

Kyoto University
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On the Chiralization of Poisson-Lie Theory with Applications in Integrable Systems

Doctoral Dissertation

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Abstract

The present thesis is based on two original papers by the author ^[ref]. Its goal is to develop a framework that *chiralizes* Poisson–Lie theory and, as a consequence, produces integrable hierarchies.

Let $\mathfrak{g}$ be a quadratic Lie algebra equipped with an ad-invariant bilinear form $(\cdot, \cdot)$. Then the dual space $\mathfrak{g}^*$ carries the Kirillov-Kostant-Souriau (KKS) Poisson structure, from which one can construct a classical R -matrix on $\mathfrak{g}$. The associated R -matrix formalism yields many integrable hierarchies.

In the language of Poisson vertex algebras (PVAs), a Lie conformal algebra (LCA) plays a role analogous to that of a Lie algebra in finite-dimensional Poisson geometry. In particular, the universal enveloping vertex algebra of an LCA L admits a classical limit, which gives rise to a PVA.

In the first part of this thesis, we recall and slightly refine the notion of a classical R -matrix on a Lie conformal algebra, as developed in the first paper. In the second part, we provide a geometric explanation of such R -matrices in terms of shifted symplectic structures. In particular, we describe the classical R -matrix via a Maurer-Cartan equation. Motivated by this geometric picture, we also define a classical R -matrix on finite and affine $\mathcal{W}$ -algebras, and construct integrable systems on them by an AKS-type argument.

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Statement of Authorship

This thesis contains original work conducted by the author unless otherwise stated.

Brief declaration of contributions, especially for co-authored chapters. If chapters are based on published or submitted papers, state each co-author's contribution explicitly.

List of Publications

Main Papers (forming the core of this thesis)

- (1) Wenda Fang, “Generalized AKS scheme of integrability via vertex algebra”, *International Mathematics Research Notices* **2025** (11), rnaf143, 1–48.
- (2) Wenda Fang, “Shifted symplectic structures and Poisson vertex algebra”, *arXiv preprint* arXiv:2601.17840 [math.QA], 2026.

Introduction

The classical R -matrix of Lie algebras first appeared in Sklyanin’s work as a “by-product” of the quantum inverse-scattering method [78]; it also appeared earlier in a paper by Michaelis [92]. Later, in [104], Semenov-Tyan-Shanskii established the connection between classical R -matrices and integrable systems. Let $\mathfrak{g}$ be a Lie algebra with a nondegenerate, ad-invariant bilinear form $(\cdot, \cdot)$. An operator $R \in \text{End}\mathfrak{g}$ is called a classical R -matrix of $\mathfrak{g}$ if the new bracket

$$[X, Y]_R := [R(X), Y] + [X, R(Y)], \quad X, Y \in \mathfrak{g},$$

turns the vector space $\mathfrak{g}$ into a Lie algebra. In [104], Semenov-Tyan-Shanskii proved a sufficient condition for $R \in \text{End}\mathfrak{g}$ to be a classical R -matrix of $\mathfrak{g}$, namely the modified Yang–Baxter equation. The (unmodified) classical Yang–Baxter equation was interpreted there as a “unitary” condition on R . These results inspired numerous works in various areas, such as quantum groups [32] and Poisson–Lie groups [79]. Additionally, Semenov-Tyan-Shanskii demonstrated the connection between the classical R -matrix and the Adler–Kostant–Symes (AKS) scheme, which is a powerful tool for constructing integrable systems such as the Toda lattice. The AKS scheme can also be used to derive the KdV hierarchy; see [1]. Furthermore, in the same paper, Semenov-Tyan-Shanskii explored the relation between the Riemann–Hilbert problem and the R -matrix method. In [98], Reyman and Semenov-Tyan-Shanskii provided a geometric interpretation of the AKS scheme and of the classical R -matrix. Classical R -matrix theory also appears in the physical understanding of integrability; see [15, 35, 36].

