathbb{Q}}$ is non-big on $X_{\mathbb{Q}}$, $H^0(X|Y, mL - kA) = 0$ for $m < ka_0$. Thus we may assume $m \geq ka_0$ and write

$$mL - kA = pa_0L + q(a_0L - A) + rL,$$

where $p \geq 0$, $q \geq 0$, and $0 \leq r < a_0$ are integers. By the properties (Y2) and (Y3), there are sections $t_1 \in H^0(X|Y, A)$ and $t_2 \in H^0(X|Y, a_0L - A)$, both of which are non-zero on each irreducible component of Y . Note that $\text{rk } H^0(X|Y, mL - kA)$ is bounded from above and that there is an injective homomorphism

$$H^0(X|Y, mL - kA) \xrightarrow{\otimes t_1^{\otimes k}} H^0(X|Y, mL)$$

for each m, k . For each fixed integer r with $0 \leq r < a_0$, there is a non-negative integer $p_0(r) \geq 0$ such that

$$\text{rk } H^0(X|Y, p_0(r)a_0L + rL) = \max_p \{\text{rk } H^0(X|Y, pa_0L + rL)\}.$$

We put

$$p_0 := \max_{0 \leq r < a_0} \{p_0(r)\}.$$

For each fixed integer p, r with $0 \leq p < p_0$ and $0 \leq r < a_0$, there is a non-negative integer $q_0(p, r) \geq 0$ such that

$$\begin{aligned} \text{rk } H^0(X|Y, pa_0L + q_0(p, r)(a_0L - A) + rL) \\ = \max_q \{\text{rk } H^0(X|Y, pa_0L + q(a_0L - A) + rL)\} \end{aligned}$$

and we put

$$q_0 := \max_{\substack{0 \leq p < p_0 \\ 0 \leq r < a_0}} \{q_0(p, r)\}.$$

Claim 1.4.5. The $\mathbb{Q}$ -algebra $\bigoplus_{m, k \geq 0} H^0(X|Y, mL - kA)_{\mathbb{Q}}$ is generated by $t_1 \otimes t_2 \in H^0(X|Y, a_0L)_{\mathbb{Q}}$, $t_2 \in H^0(X|Y, a_0L - A)_{\mathbb{Q}}$, and a $\mathbb{Q}$ -basis for the finite dimensional $\mathbb{Q}$ -vector space

$$\bigoplus_{\substack{0 \leq p \leq p_0 \\ 0 \leq q \leq q_0 \\ 0 \leq r < a_0}} H^0(X|Y, pa_0L + q(a_0L - A) + rL)_{\mathbb{Q}}.$$

Proof. If $p \geq p_0$, then the $\mathbb{Q}$ -linear maps

$$\begin{aligned} H^0(X|Y, (p-1)a_0L + q(a_0L - A) + rL)_{\mathbb{Q}} \\ \xrightarrow{\otimes (t_1 \otimes t_2)} H^0(X|Y, pa_0L + q(a_0L - A) + rL)_{\mathbb{Q}}, \end{aligned}$$

where we assume $p > p_0$, and

$$H^0(X|Y, pa_0L + (q-1)(a_0L - A) + rL)_{\mathbb{Q}} \xrightarrow{\otimes t_2} H^0(X|Y, pa_0L + q(a_0L - A) + rL)_{\mathbb{Q}},$$

where we assume $q > 0$, are both isomorphisms for any r with $0 \leq r < a_0$. Thus

$$H^0(X|Y, p_0a_0L + rL)_{\mathbb{Q}} \xrightarrow{\sim} H^0(X|Y, pa_0L + q(a_0L - A) + rL)_{\mathbb{Q}}.$$

If $0 \leq p < p_0$ and $q > q_0$, the $\mathbb{Q}$ -linear map

$$H^0(X|Y, pa_0L + (q-1)(a_0L - A) + rL)_{\mathbb{Q}} \xrightarrow{\otimes t_2} H^0(X|Y, pa_0L + q(a_0L - A) + rL)_{\mathbb{Q}}$$

is an isomorphism for $0 \leq r < a_0$ so that

$$H^0(X|Y, pa_0L + q_0(a_0L - A) + rL)_{\mathbb{Q}} \xrightarrow{\sim} H^0(X|Y, pa_0L + q(a_0L - A) + rL)_{\mathbb{Q}}.$$

Hence we obtain the claim. $\square$

If $m + k = a_0p + (a_0 + 1)q + r \geq (a_0 + 1)(p_0 + q_0 + 1)$, then $p + q \geq p_0 + q_0$ and either $p \geq p_0$ or $q \geq q_0$ holds. By the proof of Claim 1.4.5, if $p \geq p_0$, we have an isomorphism

$$H^0(X|Y, pa_0L + q(a_0L - A) + rL)_{\mathbb{Q}} \xleftarrow{\otimes (t_1 \otimes t_2)^{\otimes (p-p_0)} \otimes t_2^{\otimes q}} H^0(X|Y, p_0a_0L + rL)_{\mathbb{Q}}$$

and, if $p < p_0$, an isomorphism

$$H^0(X|Y, pa_0L + q(a_0L - A) + rL)_{\mathbb{Q}} \xleftarrow{\otimes t_2^{\otimes (q-q_0)}} H^0(X|Y, pa_0L + q_0(a_0L - A) + rL)_{\mathbb{Q}}.$$

In other words, there are non-negative integers m_0, k_0, p', q' with $0 \leq m_0 \leq (p_0 + q_0 + 1)a_0$, $k_0 = 0$ or q_0 such that

$$H^0(X|Y, mL - kA)_{\mathbb{Q}} \xleftarrow{\otimes (t_1 \otimes t_2)^{\otimes p'} \otimes t_2^{\otimes q'}} H^0(X|Y, m_0L - k_0A)_{\mathbb{Q}}$$

is an isomorphism. Now, we put $l := \text{rk } H^0(X|Y, m_0L - k_0A)$. If $l = 0$, then the assertion of Lemma 1.4.4 is trivial. Suppose that $l \geq 1$ and let $u_1, \dots, u_l \in H^0(X|Y, m_0L - k_0A)$ be a $\mathbb{Q}$ -basis for $H^0(X|Y, m_0L - k_0A)_{\mathbb{Q}}$ with

$$\|u_i\|_{\text{sub}(t_1^{\otimes k_0}), \text{quot}}^{(m_0)} \leq \lambda_{\max}(H^0(X|Y, m_0L - k_0A), \|\cdot\|_{\text{sub}(t_1^{\otimes k_0}), \text{quot}}^{(m_0)})$$

for $i = 1, \dots, l$. Since the sections

$$\begin{aligned} v_1 &:= u_1 \otimes (t_1 \otimes t_2)^{\otimes p'} \otimes t_2^{\otimes q'}, \dots, v_l \\ &:= u_l \otimes (t_1 \otimes t_2)^{\otimes p'} \otimes t_2^{\otimes q'} \in H^0(X|Y, mL - kA) \end{aligned}$$

form a $\mathbb{Q}$ -basis for $H^0(X|Y, mL - kA)_{\mathbb{Q}}$, we have

$$\lambda_{\max}(H^0(X|Y, mL - kA), \|\cdot\|_{\text{sub}(t_1^{\otimes k}), \text{quot}}^{(m)}) \leq \max_{1 \leq i \leq l} \left\{ \|v_i\|_{\text{sub}(t_1^{\otimes k}), \text{quot}}^{(m)} \right\}$$

by definition of the last successive minimum. Since $\bar{R}(X, L)$ is of bounded type,

$$\begin{aligned} \|v_i\|_{\text{sub}(t_1^{\otimes k}), \text{quot}}^{(m)} &\leq \|u_i\|_{\text{sub}(t_1^{\otimes k_0}), \text{quot}}^{(m_0)} \cdot (\max\{1, \sigma_0(t_1 \otimes t_2)\})^{m+k} \\ &\leq \lambda_{\max}(H^0(X|Y, m_0L - k_0A), \|\cdot\|_{\text{sub}(t_1^{\otimes k_0}), \text{quot}}^{(m_0)}) \\ &\quad \cdot (\max\{1, \sigma_0(t_1 \otimes t_2)\})^{m+k} \end{aligned}$$

for $i = 1, \dots, l$. Hence, there is a positive constant $P > 0$ such that the inequality

$$\lambda_{\max}(H^0(X|Y, mL - kA), \|\cdot\|_{\text{sub}(t_1^{\otimes k}), \text{quot}}^{(m)}) \leq P \cdot (\max\{1, \sigma_0(t_1 \otimes t_2)\})^{m+k}$$

holds for $m + k \gg 0$.

Next, we let $Y = \bigcup_j Y_j$ be the irreducible decomposition of Y and $\tilde{Y}_j \rightarrow Y_j$ the normalization of Y_j . Since

$$\begin{aligned} \lambda_{\min}(H^0(X|Y, mL - kA), \|\cdot\|_{\text{sub}(t_1^{\otimes k}), \text{quot}}^{(m)}) \\ \geq \min_j \left\{ \lambda_{\min}(H^0(X|\tilde{Y}_j, mL - kA), \|\cdot\|_{\text{sub}(t_1^{\otimes k}), \text{quot}}^{(m)}) \right\}, \end{aligned}$$

we may assume that Y is irreducible and normal. Let $\text{Rat}(Y)$ be the rational function field of Y . By the ‘‘Hauptidealsatz’’ of Hilbert-Furtwängler, we may assume, further, that $\text{Rat}(Y)$ is a Galois extension of the rationals $\mathbb{Q}$ and the $\mathcal{O}_Y$ -module $L|_Y$ (resp. $A|_Y$) is generated by a single section.

Claim 1.4.6. There is a positive integer $\tau' > 0$, depending only on $t_1 \otimes t_2$, t_2 , and $H^0(X|Y, m_0L - k_0A)$, such that

$$\# \text{Coker} \left(H^0(X|Y, m_0L - k_0A) \xrightarrow{\otimes(t_1 \otimes t_2)^{\otimes p'} \otimes t_2^{\otimes q'}} H^0(X|Y, mL - kA) \right)$$

divides τ'^{m+k} .

Proof. We put

$$s := (t_1 \otimes t_2)|_Y, \quad t := t_2|_Y$$

and

$$s' := \bigotimes_{\substack{\gamma \in \text{Gal}(\text{Rat}(Y)/\mathbb{Q}) \\ \gamma \neq \text{id}}} s^\gamma, \quad t' := \bigotimes_{\substack{\gamma \in \text{Gal}(\text{Rat}(Y)/\mathbb{Q}) \\ \gamma \neq \text{id}}} t^\gamma$$

for simplicity. We consider a commutative diagram

$$\begin{array}{ccccc} (m_0L - k_0A)|_Y & \xrightarrow{\otimes s^{\otimes p'} \otimes t^{\otimes q'}} & (mL - kA)|_Y & \xrightarrow{\otimes s'^{\otimes p'} \otimes t'^{\otimes q'}} & (m_0L - k_0A)|_Y \\ \uparrow & & \uparrow & & \uparrow \\ H^0(X|Y, m_0L - k_0A) & \xrightarrow{\otimes s^{\otimes p'} \otimes t^{\otimes q'}} & H^0(X|Y, mL - kA) & \longrightarrow & \Gamma \end{array}$$

where Γ is the image of $H^0(X|Y, mL - kA)$ via $(mL - kA)|_Y \rightarrow (m_0L - k_0A)|_Y$. There is a $\mathbb{Z}$ -basis $\delta_1 e_1, \dots, \delta_l e_l$ for $H^0(X|Y, m_0L - k_0A)$ such that $\delta_i \in \mathbb{Z}$ for $i = 1, \dots, l$, $\delta_i | \delta_{i+1}$ for $i = 1, \dots, l-1$, and $e_1, \dots, e_l$ form part of a $\mathbb{Z}$ -basis for $(m_0L - k_0A)|_Y$. Let $\tau' := \text{Nm}(s) \cdot \text{Nm}(t) \in \mathbb{Z}$ be the norm. We recall that $p' + q' \leq m + k$. Since $\Gamma_{\mathbb{Q}} = H^0(X|Y, m_0L - k_0A)_{\mathbb{Q}}$ holds in $(m_0L - k_0A)|_Y \otimes_{\mathbb{Z}} \mathbb{Q}$, $e_1, \dots, e_l$ is a $\mathbb{Q}$ -basis for $\Gamma_{\mathbb{Q}}$ and there is a $\mathbb{Z}$ -basis $f_1, \dots, f_l$ for Γ such that $\delta_l \cdot \tau'^{m+k} \cdot f_i \in \text{Im}(H^0(X|Y, m_0L - k_0A) \rightarrow \Gamma)$ for $i = 1, \dots, l$. Hence we obtain

$$\begin{aligned} \# \text{Coker}(H^0(X|Y, m_0L - k_0A) \rightarrow H^0(X|Y, mL - kA)) \\ | \# \text{Coker}(H^0(X|Y, m_0L - k_0A) \rightarrow \Gamma) | |\delta_l|^l \cdot |\tau'|^{l(m+k)} \end{aligned}$$

as desired. $\square$

Let $v \in H^0(X|Y, mL - kA)$ be a non-zero section with

$$\|v\|_{\text{sub}(t_1^{\otimes k}), \text{quot}}^{(m)} = \lambda_{\min}(H^0(X|Y, mL - kA), \|\cdot\|_{\text{sub}(t_1^{\otimes k}), \text{quot}}^{(m)}).$$

There is a non-zero section $u \in H^0(X|Y, m_0L - k_0A)$ such that

$$\tau'^{m+k} v = u \otimes (t_1 \otimes t_2)^{\otimes p'} \otimes t_2^{\otimes q'}$$

holds in $H^0(X|Y, mL - kA)$. Since $\bar{R}(X, L)$ is of bounded type,

$$\begin{aligned} \|v\|_{\text{sub}(t_1^{\otimes k}), \text{quot}}^{(m)} &\geq \|u\|_{\text{sub}(t_1^{\otimes k_0}), \text{quot}}^{(m_0)} \cdot \tau'^{-(m+k)} \cdot (\min\{1, \tau_0(t_1 \otimes t_2)\})^{m+k} \\ &\geq \lambda_{\min}(H^0(X|Y, m_0L - k_0A), \|\cdot\|_{\text{sub}(t_1^{\otimes k_0}), \text{quot}}^{(m_0)}) \\ &\quad \cdot \left(\tau'^{-1} \cdot \min\{1, \tau_0(t_1 \otimes t_2)\}\right)^{m+k}. \end{aligned}$$

Hence, there is a positive constant $Q > 0$ such that the inequality

$$\begin{aligned} \lambda_{\min}(H^0(X|Y, mL - kA), \|\cdot\|_{\text{sub}(t_1^{\otimes k}), \text{quot}}^{(m)}) \\ \geq Q^{-1} \cdot \left(\tau'^{-1} \cdot \min\{1, \tau_0(t_1 \otimes t_2)\}\right)^{m+k} \end{aligned}$$

holds for $m+k \gg 0$. Since $t_1^{\otimes b_0}$ is a non-zero section of $A_1 = b_0A$, we conclude the proof by Claim 1.4.2. $\square$

For an invertible sheaf L on X , the *base locus* $\text{Bs}(L)$ (resp. the *stable base locus* $\mathbf{B}(L)$) of L is a subset of X defined as

$$\begin{aligned} \text{Bs}(L) &:= \{x \in X \mid s(x) = 0 \text{ for all sections } s \in H^0(X, L)\} \\ \left(\text{resp. } \mathbf{B}(L) &:= \left\{ x \in X \mid \begin{array}{l} s(x) = 0 \text{ for all sections } s \in H^0(X, mL) \\ \text{and for all } m \geq 1 \end{array} \right\} \right) \end{aligned}$$

(cf. [14, I, Definition 2.1.20]). We can endow the base locus $\text{Bs}(L)$ with the scheme structure defined by the ideal sheaf

$$\mathfrak{b}(L) := \text{Im}(\mathcal{H}^0(X, L) \otimes_{\mathbb{Z}} L^{\otimes -1} \rightarrow \mathcal{O}_X)$$

of $\mathcal{O}_X$. We call the base locus $\text{Bs}(L)$ with the above scheme structure the *base scheme* of L .

Proof of Theorem 1.2.1. We reduce Theorem 1.2.1 to Proposition 1.4.1. X is assumed to be generically smooth and normal. If $L_{\mathbb{Q}}$ is non-big on $X_{\mathbb{Q}}$, then we can apply Proposition 1.4.1 directly. Thus, we assume that $L_{\mathbb{Q}}$ is big on $X_{\mathbb{Q}}$. Let a_0 be a positive integer such that the base locus $\text{Bs}(a_0L)$ equals the stable base locus $\mathbf{B}(L)$ of L as subsets of X (cf. [14, I, Proposition 2.1.21]). Let $\mu : X' \rightarrow X$ be the blowing up with center the base scheme $\text{Bs}(a_0L)$ and decompose $a_0\mu^*L$ into its mobile part and its fixed part:

$$a_0\mu^*L = A + F.$$

Since $\mathfrak{b}(a_0L_{\mathbb{Q}}) = \mathfrak{b}(a_0L)_{\mathbb{Q}}$, $A_{\mathbb{Q}}$ is big and free and $F_{\mathbb{Q}}$ is effective on $X'_{\mathbb{Q}}$. We may assume without loss of generality that X' is generically smooth and normal.

Claim 1.4.7. $F_{\mathbb{Q}}$ is not big on $X'_{\mathbb{Q}}$.

Proof. Assume that $F_{\mathbb{Q}}$ is big on $X'_{\mathbb{Q}}$. We can take a blow up $\nu : X'' \rightarrow X'$ again, where X'' is smooth over $\text{Spec}(\mathbb{Q})$. There is a positive integer $n_0 \geq 1$ such that the mobile part of $n_0\nu^*F_{\mathbb{Q}}$ is big and free. We decompose $n_0\nu^*F_{\mathbb{Q}}$ into its mobile part A' and its fixed part F' on X'' :

$$n_0\nu^*F_{\mathbb{Q}} = A' + F'.$$

Since $n_0a_0\nu^*\mu^*L_{\mathbb{Q}} = (n_0\nu^*A_{\mathbb{Q}} + A') + F'$, $\text{Supp}(F') = \text{Bs}(n_0a_0\nu^*\mu^*L_{\mathbb{Q}}) = \nu^{-1}\text{Supp}(F_{\mathbb{Q}})$. Hence we can write

$$\begin{aligned} n_0\nu^*F_{\mathbb{Q}} &= \sum_j p_j F_j, \\ \sum_j q_j F_j &= A' + \sum_j r_j F_j, \end{aligned}$$

where F_j 's are prime Weil divisors on X'' , p_j 's are positive integers, and q_j 's and r_j 's are non-negative integers such that, if $q_j > 0$, then $r_j = 0$. Since A' is non-trivial, $q_j > 0$ for at least one j and we put $q/p := \max_j \{q_j/p_j\} > 0$. Then

$$(qn_0)a_0\nu^*\mu^*L_{\mathbb{Q}} = (qn_0\nu^*A + pA') + \sum_j (qp_j - pq_j)F_j + \sum_j pr_jF_j.$$

We note that $qp_j - pq_j \geq 0$ holds for any j and, if $qp_j - pq_j = 0$, then $q_j > 0$ and $r_j = 0$. Thus, we conclude that $\text{Bs}((qn_0)a_0\nu^*\mu^*L_{\mathbb{Q}}) \subsetneq \nu^{-1}\text{Bs}(a_0\mu^*L_{\mathbb{Q}})$, which

contradicts the fact that $\text{Bs}((qn_0)a_0L_{\mathbb{Q}}) = \text{Bs}(a_0L_{\mathbb{Q}})$ on $X_{\mathbb{Q}}$. In particular, A' must be trivial and the fixed part of $n_0\nu^*F_{\mathbb{Q}}$ equals $n_0\nu^*F_{\mathbb{Q}}$. $\square$

Since X is normal, we have $\mu_*\mathcal{O}_{X'} = \mathcal{O}_X$ and a natural isomorphism

$$H^0(X, mL) \xrightarrow{\sim} H^0(X', m\mu^*L).$$

Let us consider the normed graded $\mathbb{Z}$ -module

$$\bar{R}(X', \mu^*L) := \bigoplus_m (H^0(X', m\mu^*L), \|\cdot\|^{(m)}),$$

where the norms $\|\cdot\|^{(m)}$ are induced via the above isomorphisms. Since $\bar{R}(X, L)$ is of bounded type, $\bar{R}(X', \mu^*L)$ is also of bounded type. Hence, we apply Proposition 1.4.1 to $\bar{R}(X', \mu^*L)$ and obtain Theorem 1.2.1. $\square$

Remark 1.4.8. The statement of Theorem 1.2.1 remains true if we replace the supposition that X is generically smooth and normal with one of the following two suppositions:

- (i) Let $\mu : X' \rightarrow X$ be a generic resolution of singularities of the blow-up with center $\mathbf{B}(L)$ endowed with a suitable scheme structure as above. Then the natural homomorphisms

$$H^0(X, mL) \rightarrow H^0(X', m\mu^*L)$$

are isomorphic for all sufficiently large integers $m \gg 0$.

- (ii) X is generically smooth and $L_{\mathbb{Q}}$ is non-big on $X_{\mathbb{Q}}$.

2. Asymptotics of the arithmetic Euler characteristics

2.1. Multi-indexed version of the fundamental estimate. In this subsection, X is always assumed to be generically smooth and normal. We fix a volume form Φ_X on X_{∞} and consider the L^2 -norms with respect to Φ_X . We establish the multi-indexed version of the fundamental estimate for the arithmetic Euler characteristics (cf. [16, Theorem 1.1.1]).

Theorem 2.1.1. *Suppose that X is generically smooth and normal. Let $r \geq 1$ be a positive integer. Let $\bar{\mathbf{L}} = (\bar{\mathbf{L}}(1), \dots, \bar{\mathbf{L}}(r))$ be a family of C^{∞} -Hermitian invertible sheaves on X . Let $\bar{A}$ be a C^{∞} -Hermitian invertible sheaf on X . There are three positive constants a_0 , C , and D , depending only on $\bar{\mathbf{L}}$, $\bar{A}$, and X , such that the inequality*

$$\begin{aligned} |\hat{\chi}(H^0(X, \mathbf{a} \cdot \mathbf{L} + bA), \|\cdot\|_{\sup}^{\mathbf{a} \cdot \bar{\mathbf{L}} + b\bar{A}}) - \hat{\chi}(H^0(X, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{\sup}^{\mathbf{a} \cdot \bar{\mathbf{L}}})| \\ \leq C \|\mathbf{a}\|_1^{d-1} |b| + D \|\mathbf{a}\|_1^{d-1} \log \|\mathbf{a}\|_1 \end{aligned}$$

holds for any $\mathbf{a} \in \mathbb{Z}^r$ and for any $b \in \mathbb{Z}$ with $\|\mathbf{a}\|_1 \geq \max\{a_0, |b|\}$.

By the same arguments as in §1.2, Theorem 2.1.1 follows from the theorem below.

Theorem 2.1.2. *Suppose that X is generically smooth and normal and fix a volume form Φ_X on X_∞ . Then, there are three positive constants a_1 , C_1 and D_1 , depending only on $\bar{L}$, $\bar{A}$, and (X, Φ_X) , such that the inequality*

$$\begin{aligned} |\hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L} + bA), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{L} + b\bar{A}}) - \hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{L}})| \\ \leq C_1 \|\mathbf{a}\|_1^{d-1} b + D_1 \|\mathbf{a}\|_1^{d-1} \log \|\mathbf{a}\|_1 \end{aligned}$$

holds for any non-negative integers $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$ and $b \in \mathbb{Z}_{>0}$ with $\|\mathbf{a}\|_1 \geq \max\{a_1, b\}$.

Proof of Theorem 2.1.2 $\Rightarrow$ Theorem 2.1.1. First, by [15, Corollary 1.1.2], there is a positive constant D such that

$$\|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{L}} \leq \|\cdot\|_{\sup}^{\mathbf{a} \cdot \bar{L}} \leq D(1 + \|\mathbf{a}\|_1)^{d-1} \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{L}}$$

hold for $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$. Thus we may replace L^2 with $\sup$ in Theorem 2.1.2.

We may assume $b \neq 0$ in Theorem 2.1.1 since, if $b = 0$, then the assertion of Theorem 2.1.1 is trivial. For any $\epsilon \in \{\pm 1\}^r$, we put $\epsilon \bar{L} = (\epsilon(1)\bar{L}(1), \dots, \epsilon(r)\bar{L}(r))$. By Theorem 2.1.2, there are positive constants $a_1(\epsilon)$, $C_1(\epsilon)$, and $D_1(\epsilon)$, depending on ϵ and $\bar{L}$, such that the inequality

$$\begin{aligned} |\hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \epsilon \mathbf{L} \pm bA), \|\cdot\|_{\sup}^{\mathbf{a} \cdot \epsilon \bar{L} + b\bar{A}}) - \hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \epsilon \mathbf{L}), \|\cdot\|_{\sup}^{\mathbf{a} \cdot \epsilon \bar{L}})| \\ \leq C_1(\epsilon) \|\mathbf{a}\|_1^{d-1} b + D_1(\epsilon) \|\mathbf{a}\|_1^{d-1} \log \|\mathbf{a}\|_1 \end{aligned}$$

holds for $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$ and $b \in \mathbb{Z}_{>0}$ with $\|\mathbf{a}\|_1 \geq \max\{a_1(\epsilon), b\}$. We put

$$\begin{aligned} a_0 &= \max\{a_1(\epsilon) \mid \epsilon \in \{\pm 1\}^r\}, \\ C &= \max\{C_1(\epsilon) \mid \epsilon \in \{\pm 1\}^r\}, \\ D &= \min\{D_1(\epsilon) \mid \epsilon \in \{\pm 1\}^r\}. \end{aligned}$$

Then, Theorem 2.1.1 holds. □

2.2. Proof of Theorem 2.1.2. We prove Theorem 2.1.2 in this subsection. We keep the notation in the previous subsection. Suppose that X is generically smooth and normal and fix a volume form Φ_X on X_∞ . If $d = 1$, the assertion of Theorem 2.1.2 is trivial, so that we may assume $d \geq 2$.

Theorem 2.2.1. *Let $\bar{A}$ be an effective C^∞ -Hermitian invertible sheaf on X (cf. Conventions and terminology, (8)). Then there are three positive constants a_0 , C , and D , depending only on $\bar{L}$, $\bar{A}$, and (X, Φ_X) , such that the inequality*

$$\begin{aligned} \hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L} + bA), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{L} + b\bar{A}}) - \hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{L}}) \\ \geq C\|\mathbf{a}\|_1^{d-1}b + D\|\mathbf{a}\|_1^{d-1} \log \|\mathbf{a}\|_1 \end{aligned}$$

holds for $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$ and $b \in \mathbb{Z}_{>0}$ with $\|\mathbf{a}\|_1 \geq \max\{a_0, b\}$.

Proof of Theorem 2.2.1. Let $s \in \mathrm{H}^0(X, A)$ be a non-zero small section and consider the injective homomorphism

$$\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L}) \xrightarrow{\otimes s^{\otimes b}} \mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L} + bA).$$

Since s is small,

$$\|\cdot\|_{L^2, \text{sub}(s^{\otimes b})}^{\mathbf{a} \cdot \bar{L} + b\bar{A}} \leq \left(\|s\|_{\text{sup}}^{\bar{A}}\right)^b \cdot \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{L}} \leq \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{L}}$$

on $\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L})$. By the classical Hilbert-Samuel formula on $X_{\mathbb{Q}}$,

$$\mathrm{rk} \mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L} + bA) = \frac{\deg(c_1(\mathbf{a} \cdot \mathbf{L}_{\mathbb{Q}} + bA_{\mathbb{Q}})^{\cdot(d-1)})}{(d-1)!} + O(\|\mathbf{a}\|_1^{d-2})$$

and

$$\mathrm{rk} \mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L}) = \frac{\deg(c_1(\mathbf{a} \cdot \mathbf{L}_{\mathbb{Q}})^{\cdot(d-1)})}{(d-1)!} + O(\|\mathbf{a}\|_1^{d-2})$$

for any $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$ and for any $b \in \mathbb{Z}_{>0}$ with $\|\mathbf{a}\|_1 \geq b$ and $\|\mathbf{a}\|_1 \gg 0$. Thus, there is a positive constant C depending on $\mathbf{L}_{\mathbb{Q}}$, $A_{\mathbb{Q}}$, and $X_{\mathbb{Q}}$ such that the inequality

$$|\mathrm{rk} \mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L} + bA) - \mathrm{rk} \mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L})| \leq C\|\mathbf{a}\|_1^{d-2}b + O(\|\mathbf{a}\|_1^{d-2})$$

holds for $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$ and $b \in \mathbb{Z}_{>0}$ with $\|\mathbf{a}\|_1 \geq b$ and $\|\mathbf{a}\|_1 \gg 0$.

Consequently, by Theorem 1.2.5 and Lemma 1.1.2(3), there are positive constants a_0 and C' such that

$$\begin{aligned} & \hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L} + bA), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{L} + b\bar{A}}) - \hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{L}}) \\ & \geq \hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L} + bA), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{L} + b\bar{A}}) - \hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{L^2, \text{sub}(s^{\otimes b})}^{\mathbf{a} \cdot \bar{L} + b\bar{A}}) \\ & \geq -(\mathrm{rk} \mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L} + bA) - \mathrm{rk} \mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L})) \\ & \quad \cdot \log \lambda_{\max}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L} + bA), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{L} + b\bar{A}}) + O(\|\mathbf{a}\|_1^{d-1} \log \|\mathbf{a}\|_1) \\ & \geq -C'\|\mathbf{a}\|_1^{d-1}b + O(\|\mathbf{a}\|_1^{d-1} \log \|\mathbf{a}\|_1) \end{aligned}$$

for $\|\mathbf{a}\|_1 \geq \max\{a_0, b\}$. □

Theorem 2.2.2. *Let $\bar{A}$ be a C^∞ -Hermitian invertible sheaf on X . Then there are three positive constants a_0 , C and D , depending only on $\bar{L}$, $\bar{A}$, and (X, Φ_X) , such that*

$$\begin{aligned} \hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L} + bA), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{L} + b\bar{A}}) - \hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{L}}) \\ \leq C\|\mathbf{a}\|_1^{d-1}b + D\|\mathbf{a}\|_1^{d-1} \log \|\mathbf{a}\|_1 \end{aligned}$$

holds for $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$ and $b \in \mathbb{Z}_{>0}$ with $\|\mathbf{a}\|_1 \geq \max\{a_0, b\}$.

We start the proof of Theorem 2.2.2 with the following claim.

Claim 2.2.3. Let $\bar{A}'$ be another C^∞ -Hermitian invertible sheaf on X such that $\bar{A}' - \bar{A}$ is effective. If Theorem 2.2.2 holds for $\bar{L}$ and $\bar{A}'$, then it also holds for $\bar{L}$ and $\bar{A}$.