Vertex algebras, introduced by Borchers [21], may be viewed as an axiomatic framework for two-dimensional chiral conformal field theory. In the two-dimensional case, conformal symmetry simplifies the state space of the quantum field theory: typically, it becomes a highest weight module for certain infinite-dimensional Lie algebras, such as Kac–Moody algebras; see [19, 101]. Vertex algebras naturally give rise to Lie conformal algebras, and conversely one can reconstruct vertex algebras from Lie conformal algebras (for precise definitions, see Section 1). From an algebraic viewpoint, vertex (operator) algebras are analogous to both Lie algebras and commutative associative algebras. It is therefore natural to generalize the classical R -matrix to the setting of vertex algebras. X. Xu introduced a classical R -matrix for vertex operator algebras [111], a special class of vertex algebras equipped with a conformal vector. However, constructing examples using Xu’s definition appears to be quite difficult. In contrast, in our construction it is relatively easy to produce examples. Poisson vertex algebras and R -matrices were also

discussed in [46]. De Sole, Kac, and Valeri studied the connections between $\mathcal{W}$ -algebras and R -matrices in [43, 44].

As an analogue of Semenov-Tyan-Shanskii's work [104], we obtain a similar result, which we call the *Factorization Theorem*, and which is presented in Section 2. Furthermore, in Section 1, we prove that if L is a Lie conformal algebra and an endomorphism $R_L \in \text{End}(L)$ satisfies the modified Yang–Baxter equation (12) for Lie conformal algebras, then R_L is a classical R -matrix of L . In particular, the Factorization Theorem provides an effective method for constructing R -matrices for affine Lie conformal algebras. For example, let $\mathfrak{g}$ be a Lie algebra with a nondegenerate ad-invariant bilinear form $(\cdot, \cdot)$ and a vector space decomposition

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b},$$

where $\mathfrak{a}$ is a Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{b}$ is an isotropic subalgebra of $\mathfrak{g}$. This decomposition automatically induces a factorization of the corresponding Lie conformal algebra,

$$v_k(\mathfrak{g}) = v_k(\mathfrak{a}) \oplus v'_k(\mathfrak{b})$$

(see Example 1.1.3 and Theorem 2.2.3 for definitions). Let P_+ and P_- denote the projections from $v_k(\mathfrak{g})$ onto $v_k(\mathfrak{a})$ and $v'_k(\mathfrak{b})$, respectively. Then the operator

$$R_L = \frac{1}{2}(P_+ - P_-)$$

is a classical R -matrix of the Lie conformal algebra $v_k(\mathfrak{g})$. A similar construction for Lie algebras appeared earlier; see [104, 82]. One application of R -matrices for Lie conformal algebras is that they can be used to construct new vertex algebras; explicit examples are given in Section 2.3. A natural idea is to construct factorizable R -matrices (see Theorem 2.2.3 and Corollary 2.2.6) via Manin triples of Lie algebras, which suggests a connection between our theory and quantum groups.

From the perspective of mathematical physics, a Poisson vertex algebra (PVA) can be obtained as the (quasi-)classical limit of a vertex algebra [37, 75, 84]. More mathematically, a PVA is the Poisson analogue of a Lie conformal algebra. Since every vertex algebra carries a natural filtration and its structure is quite complicated, in many situations one instead studies the geometry of the associated *vertex Poisson scheme*, i.e. the spectrum of the associated Poisson vertex algebra; see [5, 6, 7, 60].

One source of PVAs comes from the following problem. A longstanding question, stated by A. Belavin in [16], concerns the connection between $\mathcal{W}$ -algebras and integrable systems, particularly the KdV equation; related ideas already appeared in earlier work, e.g. [61]. This problem was resolved in a series of papers by Kac and collaborators (see [40]). In particular, in [12] they showed that Poisson vertex algebras play an important role in infinite-dimensional Hamiltonian integrable systems. Several connections between PVAs, Dubrovin–Novikov theory [48, 50], and Dubrovin–Zhang theory [51] were indicated in [26].