Proof. This is clear since, by Theorem 2.2.1, we know that there are positive constants C' and D' with

$$\begin{aligned} \hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L} + bA), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{L} + b\bar{A}}) \\ \leq \hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L} + bA'), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{L} + b\bar{A}'} + C'\|\mathbf{a}\|_1^{d-1}b + D'\|\mathbf{a}\|_1^{d-1} \log \|\mathbf{a}\|_1 \end{aligned}$$

for $\|\mathbf{a}\|_1 \geq b$ and $\|\mathbf{a}\|_1 \gg 0$. Thus

$$\begin{aligned} \hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L} + bA), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{L} + b\bar{A}}) - \hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{L}}) \\ \leq \hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L} + bA'), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{L} + b\bar{A}'} - \hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{L}}) \\ + C'\|\mathbf{a}\|_1^{d-1}b + D'\|\mathbf{a}\|_1^{d-1} \log \|\mathbf{a}\|_1 \end{aligned}$$

for $\|\mathbf{a}\|_1 \geq b$ and $\|\mathbf{a}\|_1 \gg 0$. $\square$

Therefore, we may assume that the following five conditions are satisfied [16, Claim 1.3.2.4]:

- (i) A is ample on X and $A_{\mathbb{Q}}$ is very ample on $X_{\mathbb{Q}}$.
- (ii) $\mathbf{L}(i)_{\mathbb{Q}} + A_{\mathbb{Q}}$ is ample on $X_{\mathbb{Q}}$ for $i = 1, \dots, r$.
- (iii) The first Chern forms $c_1(\bar{A})$ and $c_1(\bar{L}(i) + \bar{A})$ for $i = 1, \dots, r$ are positive pointwise on X_∞ .
- (iv) There is a non-zero small section $s \in \mathrm{H}^0(X, A)$ such that $\mathrm{div}(s)$ is generically smooth.
- (v) There are a positive integer $n \geq 1$ and non-zero small sections $t_i \in \mathrm{H}^0(X, nA - \mathbf{L}(i))$, $i = 1, \dots, r$, such that t_i is non-zero on each irreducible component of $\mathrm{div}(s)$ for $i = 1, \dots, r$.

In fact, there is an ample C^∞ -Hermitian invertible sheaf $\bar{A}'$ such that A' is very ample on X , $\bar{A}' \geq \bar{A}$, and $\bar{L}(i) + \bar{A}'$ are all ample for $i = 1, \dots, r$. Since A' is very ample, there is a non-zero section $s \in \mathrm{H}^0(X, A')$ such that $\mathrm{div}(s)$ is generically smooth. Moreover, we can find a positive integer $n \geq 1$ and non-zero sections $t_i \in \mathrm{H}^0(X, nA' - \mathbf{L}(i))$, $i = 1, \dots, r$, such that t_i is non-zero

on each irreducible component of $\operatorname{div}(s)$ for $i = 1, \dots, r$. Let $\lambda \geq 0$ be a non-negative real number such that $e^{-\lambda} \cdot \|s\|_{\sup}^{\bar{A}'} \leq 1$ and $e^{-n\lambda} \cdot \|t_i\|_{\sup}^{n\bar{A}' - \bar{L}(i)} \leq 1$ for $i = 1, \dots, r$. Then the ample C^∞ -Hermitian invertible sheaf $\bar{A}'' := (\bar{A}')^\lambda$ satisfies the above five conditions and $\bar{A}'' \geq \bar{A}$.

Moreover, by the same argument as in [16, Claim 1.3.2.3], we may assume the following condition.

- (vi) Φ_X is given by the *normalized* volume form associated with the positive $(1, 1)$ -form $c_1(\bar{L}(1) + \dots + \bar{L}(r) + r\bar{A})$: i.e.

$$\Phi_X = \frac{c_1(\bar{L}(1) + \dots + \bar{L}(r) + r\bar{A})^{\wedge(d-1)}}{\int_{X_\infty} c_1(\bar{L}(1) + \dots + \bar{L}(r) + r\bar{A})^{\wedge(d-1)}}.$$

We put $X^1 = \operatorname{div}(s)$ and consider the exact sequence

$$0 \rightarrow H^0(X, \mathbf{a} \cdot \mathbf{L}) \xrightarrow{\otimes s^{\otimes b}} H^0(X, \mathbf{a} \cdot \mathbf{L} + bA) \rightarrow H^0(X|bX^1, \mathbf{a} \cdot \mathbf{L} + bA) \rightarrow 0.$$

Since the arithmetic degree is additive, we can take positive constants a_1 and C_1 with the property that

$$\begin{aligned} & |\hat{\chi}(H^0(X, \mathbf{a} \cdot \mathbf{L} + bA), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{L} + b\bar{A}}) - \hat{\chi}(H^0(X, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{L^2, \operatorname{sub}(s^{\otimes b})}^{\mathbf{a} \cdot \bar{L} + b\bar{A}}) \\ & - \hat{\chi}(H^0(X|bX^1, \mathbf{a} \cdot \mathbf{L} + bA), \|\cdot\|_{L^2, \operatorname{quot}}^{\mathbf{a} \cdot \bar{L} + b\bar{A}})| \leq C_1 \|\mathbf{a}\|_1^{d-1} \log \|\mathbf{a}\|_1 \end{aligned}$$

for any $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$ with $\|\mathbf{a}\|_1 \geq \max\{a_1, b\}$. Now we need the following two lemmas: Lemmas 2.2.4 and 2.2.5. The first lemma is nothing but [16, Corollary 1.2.2].

Lemma 2.2.4. *There are positive constants a_2 and C_2 , depending only on $\bar{L}$, $\bar{A}$, and (X, Φ_X) , such that*

$$\hat{\chi}(H^0(X, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{L^2, \operatorname{sub}(s^{\otimes b})}^{\mathbf{a} \cdot \bar{L} + b\bar{A}}) - \hat{\chi}(H^0(X, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{L}}) \leq C_2 \|\mathbf{a}\|_1^{d-1} b$$

hold for $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$ and $b \in \mathbb{Z}_{>0}$ with $\|\mathbf{a}\|_1 \geq \max\{a_2, b\}$.

The second lemma is an easy corollary of [16, Lemma 1.3.4].

Lemma 2.2.5. *There are positive constants a_3 , C_3 and C'_3 , depending only on $\mathbf{L}$, $\bar{A}$, and (X, Φ_X) , such that*

$$\hat{\chi}(H^0(X|bX^1, \mathbf{a} \cdot \mathbf{L} + bA), \|\cdot\|_{L^2, \operatorname{quot}}^{\mathbf{a} \cdot \bar{L} + b\bar{A}}) \leq C_3 \|\mathbf{a}\|^{d-1} b + C'_3 \|\mathbf{a}\|_1^{d-1} \log \|\mathbf{a}\|_1$$

holds for any $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$ and $b \in \mathbb{Z}_{>0}$ with $\|\mathbf{a}\|_1 \geq \max\{a_3, b\}$.

Proof of Lemma 2.2.5. By Lemma 1.1.1,

$$\begin{aligned} & \hat{\chi}(H^0(X|bX^1, \mathbf{a} \cdot \mathbf{L} + bA), \|\cdot\|_{L^2, \operatorname{quot}}^{\mathbf{a} \cdot \bar{L} + b\bar{A}}) \\ & \leq \hat{h}^0(H^0(X|bX^1, \mathbf{a} \cdot \mathbf{L} + bA), \|\cdot\|_{L^2, \operatorname{quot}}^{\mathbf{a} \cdot \bar{L} + b\bar{A}}) + O(\|\mathbf{a}\|_1^{d-1}) \end{aligned}$$

for $\|\mathbf{a}\|_1 \gg 0$. Then it is easy to see that we can apply [16, Lemma 1.3.4] to our situation. $\square$

Proof of Theorem 2.2.2. Combining these lemmas, we conclude the proof of Theorem 2.2.2. By Lemmas 2.2.4 and 2.2.5, we obtain

$$\begin{aligned} & \hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L} + bA), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{\mathbf{L}} + b\bar{A}}) - \hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{\mathbf{L}}}) \\ & \leq \hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{L^2, \mathrm{sub}(s \otimes b)}^{\mathbf{a} \cdot \bar{\mathbf{L}} + b\bar{A}}) - \hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{\mathbf{L}}}) \\ & \quad + \hat{\chi}(\mathrm{H}^0(X|bX^1, \mathbf{a} \cdot \mathbf{L} + bA), \|\cdot\|_{L^2, \mathrm{quot}}^{\mathbf{a} \cdot \bar{\mathbf{L}} + b\bar{A}}) + C_1 \|\mathbf{a}\|_1^{d-1} \log \|\mathbf{a}\|_1 \\ & \leq (C_2 + C_3) \|\mathbf{a}\|_1^{d-1} b + (C_1 + C'_3) \|\mathbf{a}\|_1^{d-1} \log \|\mathbf{a}\|_1 \end{aligned}$$

for $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$ and $b \in \mathbb{Z}_{>0}$ with $\|\mathbf{a}\|_1 \geq \max\{a_1, a_2, a_3, b\}$. $\square$

Proof of Theorem 2.1.2. We can write $\bar{A} = \bar{A}' - \bar{A}''$ with two effective C^∞ -Hermitian invertible sheaves, $\bar{A}'$ and $\bar{A}''$, on X . Then by Theorems 2.2.1 and 2.2.2, there are positive constants a_0, C, D, C' , and D' such that

$$\begin{aligned} & |\hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L} + bA'), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{\mathbf{L}} + b\bar{A}'} - \hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{\mathbf{L}}})| \\ & \leq C \|\mathbf{a}\|_1^{d-1} b + D \|\mathbf{a}\|_1^{d-1} \log \|\mathbf{a}\|_1 \end{aligned}$$

and

$$\begin{aligned} & |\hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L} + bA), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{\mathbf{L}} + b\bar{A}}) - \hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L} + bA'), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{\mathbf{L}} + b\bar{A}'})| \\ & \leq C' \|\mathbf{a}\|_1^{d-1} b + D' \|\mathbf{a}\|_1^{d-1} \log \|\mathbf{a}\|_1 \end{aligned}$$

hold for $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$ and $b \in \mathbb{Z}_{>0}$ with $\|\mathbf{a}\|_1 \geq \max\{a_0, b\}$. Thus we obtain

$$\begin{aligned} & |\hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L} + bA), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{\mathbf{L}} + b\bar{A}}) - \hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{\mathbf{L}}})| \\ & \leq |\hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L} + bA), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{\mathbf{L}} + b\bar{A}}) \\ & \quad - \hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L} + bA'), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{\mathbf{L}} + b\bar{A}'})| \\ & \quad + |\hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L} + bA'), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{\mathbf{L}} + b\bar{A}'} - \hat{\chi}(\mathrm{H}^0(X, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{L^2}^{\mathbf{a} \cdot \bar{\mathbf{L}}})| \\ & \leq (C + C') \|\mathbf{a}\|_1^{d-1} b + (D + D') \|\mathbf{a}\|_1^{d-1} \log \|\mathbf{a}\|_1 \end{aligned}$$

as desired. $\square$

Remark 2.2.6. Let X be a generically smooth normal projective arithmetic variety of dimension $d \geq 2$. If $d = 2$, we assume further that X is regular. If $r = 1$, we can determine the constant C in Theorem 2.2.2. Actually, Yuan proves the following theorem [22, Proposition 2.14 and Theorem 2.2].

Theorem 2.2.7 (Yuan). *Let $\bar{L}, \bar{A}$ be two C^∞ -Hermitian invertible sheaves on X . Suppose that $\bar{A}$ and $\bar{L} + \bar{A}$ are ample and let Φ_X be the normalized volume form on X_∞ associated with the positive $(1, 1)$ -form $c_1(\bar{L} + \bar{A})$. Then,*

there is a positive integer $p \geq 1$ such that

$$\begin{aligned} \hat{\chi}(\mathrm{H}^0(X, apL + bpA), \|\cdot\|_{L^2}^{ap\bar{L}+bp\bar{A}}) - \hat{\chi}(\mathrm{H}^0(X, apL), \|\cdot\|_{L^2}^{ap\bar{L}}) \\ \leq \frac{\widehat{\deg}(\hat{c}_1(\bar{L} + \bar{A})^{(d-1)} \cdot \hat{c}_1(\bar{A}))}{(d-1)!} p^d a^{d-1} b + O(a^{d-1} \log a) \end{aligned}$$

for any $a, b \in \mathbb{Z}_{>0}$ with $a \geq b$ and $a \gg 0$.

Corollary 2.2.8 (Yuan). *Under the assumptions in Theorem 2.2.7. There is a positive integer $p \geq 1$ such that*

$$\begin{aligned} \hat{\chi}(\mathrm{H}^0(X, mpL), \|\cdot\|_{L^2}^{mp\bar{L}}) \\ \geq \frac{\widehat{\deg}(\hat{c}_1(\bar{L} + \bar{A})^d) - d \widehat{\deg}(\hat{c}_1(\bar{L} + \bar{A})^{(d-1)} \cdot \hat{c}_1(\bar{A}))}{d!} (pm)^d \\ + O(m^{d-1} \log m) \end{aligned}$$

for any integer $m \gg 0$.

3. Continuity of the arithmetic $\hat{\chi}$ -volume function

3.1. Birational invariance. In this subsection, we prove that the arithmetic Euler characteristics are asymptotically birationally invariant.

Theorem 3.1.1. *Let $\pi : X' \rightarrow X$ be a birational morphism of projective arithmetic varieties. Let $\bar{L}$ and $\bar{E}$ be two C^∞ -Hermitian invertible sheaves on X .*

- (1) *There are positive constants m_0 and C , depending only on $\bar{L}$, $\bar{E}$, and π , such that the inequality*

$$\begin{aligned} |\hat{\chi}(\mathrm{H}^0(X', \pi^*(mL + E)), \|\cdot\|_{\sup}^{\pi^*(m\bar{L} + \bar{E})}) - \hat{\chi}(\mathrm{H}^0(X, mL + E), \|\cdot\|_{\sup}^{m\bar{L} + \bar{E}})| \\ \leq Cm^{d-1} \log m \end{aligned}$$

holds for any $m \in \mathbb{Z}_{\geq 0}$ with $m \geq m_0$.

- (2) *There are positive constants m'_0 and C' , depending only on $\bar{L}$, $\bar{E}$, and π , such that the inequality*

$$\begin{aligned} |\hat{h}^1(\mathrm{H}^0(X', \pi^*(mL + E)), \|\cdot\|_{\sup}^{\pi^*(m\bar{L} + \bar{E})}) - \hat{h}^1(\mathrm{H}^0(X, mL + E), \|\cdot\|_{\sup}^{m\bar{L} + \bar{E}})| \\ \leq C'm^{d-1} \log m \end{aligned}$$

holds for any $m \in \mathbb{Z}_{\geq 0}$ with $m \geq m'_0$.

Proof. (1) We divide the proof into four steps.

Step 1. Let $\mu : \tilde{X}' \rightarrow X'$ be a generic resolution of singularities and fix a volume form $\Phi_{\tilde{X}'}$ on $\tilde{X}'_\infty$. We define the L^2 -norms on $H^0(X, mL + E)_\mathbb{R}$ and $H^0(X', \pi^*(mL + E))_\mathbb{R}$ by using the push-forward measure induced from $\Phi_{\tilde{X}'}$; i.e., for $t \in H^0(X, mL + E)_\mathbb{R}$ and for $t' \in H^0(X', \pi^*(mL + E))_\mathbb{R}$, the L^2 -norms are defined as

$$\|t\|_{L^2}^{m\bar{L}+\bar{E}} := \left(\int_{\tilde{X}'_\infty} \mu^* \pi^* (|t|_{m\bar{L}+\bar{E}}^2) \Phi_{\tilde{X}'} \right)^{\frac{1}{2}}$$

and

$$\|t'\|_{L^2}^{\pi^*(m\bar{L}+\bar{E})} := \left(\int_{\tilde{X}'_\infty} \mu^* (|t'|_{\pi^*(m\bar{L}+\bar{E})}^2) \Phi_{\tilde{X}'} \right)^{\frac{1}{2}},$$

respectively. By Gromov's inequality [15, Corollary 1.1.2] on $\tilde{X}'_\infty$, it suffices to show that Theorem 3.1.1 holds true for the L^2 -norms in place of the supremum-norms.

Step 2. Suppose that X' is normal. Consider the natural homomorphism

$$H^0(X, mL + E) \rightarrow H^0(X', \pi^*(mL + E))$$

and apply Theorem 1.2.5 and Lemma 1.1.2(3) to this case. There are positive constants m_0 and C , depending on $\bar{L}$ and π , such that

$$\begin{aligned} \hat{\chi}(H^0(X', \pi^*(mL + E)), \|\cdot\|_{L^2}^{\pi^*(m\bar{L}+\bar{E})}) - \hat{\chi}(H^0(X, mL + E), \|\cdot\|_{L^2}^{m\bar{L}+\bar{E}}) \\ \geq -Cm^{d-1} \log m \end{aligned}$$

holds for any $m \in \mathbb{Z}_{\geq 0}$ with $m \geq m_0$.

Step 3. This step is due to Yuan [22, Proposition 2.8]. For reader's convenience, we would like to recall the proof briefly. We do not need to assume that X' is normal in this step. We can reduce the proof to the case that $X' \rightarrow X$ is finite by the Stein factorization. Let $\bar{A}$ be a C^∞ -Hermitian invertible sheaf on X with the following two properties:

- (i) $L_\mathbb{Q} + A_\mathbb{Q}$ is ample on $X_\mathbb{Q}$ and $\bar{L} + \bar{A}$ is vertically nef.
- (ii) There is a non-zero small section $s \in H^0(X, A)$ that does not vanish at any associated point of the coherent sheaf $\pi_* \mathcal{O}_{X'}/\mathcal{O}_X$.

The commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & m(L + A) + E & \longrightarrow & m(L + A) + E + \pi_* \mathcal{O}_{X'} & \longrightarrow & m(L + A) + E + \pi_* \mathcal{O}_{X'}/\mathcal{O}_X \longrightarrow 0 \\ & & \uparrow \otimes s \otimes m & & \uparrow \otimes s \otimes m & & \uparrow \otimes s \otimes m \\ 0 & \longrightarrow & mL + E & \longrightarrow & mL + E + \pi_* \mathcal{O}_{X'} & \longrightarrow & mL + E + \pi_* \mathcal{O}_{X'}/\mathcal{O}_X \longrightarrow 0 \end{array}$$

yields an injective homomorphism

$$\begin{aligned} & H^0(X', \pi^*(mL + E)) / H^0(X, mL + E) \\ & \xrightarrow{\otimes s^{\otimes m}} H^0(X', \pi^*(m(L + A) + E)) / H^0(X, m(L + A) + E). \end{aligned}$$

Since $\bar{L} + \bar{A}$ (resp. $\pi^*(\bar{L} + \bar{A})$) has the property (i), we obtain

$$\begin{aligned} & \hat{\chi}(H^0(X', \pi^*(mL + E)), \|\cdot\|_{L^2}^{\pi^*(m\bar{L} + \bar{E})}) - \hat{\chi}(H^0(X, mL + E), \|\cdot\|_{L^2}^{m\bar{L} + \bar{E}}) \\ & = \widehat{\deg}(H^0(X', \pi^*(mL + E)) / H^0(X, mL + E), h_{L^2, \text{quot}}^{\pi^*(m\bar{L} + \bar{E})}) + O(m^{d-1} \log m) \\ & \leq \widehat{\deg}(H^0(X', \pi^*(mL + E)) / H^0(X, mL + E), h_{L^2, \text{quot}, \text{sub}}^{\pi^*(m(\bar{L} + \bar{A}) + \bar{E})}) \\ & \quad + O(m^{d-1} \log m) \\ & \leq \widehat{\deg}(H^0(X', \pi^*(m(L + A) + E)) / H^0(X, m(L + A) + E), h_{L^2, \text{quot}}^{\pi^*(m(\bar{L} + \bar{A}) + \bar{E})}) \\ & \quad + O(m^{d-1} \log m) \\ & = \widehat{\deg}(H^0(X', \pi^*(m(L + A) + E)), h_{L^2}^{\pi^*(m(\bar{L} + \bar{A}) + \bar{E})}) \\ & \quad - \widehat{\deg}(H^0(X, m(L + A) + E), h_{L^2}^{m(\bar{L} + \bar{A}) + \bar{E}}) + O(m^{d-1} \log m) \\ & = (\widehat{\deg}(\hat{c}_1(\pi^*(L + A))^{\cdot d}) - \widehat{\deg}(\hat{c}_1(L + A)^{\cdot d}))m^d + O(m^{d-1} \log m) \\ & = O(m^{d-1} \log m) \end{aligned}$$

by Lemma 1.1.2 (3) and [26, Theorem (1.4)]. To sum up, we have

$$\begin{aligned} \hat{\chi}(H^0(X', \pi^*(mL + E)), \|\cdot\|_{L^2}^{\pi^*(m\bar{L} + \bar{E})}) & \leq \hat{\chi}(H^0(X, mL + E), \|\cdot\|_{L^2}^{m\bar{L} + \bar{E}}) \\ & \quad + O(m^{d-1} \log m) \end{aligned}$$

for any birational projective morphism $\pi : X' \rightarrow X$.

Step 4. In general, let $\nu : X'' \rightarrow X'$ be the normalization of X' . Then we have

$$\begin{aligned} & \hat{\chi}(H^0(X', \pi^*(mL + E)), \|\cdot\|_{L^2}^{\pi^*(m\bar{L} + \bar{E})}) \\ & \geq \hat{\chi}(H^0(X'', \nu^*\pi^*(mL + E)), \|\cdot\|_{L^2}^{\nu^*\pi^*(m\bar{L} + \bar{E})}) \\ & \quad + O(m^{d-1} \log m) \\ & \geq \hat{\chi}(H^0(X, mL + E), \|\cdot\|_{L^2}^{m\bar{L} + \bar{E}}) + O(m^{d-1} \log m) \end{aligned}$$

as desired. Hence, we complete the proof of Theorem 3.1.1(1).

(2) By Lemma 1.1.1,

$$\begin{aligned} & |\hat{h}^1(H^0(X', \pi^*(mL + E)), \|\cdot\|_{\sup}^{\pi^*(m\bar{L} + \bar{E})}) - \hat{h}^1(H^0(X, mL + E), \|\cdot\|_{\sup}^{m\bar{L} + \bar{E}})| \\ & \leq |\hat{h}^0(H^0(X', \pi^*(mL + E)), \|\cdot\|_{\sup}^{\pi^*(m\bar{L} + \bar{E})}) - \hat{h}^0(H^0(X, mL + E), \|\cdot\|_{\sup}^{m\bar{L} + \bar{E}})| \\ & \quad + |\hat{\chi}(H^0(X', \pi^*(mL + E)), \|\cdot\|_{\sup}^{\pi^*(m\bar{L} + \bar{E})}) - \hat{\chi}(H^0(X, mL + E), \|\cdot\|_{\sup}^{m\bar{L} + \bar{E}})| \\ & \quad + O(m^{d-1} \log m). \end{aligned}$$

We note that, in the proof of [15, Theorem 4.3], we can replace $o(m^d)$ with $O(m^{d-1} \log m)$. Thus, by [15, Theorem 4.3] and (1), we have the assertion. $\square$

3.2. Arithmetic volume functions. Let X be a projective arithmetic variety of dimension d . As a consequence of Theorem 1.2.5, we have the following.

Proposition 3.2.1. *Suppose that X is normal. Let $r > 0$ be a positive integer. Let $\bar{\mathbf{L}} = (\bar{\mathbf{L}}(1), \dots, \bar{\mathbf{L}}(r))$ be a family of C^∞ -Hermitian invertible sheaves on X and Y a closed arithmetic subvariety of X of dimension d' .*

- (1) *There are positive constants a_0 and C , depending only on $\bar{\mathbf{L}}$, X , and Y , such that the inequality*

$$|\hat{\chi}(H^0(X|Y, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{\sup, \text{quot}}^{\mathbf{a} \cdot \bar{\mathbf{L}}})| \leq C \|\mathbf{a}\|_1^{d'}$$

holds for any $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$ with $\|\mathbf{a}\|_1 \geq a_0$.

- (2) *There are positive constants a'_0 and C' , depending only on $\bar{\mathbf{L}}$, X , and Y , such that the inequality*

$$\hat{h}^1(H^0(X|Y, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{\sup, \text{quot}}^{\mathbf{a} \cdot \bar{\mathbf{L}}}) \leq C' \|\mathbf{a}\|_1^{d'}$$

holds for any $\mathbf{a} \in \mathbb{Z}_{\geq 0}^r$ with $\|\mathbf{a}\|_1 \geq a'_0$.

Proof. (1) Let $\tilde{Y} \rightarrow Y$ be a generic resolution of singularities. Let $\bar{A}$ be an ample C^∞ -Hermitian invertible sheaf on $\tilde{Y}$ such that $\bar{A} - \bar{\mathbf{L}}(i)|_{\tilde{Y}}$ is effective on $\tilde{Y}$ for $i = 1, \dots, r$. There is a positive constant C with the property that

$$\hat{h}^0(H^0(\tilde{Y}, mA), \|\cdot\|_{\sup}^{m\bar{A}}) \leq Cm^{d'}$$

for any integer $m \geq 1$. Then, by Lemma 1.1.1,

$$\begin{aligned} & \hat{\chi}(H^0(X|Y, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{\sup, \text{quot}}^{\mathbf{a} \cdot \bar{\mathbf{L}}}) \\ & \leq \hat{h}^0(H^0(X|Y, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{\sup, \text{quot}}^{\mathbf{a} \cdot \bar{\mathbf{L}}}) + O(\|\mathbf{a}\|_1^{d'-1} \log \|\mathbf{a}\|_1) \\ & \leq \hat{h}^0(H^0(Y, \mathbf{a} \cdot \mathbf{L}|_Y), \|\cdot\|_{\sup}^{\mathbf{a} \cdot \bar{\mathbf{L}}|_Y}) + O(\|\mathbf{a}\|_1^{d'-1} \log \|\mathbf{a}\|_1) \\ & \leq \hat{h}^0(H^0(\tilde{Y}, \mathbf{a} \cdot \mathbf{L}|_{\tilde{Y}}), \|\cdot\|_{\sup}^{\mathbf{a} \cdot \bar{\mathbf{L}}|_{\tilde{Y}}}) + O(\|\mathbf{a}\|_1^{d'-1} \log \|\mathbf{a}\|_1) \\ & \leq \hat{h}^0(H^0(\tilde{Y}, \|\mathbf{a}\|_1 \bar{A}), \|\cdot\|_{\sup}^{\|\mathbf{a}\|_1 \bar{A}}) + O(\|\mathbf{a}\|_1^{d'-1} \log \|\mathbf{a}\|_1) \\ & \leq C \|\mathbf{a}\|_1^{d'} + O(\|\mathbf{a}\|_1^{d'-1} \log \|\mathbf{a}\|_1). \end{aligned}$$

On the other hand, by Lemma 1.1.2(1),

$$\begin{aligned} & \hat{\chi}(H^0(X|Y, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{\sup, \text{quot}}^{\mathbf{a} \cdot \bar{\mathbf{L}}}) \\ & \geq -\text{rk } H^0(X|Y, \mathbf{a} \cdot \mathbf{L}) \log(\lambda_{\max}(H^0(X|Y, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{\sup, \text{quot}}^{\mathbf{a} \cdot \bar{\mathbf{L}}})/2) \\ & \quad + O(\|\mathbf{a}\|_1^{d'-1} \log \|\mathbf{a}\|_1) \\ & \geq -\text{rk } H^0(Y, \mathbf{a} \cdot \mathbf{L}|_Y) \cdot \max \left\{ 0, \log \lambda_{\max}(H^0(X, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{\sup}^{\mathbf{a} \cdot \bar{\mathbf{L}}}) \right\} \\ & \quad + O(\|\mathbf{a}\|_1^{d'-1} \log \|\mathbf{a}\|_1). \end{aligned}$$

Since X is normal, we can find a positive constant C' such that

$$\max \left\{ 0, \log \lambda_{\max}(H^0(X, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{\sup}^{\mathbf{a} \cdot \bar{\mathbf{L}}}) \right\} \leq C' \|\mathbf{a}\|_1$$

for $\|\mathbf{a}\|_1 \gg 0$ by using Theorem 1.2.5. Moreover, there is a positive constant C'' with

$$\text{rk } H^0(Y, \mathbf{a} \cdot \mathbf{L}|_Y) \leq C'' \|\mathbf{a}\|_1^{d'-2}.$$

Hence, we have (1).

(2) By Lemma 1.1.1 and (1), there is a constant C''' , depending only on $\bar{\mathbf{L}}$, X , and Y , such that

$$\begin{aligned} & \hat{h}^1(H^0(X|Y, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{\sup, \text{quot}}^{\mathbf{a} \cdot \bar{\mathbf{L}}}) \\ & = \hat{h}^0(H^0(X|Y, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{\sup, \text{quot}}^{\mathbf{a} \cdot \bar{\mathbf{L}}}) - \hat{\chi}(H^0(X|Y, \mathbf{a} \cdot \mathbf{L}), \|\cdot\|_{\sup, \text{quot}}^{\mathbf{a} \cdot \bar{\mathbf{L}}}) \\ & \quad + O(\|\mathbf{a}\|_1^{d'-1} \log \|\mathbf{a}\|_1) \\ & \leq C''' \|\mathbf{a}\|_1^{d'} + O(\|\mathbf{a}\|_1^{d'-1} \log \|\mathbf{a}\|_1) \end{aligned}$$

hold for $\|\mathbf{a}\|_1 \gg 0$. □

By Proposition 3.2.1 and Theorem 3.1.1, one finds the following.

Corollary 3.2.2. *Let $\bar{\mathbf{L}}$ and $\bar{\mathbf{E}}$ be two C^∞ -Hermitian invertible sheaves on X . There are positive constants C , C' , and C'' , depending only on $\bar{\mathbf{L}}$, $\bar{\mathbf{E}}$, and X , such that the inequalities*

$$\begin{aligned} & |\hat{\chi}(H^0(X, mL + E), \|\cdot\|_{\sup}^{m\bar{\mathbf{L}} + \bar{\mathbf{E}}})| \leq Cm^d, \\ & \hat{h}^0(H^0(X, mL + E), \|\cdot\|_{\sup}^{m\bar{\mathbf{L}} + \bar{\mathbf{E}}}) \leq C'm^d, \end{aligned}$$

and

$$\hat{h}^1(H^0(X, mL + E), \|\cdot\|_{\sup}^{m\bar{\mathbf{L}} + \bar{\mathbf{E}}}) \leq C''m^d$$

hold.

In the rest of this paper, we also treat the C^0 -Hermitian invertible sheaves and $\bar{\mathbf{L}}$ denotes a C^0 -Hermitian invertible sheaf on X unless otherwise stated. First, we recall the Stone-Weierstrass theorem in the following form.