Another goal of this thesis is to generalize the AKS scheme using the classical R -matrix introduced above. Following [104], we recall the key observation underlying the classical AKS scheme and its relation to classical R -matrices; see also [76, 20, 1]. For a Lie algebra $\mathfrak{g}$ with a nondegenerate, ad-invariant bilinear form $(\cdot, \cdot)$ and a classical R -matrix on $\mathfrak{g}$, there exist two Poisson structures on $\mathfrak{g}^*$, induced by the Lie brackets $[\cdot, \cdot]$ and $[\cdot, \cdot]_R$. These are the (Lie-)Poisson–Kirillov–Kostant–Souriau Poisson structures. Denote the corresponding Poisson brackets by $\{\cdot, \cdot\}$ and $\{\cdot, \cdot\}_R$. The key observation of the AKS scheme is that elements in the Poisson center with respect to $\{\cdot, \cdot\}$ Poisson-commute with respect to $\{\cdot, \cdot\}_R$. Therefore one obtains a family of first integrals. Restricting to algebraic functions on $\mathfrak{g}^*$, we identify $\mathbb{C}[\mathfrak{g}^*]$ with $S(\mathfrak{g})$ as Poisson algebras. Physically, for the dynamical system on $\mathfrak{g}^*$, the functions describing physical observables lie in $S(\mathfrak{g})$. This observation provides an effective method for constructing integrable systems. In addition, it has been used to explain various aspects of soliton equations; see [107].

Poisson vertex algebras offer a natural framework for infinite-dimensional (field-theoretic) integrable systems; see Section 2. Let L be a Lie conformal algebra with λ -bracket $\{\cdot_\lambda \cdot\}$ and let R_L be a classical R -matrix of L . Then L carries two λ -brackets, namely $\{\cdot_\lambda \cdot\}$ and $\{\cdot_\lambda \cdot\}_{R_L}$; see Section 2.2. Consider the Poisson vertex algebra $\mathcal{V} := S^\bullet(L)$ and extend these λ -brackets using the Master Formula (5). This yields two λ -brackets on $\mathcal{V}$.

INVOLUTION THEOREM. Let L be a Lie conformal algebra with λ -bracket $\{\cdot_\lambda \cdot\}$, and let R_L be a classical R -matrix of L . Define $\mathcal{V} := S^\bullet(L)$ and let $\mathfrak{z}(\mathcal{V})$ denote the center of $\mathcal{V}$ with respect to $\{\cdot_\lambda \cdot\}$. Then, for any $f, g \in \mathfrak{z}(\mathcal{V})$, we have $\{f_\lambda g\}_{R_L} = 0$.

This provides a method to construct Liouville integrable systems. Let $\mathcal{V}$, $\{\cdot_\lambda \cdot\}$, and $\{\cdot_\lambda \cdot\}_{R_L}$ be as in the Involution Theorem. Choose a family of linearly independent polynomials $h_n \in \mathfrak{z}(\mathcal{V})$, $n = 1, 2, \dots$, and take the local functionals $\int h_n \in \mathcal{V}/\partial\mathcal{V}$ as Hamiltonians. This construction yields nontrivial integrable systems for the twisted λ -bracket $\{\cdot_\lambda \cdot\}_{R_L}$.

Due to the existence of the Feigin–Frenkel center [57], in the affine vertex algebra case we obtain the following corollary.

Corollary 1. Let $k = -h^\vee$ be the critical level of the affine vertex algebra $V^k(\mathfrak{g})$. Write $\mathfrak{z}(\widehat{\mathfrak{g}}) \subset V^{-h^\vee}(\mathfrak{g})$ for the Feigin–Frenkel center, and set

$$\mathcal{V} := \mathrm{gr}^F V^k(\mathfrak{g}) \cong S^{\bullet'}(v_0(\mathfrak{g}))$$

as Poisson vertex algebras (see Remark 1.1.6). If R_L is a classical R -matrix for the LCA $v_0(\mathfrak{g})$, then for all $f, g \in \mathrm{gr}^F \mathfrak{z}(\widehat{\mathfrak{g}})$ we have

$$\{f_\lambda g\}_{R_L} = 0.$$

Here we give a concrete example. Consider the affine vertex algebra $V^{-2}(\mathfrak{sl}_2)$ and the classical R -matrix of $v_0(\mathfrak{sl}_2)$ associated with the Borel decomposition (see Definition 2.2.7).