Lemma 3.2.3 (Stone-Weierstrass). *Let V be a complex projective variety and $\bar{L} = (L, h^{\bar{L}})$ a C^0 -Hermitian invertible sheaf on V . Let $\epsilon > 0$ be a positive real number. Then, there is a C^0 -function $\phi : V \rightarrow \mathbb{R}$ on V such that $\|\phi\|_{\sup} \leq \epsilon$ and $\bar{L}^\phi$ is C^∞ (see Conventions and terminology, (5)).*

Proof. We can embed V into a complex projective space P . The restrictions of C^∞ -functions on P are C^∞ on V in the sense of Conventions and terminology, (5). Let $\{U_i\}$ be a finite open cover of P such that the restrictions $L|_{U_i \cap V}$ are all trivial and the closures $\bar{U}_i$ are all compact. We can find a partition of unity $\{\rho_i\}$ subordinate to $\{U_i\}$. By the Stone-Weierstrass theorem [5, Chapitre X, §4, n° 2, Théorème 3], we can approximate the metric on each $U_i \cap V$ and obtain the assertion by standard arguments. $\square$

Corollary 3.2.4. *Let $\pi : X' \rightarrow X$ be a birational morphism of projective arithmetic varieties and $\bar{L}$ and $\bar{E}$ two C^0 -Hermitian invertible sheaves on X . Then,*

$$\begin{aligned} \limsup_{m \rightarrow \infty} \frac{\hat{\chi}(\mathrm{H}^0(X', \pi^*(mL + E)), \|\cdot\|_{\sup}^{\pi^*(m\bar{L} + \bar{E})})}{m^d/d!} \\ = \limsup_{m \rightarrow \infty} \frac{\hat{\chi}(\mathrm{H}^0(X, mL + E), \|\cdot\|_{\sup}^{m\bar{L} + \bar{E}})}{m^d/d!} \end{aligned}$$

and

$$\begin{aligned} \liminf_{m \rightarrow \infty} \frac{\hat{\chi}(\mathrm{H}^0(X', \pi^*(mL + E)), \|\cdot\|_{\sup}^{\pi^*(m\bar{L} + \bar{E})})}{m^d/d!} \\ = \liminf_{m \rightarrow \infty} \frac{\hat{\chi}(\mathrm{H}^0(X, mL + E), \|\cdot\|_{\sup}^{m\bar{L} + \bar{E}})}{m^d/d!}. \end{aligned}$$

Proof. Fix an arbitrarily small positive real number $\epsilon > 0$. By Lemma 3.2.3, there are two C^0 -functions $\phi, \psi : X_\infty \rightarrow \mathbb{R}$ such that $\max\{\|\phi\|_{\sup}, \|\psi\|_{\sup}\} \leq \epsilon$ and $\bar{L}^\phi$ and $\bar{E}^\psi$ are both C^∞ . By Theorem 3.1.1, there is a positive constant $C(\phi, \psi) > 0$, depending on $\bar{L}^\phi$, $\bar{E}^\psi$, and π , with the property that

$$\begin{aligned} |\hat{\chi}(\mathrm{H}^0(X', \pi^*(mL + E)), \|\cdot\|_{\sup}^{\pi^*(m\bar{L}^\phi + \bar{E}^\psi)}) - \hat{\chi}(\mathrm{H}^0(X, mL + E), \|\cdot\|_{\sup}^{m\bar{L}^\phi + \bar{E}^\psi})| \\ \leq C(\phi, \psi)m^{d-1} \log m \end{aligned}$$

for any sufficiently large $m \geq 1$. Moreover, we can find a positive constant $D > 0$, depending only on L , E , and π , such that

$$\mathrm{rk} \mathrm{H}^0(X, mL + E) \leq \mathrm{rk} \mathrm{H}^0(X', \pi^*(mL + E)) \leq Dm^{d-1}$$

for any $m \geq 1$. Then,

$$\begin{aligned}
 & |\hat{\chi}(\mathbf{H}^0(X', \pi^*(mL + E)), \|\cdot\|_{\sup}^{\pi^*(m\bar{L} + \bar{E})}) - \hat{\chi}(\mathbf{H}^0(X, mL + E), \|\cdot\|_{\sup}^{m\bar{L} + \bar{E}})| \\
 & \leq |\hat{\chi}(\mathbf{H}^0(X', \pi^*(mL + E)), \|\cdot\|_{\sup}^{\pi^*(m\bar{L} + \bar{E})}) \\
 & \quad - \hat{\chi}(\mathbf{H}^0(X', \pi^*(mL + E)), \|\cdot\|_{\sup}^{\pi^*(m\bar{L}^\phi + \bar{E}^\psi)})| \\
 & \quad + |\hat{\chi}(\mathbf{H}^0(X', \pi^*(mL + E)), \|\cdot\|_{\sup}^{\pi^*(m\bar{L}^\phi + \bar{E}^\psi)}) \\
 & \quad - \hat{\chi}(\mathbf{H}^0(X, mL + E), \|\cdot\|_{\sup}^{m\bar{L}^\phi + \bar{E}^\psi})| \\
 & \quad + |\hat{\chi}(\mathbf{H}^0(X, mL + E), \|\cdot\|_{\sup}^{m\bar{L}^\phi + \bar{E}^\psi}) - \hat{\chi}(\mathbf{H}^0(X, mL + E), \|\cdot\|_{\sup}^{m\bar{L} + \bar{E}})| \\
 & \leq 2Dm^{d-1}(m+1) \cdot \epsilon + C(\phi, \psi)m^{d-1} \log m.
 \end{aligned}$$

Thus

$$\left| \limsup_{m \rightarrow \infty} \frac{\hat{\chi}(\mathbf{H}^0(X', \pi^*(mL + E)), \|\cdot\|_{\sup}^{\pi^*(m\bar{L} + \bar{E})})}{m^d/d!} - \limsup_{m \rightarrow \infty} \frac{\hat{\chi}(\mathbf{H}^0(X, mL + E), \|\cdot\|_{\sup}^{m\bar{L} + \bar{E}})}{m^d/d!} \right| \leq 2D(d!) \cdot \epsilon$$

and

$$\left| \liminf_{m \rightarrow \infty} \frac{\hat{\chi}(\mathbf{H}^0(X', \pi^*(mL + E)), \|\cdot\|_{\sup}^{\pi^*(m\bar{L} + \bar{E})})}{m^d/d!} - \liminf_{m \rightarrow \infty} \frac{\hat{\chi}(\mathbf{H}^0(X, mL + E), \|\cdot\|_{\sup}^{m\bar{L} + \bar{E}})}{m^d/d!} \right| \leq 2D(d!) \cdot \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we obtain the assertion. $\square$

Now we define the three types of arithmetic volume functions.

Definition 3.2.1. Let $\bar{L}$ be a C^0 -Hermitian invertible sheaf on X . We define the three types of arithmetic volume functions as follows:

$$\begin{aligned}
 \widehat{\text{vol}}^{\hat{\chi}}(\bar{L}) &:= \limsup_{m \rightarrow \infty} \frac{\hat{\chi}(\mathbf{H}^0(X, mL), \|\cdot\|_{\sup}^{m\bar{L}})}{m^d/d!}, \\
 \widehat{\text{vol}}^0(\bar{L}) &:= \limsup_{m \rightarrow \infty} \frac{\hat{h}^0(\mathbf{H}^0(X, mL), \|\cdot\|_{\sup}^{m\bar{L}})}{m^d/d!}, \\
 \widehat{\text{vol}}^1(\bar{L}) &:= \liminf_{m \rightarrow \infty} \frac{\hat{h}^1(\mathbf{H}^0(X, mL), \|\cdot\|_{\sup}^{m\bar{L}})}{m^d/d!}.
 \end{aligned}$$

The following result is due to Huayi Chen.

Theorem 3.2.5. Let $\bar{L}$ be a C^0 -Hermitian invertible sheaf on X . In Definition 3.2.1, all of the right-hand sides of the equalities actually converge and

the following equalities hold:

$$\begin{aligned}\widehat{\text{vol}}^{\hat{\chi}}(\bar{L}) &= \lim_{m \rightarrow \infty} \frac{\hat{\chi}(\text{H}^0(X, mL), \|\cdot\|_{\text{sup}}^{m\bar{L}})}{m^d/d!}, \\ \widehat{\text{vol}}^0(\bar{L}) &= \lim_{m \rightarrow \infty} \frac{\hat{h}^0(\text{H}^0(X, mL), \|\cdot\|_{\text{sup}}^{m\bar{L}})}{m^d/d!}, \\ \widehat{\text{vol}}^1(\bar{L}) &= \lim_{m \rightarrow \infty} \frac{\hat{h}^1(\text{H}^0(X, mL), \|\cdot\|_{\text{sup}}^{m\bar{L}})}{m^d/d!}.\end{aligned}$$

Proof. We can find the second formula in [6], [23] and the last formula follows from the first and the second formulae, so that it suffices to show the first formula here. By Corollary 3.2.4, we may assume, without loss of generality, that X is generically smooth and normal. Let $\epsilon > 0$ be an arbitrarily small, positive, real number. By Lemma 3.2.3, we take a C^0 -function ϕ on X such that $\|\phi\|_{\text{sup}} \leq \epsilon$ and $\bar{L}^\phi$ is C^∞ . Then, we know

$$|\log \lambda_{\max}(\text{H}^0(X, mL), \|\cdot\|_{\text{sup}}^{m\bar{L}}) - \log \lambda_{\max}(\text{H}^0(X, mL), \|\cdot\|_{\text{sup}}^{m\bar{L}^\phi})| \leq m\|\phi\|_{\text{sup}}.$$

First, we assume that $L_{\mathbb{Q}}$ is not big on $X_{\mathbb{Q}}$. If $\text{H}^0(X, mL)$ is non-zero, then

$$\begin{aligned}-\frac{\log(\lambda_{\max}(\text{H}^0(X, mL), \|\cdot\|_{\text{sup}}^{m\bar{L}})/2)}{m/d} - \frac{\log(\text{rk } \text{H}^0(X, mL))}{m/d} \\ \leq \frac{\hat{\chi}(\text{H}^0(X, mL), \|\cdot\|_{\text{sup}}^{m\bar{L}})}{\text{rk } \text{H}^0(X, mL)} \cdot \frac{1}{m/d} \\ \leq -\frac{\log(\lambda_{\min}(\text{H}^0(X, mL), \|\cdot\|_{\text{sup}}^{m\bar{L}})/2)}{m/d}\end{aligned}$$

by the Minkowski convex-body-theorem (Lemma 1.1.2(1)) and

$$\left| \frac{\hat{\chi}(\text{H}^0(X, mL), \|\cdot\|_{\text{sup}}^{m\bar{L}})}{\text{rk } \text{H}^0(X, mL)} \cdot \frac{1}{m/d} \right|$$

is bounded by a constant by Theorem 1.2.5, by [4, Proposition 2.6], and by the estimates above. Therefore,

$$\begin{aligned}\frac{\hat{\chi}(\text{H}^0(X, mL), \|\cdot\|_{\text{sup}}^{m\bar{L}})}{m^d/d!} \\ = \begin{cases} \frac{\text{rk } \text{H}^0(X, mL)}{m^{d-1}/(d-1)!} \cdot \frac{\hat{\chi}(\text{H}^0(X, mL), \|\cdot\|_{\text{sup}}^{m\bar{L}})}{\text{rk } \text{H}^0(X, mL)} \cdot \frac{1}{m/d} & \text{if } \text{H}^0(X, mL) \text{ is non-zero,} \\ 0 & \text{if } \text{H}^0(X, mL) \text{ is zero} \end{cases}\end{aligned}$$

converges to zero as $m \rightarrow \infty$.

Next, assume that $L_{\mathbb{Q}}$ is big on $X_{\mathbb{Q}}$. Then $\text{H}^0(X, mL)$ is non-zero for all sufficiently large m . We fix a normalized volume form Φ_X on X_{∞} and consider

the L^2 -metric with respect to Φ_X . By Gromov's inequality [15, Corollary 1.1.2], one knows

$$\limsup_{m \rightarrow \infty} \frac{\hat{\chi}(H^0(X, mL), \|\cdot\|_{\sup}^{m\bar{L}^\phi})}{m^d/d!} = \limsup_{m \rightarrow \infty} \frac{\widehat{\deg}(H^0(X, mL), h_{L^2}^{m\bar{L}^\phi})}{m^d/d!}$$

and

$$\liminf_{m \rightarrow \infty} \frac{\hat{\chi}(H^0(X, mL), \|\cdot\|_{\sup}^{m\bar{L}^\phi})}{m^d/d!} = \liminf_{m \rightarrow \infty} \frac{\widehat{\deg}(H^0(X, mL), h_{L^2}^{m\bar{L}^\phi})}{m^d/d!}.$$

The convergence theorem of Chen [6, Theorem 2.11], together with the boundedness theorem (Theorem 1.2.3), implies that

$$\frac{\widehat{\deg}(H^0(X, mL), h_{L^2}^{m\bar{L}^\phi})}{\operatorname{rk} H^0(X, mL)} \cdot \frac{1}{m/d}$$

converges as $m \rightarrow \infty$ (cf. [7, Proposition 1.2.9]), so that

$$\limsup_{m \rightarrow \infty} \frac{\widehat{\deg}(H^0(X, mL), h_{L^2}^{m\bar{L}^\phi})}{m^d/d!} = \liminf_{m \rightarrow \infty} \frac{\widehat{\deg}(H^0(X, mL), h_{L^2}^{m\bar{L}^\phi})}{m^d/d!}.$$

Hence,

$$\begin{aligned} & \left| \limsup_{m \rightarrow \infty} \frac{\hat{\chi}(H^0(X, mL), \|\cdot\|_{\sup}^{m\bar{L}})}{m^d/d!} - \liminf_{m \rightarrow \infty} \frac{\hat{\chi}(H^0(X, mL), \|\cdot\|_{\sup}^{m\bar{L}})}{m^d/d!} \right| \\ & \leq \left| \limsup_{m \rightarrow \infty} \frac{\hat{\chi}(H^0(X, mL), \|\cdot\|_{\sup}^{m\bar{L}})}{m^d/d!} - \limsup_{m \rightarrow \infty} \frac{\hat{\chi}(H^0(X, mL), \|\cdot\|_{\sup}^{m\bar{L}^\phi})}{m^d/d!} \right| \\ & \quad + \left| \liminf_{m \rightarrow \infty} \frac{\hat{\chi}(H^0(X, mL), \|\cdot\|_{\sup}^{m\bar{L}^\phi})}{m^d/d!} - \liminf_{m \rightarrow \infty} \frac{\hat{\chi}(H^0(X, mL), \|\cdot\|_{\sup}^{m\bar{L}})}{m^d/d!} \right| \\ & \leq 2d \operatorname{vol}(L_{\mathbb{Q}}) \cdot \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we conclude (see also [4, Theorem 3.1]). $\square$

Proposition 3.2.6. *Let $\bar{L}$ be a C^0 -Hermitian invertible sheaf on X .*

(1) *For any real number λ ,*

$$\widehat{\operatorname{vol}}^{\hat{\chi}}(\bar{L}^\lambda) = \widehat{\operatorname{vol}}^{\hat{\chi}}(\bar{L}) + d \operatorname{vol}(L_{\mathbb{Q}}) \cdot \lambda.$$

(2) *For any C^0 -function ϕ on X ,*

$$|\widehat{\operatorname{vol}}^{\hat{\chi}}(\bar{L}^\phi) - \widehat{\operatorname{vol}}^{\hat{\chi}}(\bar{L})| \leq d \operatorname{vol}(L_{\mathbb{Q}}) \cdot \|\phi\|_{\sup}.$$

(3) *We have $0 \leq \widehat{\operatorname{vol}}^0(\bar{L}) < \infty$, $0 \leq \widehat{\operatorname{vol}}^1(\bar{L}) < \infty$, and*

$$\widehat{\operatorname{vol}}^{\hat{\chi}}(\bar{L}) = \widehat{\operatorname{vol}}^0(\bar{L}) - \widehat{\operatorname{vol}}^1(\bar{L}).$$

(For the definitions of the C^0 -Hermitian invertible sheaves $\bar{L}^\lambda$ and $\bar{L}^\phi$, see Conventions and terminology, (6).)

Proof. (1) is clear since

$$\hat{\chi}(\mathrm{H}^0(X, mL), \|\cdot\|_{\sup}^{m\bar{L}^\lambda}) = \hat{\chi}(\mathrm{H}^0(X, mL), \|\cdot\|_{\sup}^{m\bar{L}}) + \mathrm{rk} \mathrm{H}^0(X, mL) \cdot m\lambda.$$

Moreover, since $-\|\phi\|_{\sup} \leq \phi \leq \|\phi\|_{\sup}$, we have (2).

(3) $0 \leq \widehat{\mathrm{vol}}^0(\bar{L}) < \infty$ is due to Moriwaki [16]. There is a C^0 -function ϕ on X such that $\bar{L}^\phi$ is C^∞ . Since

$$|\widehat{\mathrm{vol}}^{\hat{\chi}}(\bar{L}) - \widehat{\mathrm{vol}}^{\hat{\chi}}(\bar{L}^\phi)| \leq d \mathrm{vol}(L_{\mathbb{Q}}) \cdot \|\phi\|_{\sup}$$

and $|\widehat{\mathrm{vol}}^{\hat{\chi}}(\bar{L}^\phi)| < \infty$ by Corollary 3.2.2, we have $|\widehat{\mathrm{vol}}^{\hat{\chi}}(\bar{L})| < \infty$. The latter half is due to Gillet and Soulé (cf. Lemma 1.1.1). $\square$

Definition 3.2.2. Suppose that X is normal. Let $\bar{L}$ be a C^0 -Hermitian invertible sheaf on X . We define the last and the first asymptotic logarithmic successive minima of $\bar{L}$ as

$$\Lambda_{\max}(\bar{L}) := \liminf_{m \rightarrow \infty} \frac{\log \lambda_{\max}(\mathrm{H}^0(X, mL), \|\cdot\|_{\sup}^{m\bar{L}})}{m}$$

and

$$\Lambda_{\min}(\bar{L}) := \liminf_{m \rightarrow \infty} \frac{\log \lambda_{\min}(\mathrm{H}^0(X, mL), \|\cdot\|_{\sup}^{m\bar{L}})}{m}$$

respectively. Note that the convergence of the sequence defining $\Lambda_{\min}(\bar{L})$ has been known (see for example [4, Lemma 1.4]) and that, if $H^0(X, mL) = 0$ for all integers $m \geq 1$, then both of the asymptotic logarithmic successive minima of $\bar{L}$ are defined to be zero: $\Lambda_{\max}(\bar{L}) = \Lambda_{\min}(\bar{L}) = 0$.