Applying our construction, one obtains the system

$$\begin{aligned}\frac{df}{dt_2} &= 0, & \frac{dh}{dt_2} &= -8fe(h'^2 + 4f'e' + 2ef'' + hh'' + 2fe''), \\ \frac{de}{dt_2} &= 4he(h'^2 + 4f'e' + 2ef'' + hh'' + 2fe''),\end{aligned}$$

where primes denote derivatives with respect to the spatial variable x , and this system is Liouville integrable. In Example 2.3.5, we show that one solution of this system is related to the one-soliton solution of KdV. In the same example, we also show that the solutions of this system are in bijection with the points of $\mathbb{CP}^1$.

Our approach extends the R -matrix method from Lie algebras to vertex algebras; in suitable special cases, it recovers the classical situation. It admits a geometric interpretation in the spirit of the classical AKS scheme and naturally suggests extensions toward classical $\mathcal{W}$ -algebras and the Khesin–Zakharevich Poisson–Lie group [88]. It would also be interesting to consider the corresponding quantization problem.

In the second part of this thesis, we introduce shifted symplectic structures and describe PVAs via shifted symplectic geometry. The theory of graded geometry plays an important role in many areas of mathematics and physics. Differential graded (dg) symplectic manifolds are central to the AKSZ formalism and the Poisson σ -model [3]. Within this framework, one associates a topological field theory to a dg-symplectic manifold; classical examples include the Poisson σ -model and Chern–Simons theory. Graded-geometric methods also provide a uniform approach to a variety of Poisson–Lie-theoretic structures, such as Courant algebroids [105] and generalized complex geometry [66], and they are useful in deformation quantization [77]. In recent years there has been intensive work on classical r -matrices, graded manifolds, and quasi-Poisson structures [2], as well as twisted Poisson σ -models [33, 72, 71]. Moreover, the relationship between graded manifolds and PVAs was investigated in Ryo Hayashi’s PhD thesis [68]. The seminal paper of Pantev, Toën, Vaquié, and Vezzosi on shifted symplectic structures has also profoundly influenced derived algebraic geometry.¹

In earlier work, the author introduced a classical R -matrix for Lie conformal algebras [55], viewed as a chiral analogue of the finite-dimensional classical R -matrix, and established an algebraic realization of the (modified) classical Yang–Baxter equation in this conformal setting. In [88], Malikov and Linshaw gave an example of a chiral Poisson group. Motivated by these results, it is natural to regard PVAs as a chiralization of Poisson–Lie theory. In parallel, Etingof and Kazhdan developed a series of papers on the quantization of Lie bialgebras [53], while Haisheng Li and collaborators have investigated vertex coalgebras, vertex bialgebras, and quantum vertex algebras [67]. We also note that in a talk at IHES [100], Pavel Safronov discussed the Courant σ -model and chiral Lie algebroids. From the perspective of Hamiltonian PDE theory, another key motivation is the quantization of Hamiltonian structures. A notable success in this direction

¹Often cited as “PTVV”; see [96].

via PVAs appeared in Zhe Wang's recent work [109], which is related to a conjecture of Dubrovin in symplectic field theory (SFT) [52]. Independently, Bazhanov and collaborators developed an alternative quantization approach for the KdV hierarchy and its Hamiltonian structures—see, for example, [14] on quantum KdV, the quantum inverse-scattering method, and the Yang–Baxter equation. In [22], A. Buryak studied quantum KdV and double ramification cycles.

This part of the thesis is inspired by the works of Getzler [64], Liu–Zhang [90], and Arakawa [4]. For a scheme X , we reinterpret PVA structures on X_∞ as degree-two functions on the jet of the shifted cotangent bundle $T^*[1]X$, in a way that can be viewed as a chiral analogue of Schwarz's equivalence between Poisson manifolds and degree-one symplectic N -manifolds [102]. Liu and Zhang [90] showed that this framework naturally accommodates deformations of Hamiltonian structures, notably Miura transformations (see also [26]). Using the same formalism, Zhang, Liu, and Wang [89] resolved the Dubrovin–Zhang conjecture [51]. Our viewpoint unifies the Kac Hamiltonian formalism with the Dubrovin–Zhang variational approach and yields a uniform geometric derivation of the classical R -matrix for Lie conformal algebras. We refer to this procedure of chiralizing classical Lie-theoretic data via the QP -formalism as *QP-chiralization*.