Since there is a C^0 -function ϕ on X such that $\bar{L}^\phi$ is C^∞ and that

$$|\Lambda_{\max}(\bar{L}) - \Lambda_{\max}(\bar{L}^\phi)| \leq \|\phi\|_{\sup},$$

one has

$$-\infty < \Lambda_{\min}(\bar{L}) \leq \Lambda_{\max}(\bar{L}) < \infty$$

by Theorem 1.2.5 and by [4, Proposition 2.6]. By use of Lemma 1.1.2 we have an asymptotic version of the Minkowski convex-body-theorem as follows (cf. [10, Ch. 2, §9, Theorems 1 and 2]).

Theorem 3.2.7. Suppose that X is normal. Let $\bar{L}, \bar{M}$ be two C^0 -Hermitian invertible sheaves on X .

- (1) If $\Lambda_{\max}(\bar{L}) < 0$, then $\widehat{\mathrm{vol}}^1(\bar{L}) = 0$. In particular, for any $\bar{L}$, there is a positive real number $\lambda > 0$ such that $\widehat{\mathrm{vol}}^1(\bar{L}^\lambda) = 0$.
- (2) We have the inequalities

$$-d \mathrm{vol}(L_{\mathbb{Q}}) \cdot \Lambda_{\max}(\bar{L}) \leq \widehat{\mathrm{vol}}^{\hat{\chi}}(\bar{L}) \leq -d \mathrm{vol}(L_{\mathbb{Q}}) \cdot \Lambda_{\min}(\bar{L}).$$

(3) If $\bar{L} - \bar{M} \geq 0$ (Conventions and terminology, (8)), then

$$\widehat{\text{vol}}^{\hat{\chi}}(\bar{L}) - \widehat{\text{vol}}^{\hat{\chi}}(\bar{M}) \geq -d(\text{vol}(L_{\mathbb{Q}}) - \text{vol}(M_{\mathbb{Q}})) \Lambda_{\max}(\bar{L}).$$

In particular, if $\bar{L} \geq \bar{M}$ and $\text{vol}(L_{\mathbb{Q}}) = \text{vol}(M_{\mathbb{Q}})$, then $\widehat{\text{vol}}^{\hat{\chi}}(\bar{L}) \geq \widehat{\text{vol}}^{\hat{\chi}}(\bar{M})$.

Proof. (1) follows from [15, Proposition 2.1(5)]. (2) is an immediate consequence of Lemma 1.1.2(1).

(3) Let $\mu : \tilde{X} \rightarrow X$ be the normalization of a generic resolution of singularities. Since X is normal, $\Lambda_{\max}(\mu^* \bar{L}) = \Lambda_{\max}(\bar{L})$. Thus we may assume that X is generically smooth and normal by Theorem 3.1.1. We fix a volume form Φ_X on X_{∞} and consider the L^2 -norms with respect to Φ_X . Since $m\bar{L} - m\bar{M} \geq 0$ for any $m \in \mathbb{Z}_{>0}$, we have an injective homomorphism

$$H^0(X, mM) \rightarrow H^0(X, mL)$$

and an inequality

$$\begin{aligned} & \hat{\chi}(H^0(X, mL), \|\cdot\|_{L^2}^{m\bar{L}}) - \hat{\chi}(H^0(X, mM), \|\cdot\|_{L^2}^{m\bar{M}}) \\ & \geq -(\text{rk } H^0(X, mL) - \text{rk } H^0(X, mM)) \log \lambda_{\max}(H^0(X, mL), \|\cdot\|_{L^2}^{m\bar{L}}) \end{aligned}$$

by Lemma 1.1.2(3). By [15, Corollary 1.1.2], one can find a constant $D > 0$ with

$$\|\cdot\|_{L^2}^{m\bar{L}} \leq \|\cdot\|_{\text{sup}}^{m\bar{L}} \leq D(1+m)^{d-1} \|\cdot\|_{L^2}^{m\bar{L}}.$$

Thus

$$\log \lambda_{\max}(H^0(X, mL), \|\cdot\|_{L^2}^{m\bar{L}}) = \log \lambda_{\max}(H^0(X, mL), \|\cdot\|_{\text{sup}}^{m\bar{L}}) + O(\log m)$$

and

$$\liminf_{m \rightarrow \infty} \frac{\log \lambda_{\max}(H^0(X, mL), \|\cdot\|_{L^2}^{m\bar{L}})}{m} = \Lambda_{\max}(\bar{L}).$$

On the other hand, by Theorem 3.2.5, one has

$$\widehat{\text{vol}}^{\hat{\chi}}(\bar{M}) = \liminf_{m \rightarrow \infty} \frac{\hat{\chi}(H^0(X, mM), \|\cdot\|_{L^2}^{m\bar{M}})}{m^d/d!}$$

and concludes the proof. $\square$

3.3. Homogeneity of the arithmetic volume functions. Let X be a projective arithmetic variety of dimension d and $\bar{L}$ a C^0 -Hermitian invertible sheaf on X . In this subsection, we prove that the arithmetic $\hat{\chi}$ -volume function is positively homogeneous on $\widehat{\text{Pic}}(X; C^0)$ (Conventions and terminology, (2)) and extends uniquely to a positively homogeneous function

$$\widehat{\text{vol}}^{\hat{\chi}} : \widehat{\text{Pic}}(X; C^0)_{\mathbb{Q}} \rightarrow \mathbb{R}$$

on $\widehat{\text{Pic}}(X; C^0)_{\mathbb{Q}}$ (cf. Corollary 3.3.3).

Proposition 3.3.1. *Let $\bar{L}$ and $\bar{E}$ be two C^0 -Hermitian invertible sheaves on X . Then*

$$\limsup_{m \rightarrow \infty} \frac{\hat{\chi}(H^0(X, mL + E), \|\cdot\|_{\sup}^{m\bar{L} + \bar{E}})}{m^d/d!} = \widehat{\text{vol}}^{\hat{\chi}}(\bar{L}).$$

Proof. By Theorem 3.1.1, we may assume that X is generically smooth and normal. First, we treat the case where both $\bar{L}$ and $\bar{E}$ are C^∞ -Hermitian. Then, by Theorem 2.1.1, one has

$$|\hat{\chi}(H^0(X, mL + E), \|\cdot\|_{\sup}^{m\bar{L} + \bar{E}}) - \hat{\chi}(H^0(X, mL), \|\cdot\|_{\sup}^{m\bar{L}})| \leq O(m^{d-1} \log m)$$

for $m \gg 0$. Hence,

$$\limsup_{m \rightarrow \infty} \frac{\hat{\chi}(H^0(X, mL + E), \|\cdot\|_{\sup}^{m\bar{L} + \bar{E}})}{m^d/d!} = \widehat{\text{vol}}^{\hat{\chi}}(\bar{L}).$$

Next, in general, for any positive real number $\epsilon > 0$, we can take C^0 -functions $\phi, \psi \in C^0(X)$ such that both $\bar{L}^\phi$ and $\bar{E}^\psi$ are C^∞ and

$$\max\{\|\phi\|_{\sup}, \|\psi\|_{\sup}\} \leq \epsilon$$

holds by use of the Stone-Weierstrass theorem (Lemma 3.2.3). Since

$$\left| \limsup_{m \rightarrow \infty} \frac{\hat{\chi}(H^0(X, mL + E), \|\cdot\|_{\sup}^{m\bar{L} + \bar{E}})}{m^d/d!} - \limsup_{m \rightarrow \infty} \frac{\hat{\chi}(H^0(X, mL + E), \|\cdot\|_{\sup}^{m\bar{L}^\phi + \bar{E}^\psi})}{m^d/d!} \right| \leq 2\epsilon d \text{vol}(L_{\mathbb{Q}}),$$

$$|\widehat{\text{vol}}^{\hat{\chi}}(\bar{L}) - \widehat{\text{vol}}^{\hat{\chi}}(\bar{L}^\phi)| \leq \epsilon d \text{vol}(L_{\mathbb{Q}}),$$

and

$$\limsup_{m \rightarrow \infty} \frac{\hat{\chi}(H^0(X, mL + E), \|\cdot\|_{\sup}^{m\bar{L}^\phi + \bar{E}^\psi})}{m^d/d!} = \widehat{\text{vol}}^{\hat{\chi}}(\bar{L}^\phi),$$

we obtain

$$\left| \limsup_{m \rightarrow \infty} \frac{\hat{\chi}(H^0(X, mL + E), \|\cdot\|_{\sup}^{m\bar{L} + \bar{E}})}{m^d/d!} - \widehat{\text{vol}}^{\hat{\chi}}(\bar{L}) \right| \leq 3\epsilon d \text{vol}(L_{\mathbb{Q}}).$$

We can choose $\epsilon > 0$ arbitrarily and conclude the proof. $\square$

Theorem 3.3.2. *For any positive integer $p \in \mathbb{Z}_{>0}$,*

$$\widehat{\text{vol}}^{\hat{\chi}}(p\bar{L}) = p^d \cdot \widehat{\text{vol}}^{\hat{\chi}}(\bar{L}).$$

Proof. For $k = 1, \dots, p-1$,

$$\begin{aligned} \limsup_{m \rightarrow \infty} \frac{\hat{\chi}(H^0(X, (mp+k)L), \|\cdot\|_{\sup}^{(mp+k)\bar{L}})}{(mp+k)^d/d!} \\ &= \limsup_{m \rightarrow \infty} \frac{\hat{\chi}(H^0(X, (mp+k)L), \|\cdot\|_{\sup}^{(mp+k)\bar{L}})}{(mp)^d/d!} \cdot \frac{(mp)^d}{(mp+k)^d} \\ &= \limsup_{m \rightarrow \infty} \frac{\hat{\chi}(H^0(X, mpL + kL), \|\cdot\|_{\sup}^{mp\bar{L} + k\bar{L}})}{(mp)^d/d!} \\ &= \limsup_{m \rightarrow \infty} \frac{\hat{\chi}(H^0(X, mpL), \|\cdot\|_{\sup}^{mp\bar{L}})}{(mp)^d/d!} \end{aligned}$$

by Proposition 3.3.1. Hence,

$$\limsup_{m \rightarrow \infty} \frac{\hat{\chi}(H^0(X, mpL), \|\cdot\|_{\sup}^{mp\bar{L}})}{(mp)^d/d!} = \widehat{\text{vol}}^{\hat{\chi}}(\bar{L}),$$

so that

$$\limsup_{m \rightarrow \infty} \frac{\hat{\chi}(H^0(X, mpL), \|\cdot\|_{\sup}^{mp\bar{L}})}{m^d/d!} = p^d \cdot \widehat{\text{vol}}^{\hat{\chi}}(\bar{L}). \quad \square$$

Corollary 3.3.3. *The function $\widehat{\text{vol}}^{\hat{\chi}} : \widehat{\text{Pic}}(X; C^0) \rightarrow \mathbb{R}$ (resp. $\widehat{\text{vol}}^1 : \widehat{\text{Pic}}(X; C^0) \rightarrow \mathbb{R}$) extends uniquely to a positively homogeneous function*

$$\begin{aligned} \widehat{\text{vol}}^{\hat{\chi}} : \widehat{\text{Pic}}(X; C^0)_{\mathbb{Q}} &\rightarrow \mathbb{R} \\ (\text{resp. } \widehat{\text{vol}}^1 : \widehat{\text{Pic}}(X; C^0)_{\mathbb{Q}} &\rightarrow \mathbb{R}) \end{aligned}$$

of degree d on $\widehat{\text{Pic}}(X; C^0)_{\mathbb{Q}}$.

Theorem 3.3.4. *Let $\bar{L}$ be a C^0 -Hermitian $\mathbb{Q}$ -invertible sheaf on X . If $\widehat{\text{vol}}^1(\bar{L}) > 0$, then $L_{\mathbb{Q}}$ is big on $X_{\mathbb{Q}}$.*

Proof. First, by Theorem 3.1.1 and Corollary 3.3.3, we may assume without loss of generality that X is generically smooth and normal and that $\bar{L}$ is a C^0 -Hermitian invertible sheaf on X . Next, there is a C^0 -function ϕ on X such that $\bar{L}^{\phi}$ is C^{∞} . Since

$$|\widehat{\text{vol}}^1(\bar{L}^{\phi}) - \widehat{\text{vol}}^1(\bar{L})| \leq d \text{vol}(L_{\mathbb{Q}}) \cdot \|\phi\|_{\sup},$$

either $\widehat{\text{vol}}^1(\bar{L}^{\phi}) = \widehat{\text{vol}}^1(\bar{L}) > 0$ or $\text{vol}(L_{\mathbb{Q}}) > 0$ holds. Thus we may assume, further, that $\bar{L}$ is C^{∞} .

Let $\bar{A}$ be an ample C^{∞} -Hermitian invertible sheaf on X . By [16, Theorem 1.1.1] and Theorem 2.1.1, the inequality

$$\hat{h}^1(H^0(X, mL - \bar{A}), \|\cdot\|_{\sup}^{m\bar{L} - \bar{A}}) \geq \hat{h}^1(H^0(X, mL), \|\cdot\|_{\sup}^{m\bar{L}}) + O(m^{d-1} \log m)$$

holds for any integer $m \gg 0$. Thus, for a sufficiently large m ,

$$\hat{h}^1(H^0(X, mL - A), \|\cdot\|_{\sup}^{m\bar{L}-\bar{A}}) > 0.$$

Hence, $mL_{\mathbb{Q}} - A_{\mathbb{Q}}$ is effective and $L_{\mathbb{Q}}$ is big. $\square$

Yuan introduces the notion of “strongly-bigness” in [22, Definition 1.1]. By using our results, we can define the “strongly-bigness” for C^0 -Hermitian $\mathbb{Q}$ -invertible sheaves.

Definition 3.3.1. A C^0 -Hermitian $\mathbb{Q}$ -invertible sheaf $\bar{L}$ on X is said to be *strongly big* if $\widehat{\text{vol}}^{\hat{\chi}}(\bar{L}) > 0$.

Since Proposition 3.2.6 holds also for C^0 -Hermitian $\mathbb{Q}$ -invertible sheaves, one finds the following.

Proposition 3.3.5. *Let $\bar{L}$ be a C^0 -Hermitian $\mathbb{Q}$ -invertible sheaf on X . Then the following two conditions are equivalent.*

- (1) $L_{\mathbb{Q}}$ is big on $X_{\mathbb{Q}}$.
- (2) There exists a positive real number $\lambda > 0$ such that $\bar{L}^{\lambda}$ is strongly big.

3.4. Continuity of the arithmetic volume functions. Let X be a projective arithmetic variety of dimension d . In this subsection, we prove that the arithmetic $\hat{\chi}$ -volume function $\widehat{\text{vol}}^{\hat{\chi}} : \widehat{\text{Pic}}(X; C^0)_{\mathbb{Q}} \rightarrow \mathbb{R}$ is continuous in the weak sense and extends uniquely to a function

$$\widehat{\text{vol}}^{\hat{\chi}} : \widehat{\text{Pic}}(X; C^0)_{\mathbb{R}} \rightarrow \mathbb{R}$$

that is continuous in the weak sense on $\widehat{\text{Pic}}(X; C^0)_{\mathbb{R}}$ (cf. Conventions and terminology, (2) or [16, Lemma 4.2]).

Theorem 3.4.1. *Let $V \subseteq \widehat{\text{Pic}}(X; C^0)_{\mathbb{Q}}$ be a finite dimensional $\mathbb{Q}$ -vector subspace of $\widehat{\text{Pic}}(X; C^0)_{\mathbb{Q}}$ and $\|\cdot\|_V$ a norm on $V \otimes_{\mathbb{Q}} \mathbb{R}$.*

- (1) *There is a positive constant $C_{\hat{\chi}}$, depending only on $(V, \|\cdot\|_V)$, such that the inequality*

$$|\widehat{\text{vol}}^{\hat{\chi}}(\xi^{\phi}) - \widehat{\text{vol}}^{\hat{\chi}}(\xi'^{\phi'})| \leq C_{\hat{\chi}}(\max\{\|\xi\|_V, \|\xi'\|_V\})^{d-1} \cdot (\|\xi - \xi'\|_V + \|\phi - \phi'\|_{\sup})$$

holds for any $\xi, \xi' \in V$ and for any $\phi, \phi' \in C^0(X)$.

- (2) *There is a positive constant C_1 , depending only on $(V, \|\cdot\|_V)$, such that the inequality*

$$|\widehat{\text{vol}}^1(\xi^{\phi}) - \widehat{\text{vol}}^1(\xi'^{\phi'})| \leq C_1(\max\{\|\xi\|_V, \|\xi'\|_V\})^{d-1} \cdot (\|\xi - \xi'\|_V + \|\phi - \phi'\|_{\sup})$$

holds for any $\xi, \xi' \in V$ and for any $\phi, \phi' \in C^0(X)$.

Let $r, r' > 0$ be positive integers and $\bar{L} = (\bar{L}(1), \dots, \bar{L}(r))$ and $\bar{A} = (\bar{A}(1), \dots, \bar{A}(r'))$ two families of C^{∞} -Hermitian $\mathbb{Q}$ -invertible sheaves on X .

Lemma 3.4.2. *There is a positive constant C , depending only on $\bar{L}$, $\bar{A}$, and X , such that*

$$|\widehat{\text{vol}}^{\hat{\chi}}(\mathbf{a} \cdot \bar{L} + \mathbf{b} \cdot \bar{A}) - \widehat{\text{vol}}^{\hat{\chi}}(\mathbf{a} \cdot \bar{L})| \leq C(\|\mathbf{a}\|_1 + \|\mathbf{b}\|_1)^{d-1} \cdot \|\mathbf{b}\|_1$$

holds for any $\mathbf{a} \in \mathbb{Q}^r$ and for any $\mathbf{b} \in \mathbb{Q}^{r'}$.

Proof of Lemma 3.4.2. By Theorem 3.1.1 and Corollary 3.3.3, we may assume that the following three conditions are satisfied:

- (i) X is generically smooth and normal.
- (ii) $\bar{L}(1), \dots, \bar{L}(r)$ and $\bar{A}(1), \dots, \bar{A}(r')$ are C^∞ -Hermitian invertible sheaves on X .
- (iii) $\mathbf{a} \in \mathbb{Z}^r$ and $\mathbf{b} \in \mathbb{Z}^{r'}$.

We prove the theorem by induction on r' . In the case where $r' = 1$, the lemma is nothing but Theorem 2.1.1. In fact, let $\bar{L}' = (\bar{L}(1), \dots, \bar{L}(r), \bar{\mathcal{O}}_X^{\text{can}})$ and $\mathbf{a}' = (\mathbf{a}(1), \dots, \mathbf{a}(r), \mathbf{b}(1)) \in \mathbb{Z}^{r+1}$. By Theorem 2.1.1, there are positive constants m_0 , C , and D , depending only on $\bar{L}'$, $\bar{A}(1)$, and X , such that the inequality

$$\begin{aligned} & |\hat{\chi}(\mathcal{H}^0(X, m\mathbf{a}' \cdot \bar{L}' + m\mathbf{b}(1)\bar{A}(1)), \|\cdot\|_{\sup}^{m\mathbf{a}' \cdot \bar{L}' + m\mathbf{b}(1)\bar{A}(1)}) \\ & \quad - \hat{\chi}(\mathcal{H}^0(X, m\mathbf{a}' \cdot \bar{L}'), \|\cdot\|_{\sup}^{m\mathbf{a}' \cdot \bar{L}'})| \\ & \leq Cm^d(\|\mathbf{a}\|_1 + |\mathbf{b}(1)|)^{d-1} \cdot |\mathbf{b}(1)| + Dm^{d-1}\|\mathbf{a}'\|_1 \log(m\|\mathbf{a}'\|_1) \end{aligned}$$

holds for any integers $m \in \mathbb{Z}_{\geq 0}$, $\mathbf{a} \in \mathbb{Z}^r$, and $\mathbf{b}(1) \in \mathbb{Z}$ with $\|\mathbf{a}\|_1 \neq 0$ and $m \geq m_0$. Thus, if $\|\mathbf{a}\|_1 \neq 0$, we find

$$|\widehat{\text{vol}}^{\hat{\chi}}(\mathbf{a} \cdot \bar{L} + \mathbf{b}(1) \cdot \bar{A}(1)) - \widehat{\text{vol}}^{\hat{\chi}}(\mathbf{a} \cdot \bar{L})| \leq C(\|\mathbf{a}\|_1 + |\mathbf{b}(1)|)^{d-1} \cdot |\mathbf{b}(1)|.$$

On the other hand, if $\|\mathbf{a}\|_1 = 0$, then the assertion is clear by Proposition 3.2.1(1).

We assume $r' \geq 2$ and put $\bar{A}' = (\bar{A}(1), \dots, \bar{A}(r' - 1))$ and $\mathbf{b}' = (\mathbf{b}(1), \dots, \mathbf{b}(r' - 1))$. By the induction hypothesis, the assertion holds for $r' - 1$ and

$$\begin{aligned} & |\widehat{\text{vol}}^{\hat{\chi}}(\mathbf{a} \cdot \bar{L} + \mathbf{b} \cdot \bar{A}) - \widehat{\text{vol}}^{\hat{\chi}}(\mathbf{a} \cdot \bar{L})| \\ & \leq |\widehat{\text{vol}}^{\hat{\chi}}(\mathbf{a} \cdot \bar{L} + \mathbf{b} \cdot \bar{A}) - \widehat{\text{vol}}^{\hat{\chi}}(\mathbf{a} \cdot \bar{L} + \mathbf{b}' \cdot \bar{A}')| \\ & \quad + |\widehat{\text{vol}}^{\hat{\chi}}(\mathbf{a} \cdot \bar{L} + \mathbf{b}' \cdot \bar{A}') - \widehat{\text{vol}}^{\hat{\chi}}(\mathbf{a} \cdot \bar{L})| \\ & \leq C(\|\mathbf{a}\|_1 + \|\mathbf{b}\|_1)^{d-1} \cdot \|\mathbf{b}(r')\| + C(\|\mathbf{a}\|_1 + \|\mathbf{b}'\|_1)^{d-1} \cdot \|\mathbf{b}'\|_1 \\ & \leq C(\|\mathbf{a}\|_1 + \|\mathbf{b}\|_1)^{d-1} \cdot \|\mathbf{b}\|_1 \end{aligned}$$

for $\mathbf{a} \in \mathbb{Z}^r$ and $\mathbf{b} \in \mathbb{Z}^{r'}$. □

By the same arguments as in the proof of [16, Corollary 3.3], we have the following.

Lemma 3.4.3. *Let $\bar{\mathbf{L}} = (\bar{\mathbf{L}}(1), \dots, \bar{\mathbf{L}}(r))$ and $\bar{\mathbf{A}} = (\bar{\mathbf{A}}(1), \dots, \bar{\mathbf{A}}(r'))$ be two families of C^0 -Hermitian $\mathbb{Q}$ -invertible sheaves on X . Then there is a positive constant C , depending only on X , $\bar{\mathbf{L}}$, and $\bar{\mathbf{A}}$, such that the inequality*

$$|\widehat{\text{vol}}^{\hat{\chi}}(\mathbf{a} \cdot \bar{\mathbf{L}} + \mathbf{b} \cdot \bar{\mathbf{A}} + \bar{\Theta}_X^{\text{can}, \phi}) - \text{vol}^{\hat{\chi}}(\mathbf{a} \cdot \bar{\mathbf{L}})| \leq C(\|\mathbf{a}\|_1 + \|\mathbf{b}\|_1)^{d-1} \cdot (\|\mathbf{b}\|_1 + \|\phi\|_{\text{sup}})$$

holds for all $\mathbf{a} \in \mathbb{Q}^r$, $\mathbf{b} \in \mathbb{Q}^{r'}$, and $\phi \in C^0(X)$.

Proof of Lemma 3.4.3. First, we suppose that both $\bar{\mathbf{L}}$ and $\bar{\mathbf{A}}$ are families of C^∞ -Hermitian $\mathbb{Q}$ -invertible sheaves on X . Then, by Lemma 3.4.2 and Proposition 3.2.6, we have the assertion. Next, we consider the general case. We may assume that either $\|\mathbf{b}\|_1 > 0$ or $\|\phi\|_{\text{sup}} > 0$ holds. We can take a constant D as

$$\max\{\text{vol}(\mathbf{a} \cdot \mathbf{L}_{\mathbb{Q}}), \text{vol}(\mathbf{a} \cdot \mathbf{L}_{\mathbb{Q}} + \mathbf{b} \cdot \mathbf{A}_{\mathbb{Q}})\} \leq D(\|\mathbf{a}\|_1 + \|\mathbf{b}\|_1)^{d-1}$$

(cf. [16, Lemma 3.1]). By Lemma 3.2.3, there are two families of C^0 -functions on X , $\boldsymbol{\psi} = (\boldsymbol{\psi}(1), \dots, \boldsymbol{\psi}(r))$ and $\boldsymbol{\psi}' = (\boldsymbol{\psi}'(1), \dots, \boldsymbol{\psi}'(r'))$, such that both $\bar{\mathbf{L}}^{\boldsymbol{\psi}} = (\bar{\mathbf{L}}(1)^{\boldsymbol{\psi}(1)}, \dots, \bar{\mathbf{L}}(r)^{\boldsymbol{\psi}(r)})$ and $\bar{\mathbf{A}}^{\boldsymbol{\psi}'} = (\bar{\mathbf{A}}(1)^{\boldsymbol{\psi}'(1)}, \dots, \bar{\mathbf{A}}(r)^{\boldsymbol{\psi}'(r)})$ are C^∞ and

$$\max\{\|\mathbf{a} \cdot \boldsymbol{\psi}\|_{\text{sup}}, \|\mathbf{a} \cdot \boldsymbol{\psi} + \mathbf{b} \cdot \boldsymbol{\psi}'\|_{\text{sup}}\} \leq \frac{\|\mathbf{b}\|_1 + \|\phi\|_{\text{sup}}}{2dD}.$$

Then we obtain

$$\begin{aligned} & |\widehat{\text{vol}}^{\hat{\chi}}(\mathbf{a} \cdot \bar{\mathbf{L}} + \mathbf{b} \cdot \bar{\mathbf{A}} + \bar{\Theta}_X^{\phi}) - \text{vol}^{\hat{\chi}}(\mathbf{a} \cdot \bar{\mathbf{L}})| \\ & \leq |\widehat{\text{vol}}^{\hat{\chi}}(\mathbf{a} \cdot \bar{\mathbf{L}} + \mathbf{b} \cdot \bar{\mathbf{A}} + \bar{\Theta}_X^{\phi}) - \text{vol}^{\hat{\chi}}(\mathbf{a} \cdot \bar{\mathbf{L}}^{\boldsymbol{\psi}} + \mathbf{b} \cdot \bar{\mathbf{A}}^{\boldsymbol{\psi}'} + \bar{\Theta}_X^{\phi})| \\ & \quad + |\widehat{\text{vol}}^{\hat{\chi}}(\mathbf{a} \cdot \bar{\mathbf{L}}^{\boldsymbol{\psi}} + \mathbf{b} \cdot \bar{\mathbf{A}}^{\boldsymbol{\psi}'} + \bar{\Theta}_X^{\phi}) - \text{vol}^{\hat{\chi}}(\mathbf{a} \cdot \bar{\mathbf{L}}^{\boldsymbol{\psi}})| \\ & \quad + |\widehat{\text{vol}}^{\hat{\chi}}(\mathbf{a} \cdot \bar{\mathbf{L}}^{\boldsymbol{\psi}}) - \text{vol}^{\hat{\chi}}(\mathbf{a} \cdot \bar{\mathbf{L}})| \\ & \leq d \text{vol}(\mathbf{a} \cdot \mathbf{L}_{\mathbb{Q}} + \mathbf{b} \cdot \mathbf{A}_{\mathbb{Q}}) \cdot \|\mathbf{a} \cdot \boldsymbol{\psi} + \mathbf{b} \cdot \boldsymbol{\psi}'\|_{\text{sup}} \\ & \quad + C(\|\mathbf{a}\|_1 + \|\mathbf{b}\|_1)^{d-1} \cdot (\|\mathbf{b}\|_1 + \|\phi\|_{\text{sup}}) + d \text{vol}(\mathbf{a} \cdot \mathbf{L}_{\mathbb{Q}}) \cdot \|\mathbf{a} \cdot \boldsymbol{\psi}\|_{\text{sup}} \\ & \leq (C+1)(\|\mathbf{a}\|_1 + \|\mathbf{b}\|_1)^{d-1} \cdot (\|\mathbf{b}\|_1 + \|\phi\|_{\text{sup}}) \end{aligned}$$

as desired. $\square$

Proof of Theorem 3.4.1. Let $v_1, \dots, v_r \in V$ be an $\mathbb{R}$ -basis for $V \otimes_{\mathbb{Q}} \mathbb{R}$ and $\|\cdot\|_1$ the L^1 -norm associated with $v_1, \dots, v_r$. Then there is a positive constant D such that

$$D^{-1}\|\cdot\|_1 \leq \|\cdot\|_V \leq D\|\cdot\|_1$$

hold on $V \otimes_{\mathbb{Q}} \mathbb{R}$. Thus it is sufficient to show the theorem for $\|\cdot\|_1$.

(1) By Lemma 3.4.3,

$$\begin{aligned} & |\widehat{\text{vol}}^{\hat{\chi}}(\xi^\phi) - \widehat{\text{vol}}^{\hat{\chi}}(\xi'^{\phi'})| \\ &= |\widehat{\text{vol}}^{\hat{\chi}}(\xi'^{\phi'} + (\xi - \xi') + \bar{\mathcal{O}}_X^{\phi - \phi'}) - \widehat{\text{vol}}^{\hat{\chi}}(\xi'^{\phi'})| \\ &\leq C(\|\xi'\|_1 + \|\xi - \xi'\|_1)^{d-1} \cdot (\|\xi - \xi'\|_1 + \|\phi - \phi'\|_{\text{sup}}) \\ &\leq 3^{d-1} C(\max\{\|\xi\|_1, \|\xi'\|_1\})^{d-1} \cdot (\|\xi - \xi'\|_1 + \|\phi - \phi'\|_{\text{sup}}). \end{aligned}$$

(2) Since

$$|\widehat{\text{vol}}^1(\xi^\phi) - \widehat{\text{vol}}^1(\xi'^{\phi'})| \leq |\widehat{\text{vol}}^0(\xi^\phi) - \widehat{\text{vol}}^0(\xi'^{\phi'})| + |\widehat{\text{vol}}^{\hat{\chi}}(\xi^\phi) - \widehat{\text{vol}}^{\hat{\chi}}(\xi'^{\phi'})|$$

by Proposition 3.2.6(3), the assertion follows from [16, Corollary 3.3] and (1). $\square$

According to [16], we define the group $\widehat{\text{Pic}}_{\mathbb{R}}(X; C^0)$ of isometry classes of C^0 -Hermitian $\mathbb{R}$ -invertible sheaves on X as follows. Let us consider the natural homomorphism $\iota : C^0(X) \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow C^0(X)$ given by $\phi \otimes x \mapsto x\phi$. Let N be the image of $\text{Ker}(\iota)$ via $C^0(X) \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow \widehat{\text{Pic}}(X; C^0)_{\mathbb{R}} = \widehat{\text{Pic}}(X; C^0) \otimes_{\mathbb{Z}} \mathbb{R}$ and define

$$\widehat{\text{Pic}}_{\mathbb{R}}(X; C^0) := \widehat{\text{Pic}}(X; C^0)_{\mathbb{R}} / N.$$

There is a natural surjective homomorphism $\widehat{\text{Pic}}(X; C^0)_{\mathbb{R}} \rightarrow \widehat{\text{Pic}}_{\mathbb{R}}(X; C^0)$ by definition. By using Theorem 3.4.1 and the same arguments as in the proof of [16, Theorem 4.4], one finds the following.

Corollary 3.4.4. (1) *The function $\widehat{\text{vol}}^{\hat{\chi}} : \widehat{\text{Pic}}(X; C^0)_{\mathbb{Q}} \rightarrow \mathbb{R}$ (resp. $\widehat{\text{vol}}^1 : \widehat{\text{Pic}}(X; C^0)_{\mathbb{Q}} \rightarrow \mathbb{R}$) extends uniquely to a function*

$$\begin{aligned} & \widehat{\text{vol}}^{\hat{\chi}} : \widehat{\text{Pic}}(X; C^0)_{\mathbb{R}} \rightarrow \mathbb{R} \\ & \text{(resp. } \widehat{\text{vol}}^1 : \widehat{\text{Pic}}(X; C^0)_{\mathbb{R}} \rightarrow \mathbb{R}) \end{aligned}$$

on $\widehat{\text{Pic}}(X; C^0)_{\mathbb{R}}$ that is positively homogeneous of degree d and continuous in the weak sense (see Conventions and terminology, (2)).

(2) *The above function $\widehat{\text{vol}}^{\hat{\chi}} : \widehat{\text{Pic}}(X; C^0)_{\mathbb{R}} \rightarrow \mathbb{R}$ (resp. $\widehat{\text{vol}}^1 : \widehat{\text{Pic}}(X; C^0)_{\mathbb{R}} \rightarrow \mathbb{R}$) passes through $\widehat{\text{Pic}}_{\mathbb{R}}(X; C^0)$.*

3.5. Applications. Let X be a projective arithmetic variety of dimension d and $\bar{L}$ a C^∞ -Hermitian $\mathbb{Q}$ -invertible sheaf on X . In this subsection, we prove the arithmetic Hilbert-Samuel formula ([1, Théorème principal], [9, Theorem 9], [26, Theorem (1.4)], and [15, Corollary 5.5]) for *vertically nef* Hermitian invertible sheaves (Conventions and terminology, (8)). Note that Moriawaki [15, Corollary 5.5] previously obtains the formula for *nef* Hermitian invertible sheaves.

Theorem 3.5.1. *If $\bar{L}$ is vertically nef, then*

$$\widehat{\text{vol}}^{\hat{\chi}}(\bar{L}) = \widehat{\deg}(\hat{c}_1(\bar{L})^{\cdot d}).$$

In particular, a vertically nef C^∞ -Hermitian $\mathbb{Q}$ -invertible sheaf $\bar{L}$ on X is strongly big if and only if $\widehat{\deg}(\hat{c}_1(\bar{L})^{\cdot d}) > 0$.

Proof. First, we suppose that $L_{\mathbb{Q}}$ is ample on $X_{\mathbb{Q}}$ and that $\bar{L}$ is vertically nef. By the homogeneity of the function $\widehat{\text{vol}}^{\hat{\chi}}$, we may suppose, further, that $\bar{L}$ is a C^∞ -Hermitian invertible sheaf on X . By virtue of the arithmetic Hilbert-Samuel formula of Zhang [26, Theorem (1.4)],

$$\hat{\chi}(H^0(X, mL), \|\cdot\|_{\sup}^{m\bar{L}}) = \frac{\widehat{\deg}(\hat{c}_1(\bar{L})^{\cdot d})}{d!} m^d + O(m^{d-1} \log m)$$

for any integer $m \gg 0$. Thus, in this case, $\widehat{\text{vol}}^{\hat{\chi}}(\bar{L}) = \widehat{\deg}(\hat{c}_1(\bar{L})^{\cdot d})$.

In general, let $\bar{A}$ be a C^∞ -Hermitian invertible sheaf on X such that $A_{\mathbb{Q}}$ is ample on $X_{\mathbb{Q}}$ and $\bar{A}$ is vertically nef. Then, for any positive rational number $\epsilon \in \mathbb{Q}_{>0}$, $\bar{L} + \epsilon\bar{A}$ satisfies the hypotheses of [26, Theorem (1.4)] and the equality

$$\widehat{\text{vol}}^{\hat{\chi}}(\bar{L} + \epsilon\bar{A}) = \widehat{\deg}(\hat{c}_1(\bar{L} + \epsilon\bar{A})^{\cdot d})$$

holds. Since the function $\widehat{\text{vol}}^{\hat{\chi}} : \widehat{\text{Pic}}(X)_{\mathbb{R}} \rightarrow \mathbb{R}$ is continuous, $\widehat{\text{vol}}^{\hat{\chi}}(\bar{L}) = \widehat{\deg}(\hat{c}_1(\bar{L})^{\cdot d})$. $\square$

Remark 3.5.2. We note that the generalized Hodge index theorem [15, Theorem 6.2] and its corollary [15, Corollary 6.3] are still valid if we replace $\widehat{\text{vol}}$ with $\widehat{\text{vol}}^{\hat{\chi}}$.

Combining our results with Corollary 2.2.8, we can generalize Yuan's theorem to Hermitian $\mathbb{Q}$ -invertible sheaves.

Theorem 3.5.3. *Let $\bar{L}$ and $\bar{M}$ be two C^∞ -Hermitian $\mathbb{Q}$ -invertible sheaves on X . Assume that $\bar{L}$ and $\bar{M}$ are both nef. Then*

$$\widehat{\text{vol}}^{\hat{\chi}}(\bar{L} - \bar{M}) \geq \widehat{\deg}(\hat{c}_1(\bar{L})^{\cdot d}) - d \widehat{\deg}(\hat{c}_1(\bar{L})^{\cdot (d-1)} \cdot \hat{c}_1(\bar{M})).$$

In particular, if $\widehat{\deg}(\hat{c}_1(\bar{L})^{\cdot d}) > d \widehat{\deg}(\hat{c}_1(\bar{L})^{\cdot (d-1)} \cdot \hat{c}_1(\bar{M}))$, then $\bar{L} - \bar{M}$ is strongly big.

Remark 3.5.4. In [17], Moriawaki defines the arithmetic self intersection number $\widehat{\deg}(\hat{c}_1(\bar{L})^{\cdot d})$ for a *vertically nef* C^0 -Hermitian $\mathbb{R}$ -invertible sheaf $\bar{L}$ on X . By use of his definition, Theorems 3.5.1 and 3.5.3 are valid for C^0 -Hermitian $\mathbb{R}$ -invertible sheaves $\bar{L}$, $\bar{M}$ on X in place of C^∞ -Hermitian $\mathbb{Q}$ -invertible sheaves. However, in this subsection, we do not treat the C^0 -Hermitian $\mathbb{R}$ -invertible sheaves for simplicity of statements (cf. Conventions and terminology, (8)).

Our function $\widehat{\text{vol}}^1$ measures the difference between the vertically nef Hermitian invertible sheaves and the nef ones. By the continuity of the function $\widehat{\text{vol}}^1$, we easily obtain the following.

Theorem 3.5.5. *Let $\bar{L}$ be a C^∞ -Hermitian $\mathbb{Q}$ -invertible sheaf on X . Then*

$$\bar{L} \text{ is nef} \quad \Rightarrow \quad \widehat{\text{vol}}^1(\bar{L}) = 0.$$

The converse to Theorem 3.5.5 is trivially false even if X is an arithmetic surface and $\bar{L}$ is vertically nef (see Theorem 3.5.7). However, we may expect that a partial converse to Theorem 3.5.5 might be true (cf. [18]).

Proposition 3.5.6. *Let X be a projective arithmetic curve and $\bar{L}$ a Hermitian invertible sheaf on X . Then, the arithmetic volumes of $\bar{L}$ are given by*

$$\begin{aligned} \widehat{\text{vol}}^{\hat{X}}(\bar{L}) &= \widehat{\deg}(\bar{L}), \\ \widehat{\text{vol}}^0(\bar{L}) &= \begin{cases} \widehat{\deg}(\bar{L}) & \text{if } \widehat{\deg}(\bar{L}) > 0, \\ 0 & \text{if } \widehat{\deg}(\bar{L}) \leq 0, \end{cases} \\ \widehat{\text{vol}}^1(\bar{L}) &= \begin{cases} 0 & \text{if } \widehat{\deg}(\bar{L}) > 0, \\ -\widehat{\deg}(\bar{L}) & \text{if } \widehat{\deg}(\bar{L}) \leq 0. \end{cases} \end{aligned}$$

Proof. It is well known that $\widehat{\text{vol}}^{\hat{X}}(\bar{L}) = \widehat{\deg}(\bar{L})$. If $\widehat{\deg}(\bar{L}) > 0$, then $\bar{L}$ is ample and

$$\widehat{\text{vol}}^0(\bar{L}) = \widehat{\deg}(\bar{L}), \quad \widehat{\text{vol}}^1(\bar{L}) = 0.$$

If $\widehat{\deg}(\bar{L}) < 0$, then $-\bar{L}$ is ample. Since $\widehat{\text{vol}}^1(\bar{L}) = \widehat{\text{vol}}^0(-\bar{L}) = -\widehat{\deg}(\bar{L})$,

$$\widehat{\text{vol}}^0(\bar{L}) = 0, \quad \widehat{\text{vol}}^1(\bar{L}) = -\widehat{\deg}(\bar{L}).$$

Lastly, we consider the case where $\widehat{\deg}(\bar{L}) = 0$. Since $\widehat{\deg}(m\bar{L}) = 0$ for any positive integer $m \geq 1$,

$$\frac{1}{2} \sum_{x \in X(\mathbb{C})} \log h_x^{\bar{L}}(s(x), s(x)) = \log \#(mL/s\mathcal{O}_X) \geq 0$$

for any non-zero section $s \in mL$. Thus, if $\|s\|_{\sup}^{m\bar{L}} \leq 1$, then s is a generator of mL and $|s|_{m\bar{L}}(x) = 1$ for $x \in X(\mathbb{C})$. Hence,

$$\#\{s \in H^0(X, mL) \mid \|s\|_{\sup}^{m\bar{L}} \leq 1\} \leq \#(\mathcal{O}_X^\times)_{\text{tor}} + 1$$

for any $m \geq 1$ and $\widehat{\text{vol}}^0(\bar{L}) = \widehat{\text{vol}}^1(\bar{L}) = 0$. □

By use of our results, we can improve the classical Hodge index theorem of Faltings-Hriljac [25, Theorem 7.1].

Theorem 3.5.7. *Let K be a number field and $X \rightarrow \operatorname{Spec}(\mathcal{O}_K)$ a projective arithmetic surface over $\operatorname{Spec}(\mathcal{O}_K)$. Suppose that the generic fiber X_K is geometrically irreducible, smooth, and of positive genus. Let $\bar{L}$ be a C^∞ -Hermitian invertible sheaf on X . If $\bar{L}$ is vertically nef and $\deg(L_K) = 0$, then there is a positive integer $m \geq 1$ and a locally constant function $\lambda \in C^\infty(X)$ such that $m\bar{L}$ is isometric to $\bar{\mathcal{O}}_X^{\operatorname{can}, \lambda}$. In particular, $\bar{L}$ is nef if and only if*

$$\sum_{\gamma: K \rightarrow \mathbb{C}} \lambda_\gamma \geq 0.$$

We would like to see an example given by Moriwaki [18], which shows that a partial converse to Theorem 3.5.5 holds on the projective spaces $\mathbb{P}_{\mathbb{Z}}^n$.

Example 3.5.1. Let $X = \mathbb{P}_{\mathbb{Z}}^n$ be the projective space of dimension $n + 1$ and $\mathbf{a} = (a_0, a_1, \dots, a_n) \in \mathbb{R}_{>0}^{n+1}$ a sequence of positive real numbers. The Hermitian matrix $\operatorname{diag}(1/a_0, 1/a_1, \dots, 1/a_n)$ defines a Hermitian metric on $\mathbb{C}^{n+1}$. Let $h_{\mathbf{a}}^{\operatorname{FS}}$ be the Fubini-Study metric on $\mathcal{O}_X(1)$ induced from the metric on $\mathbb{C}^{n+1}$ defined by $\mathbf{a}$ and consider the C^∞ -Hermitian invertible sheaf

$$\bar{D}_{\mathbf{a}} := (\mathcal{O}_X(1), h_{\mathbf{a}}^{\operatorname{FS}}).$$

Since $\mathcal{O}_X(1)$ is ample, we do not need to use our Theorem 3.5.1 in this case. Let $\varphi_{\mathbf{a}} : \mathbb{R}_{\geq 0}^{n+1} \rightarrow \mathbb{R}$ be the function given by

$$\varphi_{\mathbf{a}}(x_0, x_1, \dots, x_n) = - \sum_{i=0}^n x_i \log x_i + \sum_{i=0}^n x_i \log a_i,$$

and let $\Delta_{\mathbf{a}}$, $\Theta_{\mathbf{a}}^+$, and $\Theta_{\mathbf{a}}^-$ be the subsets of $\mathbb{R}_{\geq 0}^n$ defined as

$$\begin{aligned} \Delta_{\mathbf{a}} &= \{(x_1, \dots, x_n) \in \mathbb{R}_{\geq 0}^n \mid x_1 + \dots + x_n \leq 1\}, \\ \Theta_{\mathbf{a}}^+ &= \{(x_1, \dots, x_n) \in \Delta_{\mathbf{a}} \mid \varphi_{\mathbf{a}}(1 - x_1 - \dots - x_n, x_1, \dots, x_n) \geq 0\}, \\ \Theta_{\mathbf{a}}^- &= \{(x_1, \dots, x_n) \in \Delta_{\mathbf{a}} \mid \varphi_{\mathbf{a}}(1 - x_1 - \dots - x_n, x_1, \dots, x_n) \leq 0\}. \end{aligned}$$

One of the main results of [18] is the following.

Proposition 3.5.8. *The arithmetic volumes of $\bar{D}_{\mathbf{a}}$ can be computed as follow (the integral formulae):*

$$\begin{aligned} \widehat{\operatorname{vol}}^{\hat{\chi}}(\bar{D}_{\mathbf{a}}) &= \frac{(n+1)!}{2} \int_{\Delta_{\mathbf{a}}} \varphi_{\mathbf{a}}(1 - x_1 - \dots - x_n, x_1, \dots, x_n) dx_1 \dots dx_n \\ &= \frac{n+1}{2} \left(\frac{1}{2} + \dots + \frac{1}{n+1} \right) + \frac{1}{2} \log(a_0 \dots a_n), \\ \widehat{\operatorname{vol}}^0(\bar{D}_{\mathbf{a}}) &= \frac{(n+1)!}{2} \int_{\Theta_{\mathbf{a}}^+} \varphi_{\mathbf{a}}(1 - x_1 - \dots - x_n, x_1, \dots, x_n) dx_1 \dots dx_n, \\ \widehat{\operatorname{vol}}^1(\bar{D}_{\mathbf{a}}) &= - \frac{(n+1)!}{2} \int_{\Theta_{\mathbf{a}}^-} \varphi_{\mathbf{a}}(1 - x_1 - \dots - x_n, x_1, \dots, x_n) dx_1 \dots dx_n \end{aligned}$$

and the following four conditions are all equivalent (see also Yuan [22, §2.2, Remark]).

- (1) $\bar{D}_a$ is nef.
- (2) $a_0 \geq 1, a_1 \geq 1, \dots, a_n \geq 1$.
- (3) $\varphi_a(1 - x_1 - \dots - x_n, x_1, \dots, x_n) \geq 0$ on Δ_a .
- (4) $\widehat{\text{vol}}^1(\bar{D}_a) = 0$.

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