The following theorem is our main result and may be understood as a chiral analogue of Schwarz's classification of odd symplectic manifolds [102], the AKSZ QP formalism [3], and Roytenberg's graded-symplectic description of Courant algebroids [99].

THEOREM 2 (Correspondence Theorem). *Let X be a smooth, reduced, finitely generated $\mathbb{C}$ -scheme, and set $Y := T^*[1]X$. Then a degree-2 element $P \in \Gamma(Y_\infty, \mathcal{O}_{Y_\infty})_2$ defines a Poisson vertex algebra structure on X_∞ (i.e., it makes $\mathcal{O}_{X_\infty}$ a sheaf of PVAs) if and only if $[\int P, \int P] = 0$, where $[\cdot, \cdot]$ denotes the Lie bracket on local functionals induced by the chiral Gerstenhaber bracket; in this case, the λ -bracket corresponding to P is skew-symmetric.*

Conversely, any PVA λ -bracket on $\mathcal{O}_{X_\infty}$ determines a unique degree-2 local functional $\int P$ such that $[\int P, \int P] = 0$; equivalently, it determines P uniquely modulo total derivatives.

Example ($X = \mathbb{CP}^1$). Let $X = \mathbb{CP}^1$ with the standard affine cover U_i ($i = 1, 2$) and local coordinates u on U_1 and v on U_2 , related by $v = 1/u$ on $U_1 \cap U_2$. We restrict to scalar PVA structures corresponding to differential operators of order at most 3. The skew-symmetry of the λ -bracket implies that on U_1 any such bracket has the form

$$\{u_\lambda u\} = \frac{1}{2}(f_1)' + f_1 \lambda + \frac{c}{12}\lambda^3, \quad f_1 \in \mathcal{A}_{U_1}, \quad c \in \mathbb{C}.$$

This gives a degree-2 function

$$P = f_1 \theta \theta^{(1)} + \frac{c}{12} \theta \theta^{(3)} \in \Gamma(T^*[1]U_1, \pi_* \mathcal{O}_{Y_\infty}).$$

The coordinate change $u \mapsto v = 1/u$ induces $\theta_u = -u^{-2}\theta_v$, and requiring P to extend globally to X leads to the following classification.

(a) If $c \neq 0$, then, up to adding total derivatives and rescaling u , the only global scalar PVA structure of the above form on X_∞ is

$$\{u_\lambda u\} = u' + 2u\lambda + \frac{c}{12}\lambda^3,$$

namely the Virasoro-Magri λ -bracket with central charge c .

(b) If $c = 0$, then every global scalar PVA structure of the above form is given by

$$\{u_\lambda u\} = q'(u)u' + 2q(u)\lambda,$$

where on an affine chart $q(u)$ is a polynomial of degree at most 4, equivalently $q \in H^0(X, \mathcal{O}_X(4))$, recovering exactly the one-dimensional hydrodynamic-type Hamiltonian structures on $\mathbb{CP}^1$.

On the other hand, starting from Theorem 3.2.6, we derive the Maurer–Cartan equation for the classical R -matrix of a Lie conformal algebra, as defined in [55]. This may also be viewed as a chiral version of Drinfel’d’s and Kosmann-Schwarzbach’s results on classical R -matrices [47, 79].

THEOREM 3. *Let $(L, \partial, [\cdot_\lambda \cdot])$ be a Lie conformal algebra which is a free $\mathbb{C}[\partial]$ -module of finite rank, and let $\mathcal{V} := S^\bullet(L)$ be the corresponding free commutative differential algebra generated by L . We write $X_\infty := \text{Spec}(\mathcal{V})$, where we regard X_∞ as the arc space of a smooth, reduced, finite type scheme X over $\mathbb{C}$ (for instance, one may take $X = \mathbb{A}^\ell$, where $\ell = \text{rank}_{\mathbb{C}[\partial]} L$). Set $Y := T^*[1]X$. Suppose $R \in \text{End}_{\mathbb{C}}(L)$ satisfies $[R, \partial] = 0$. Then the (modified) classical Yang–Baxter equation for R is equivalent to the statement that the element*

$$P_R = \theta_\alpha \{u^\alpha_\partial u^\beta\}_{R, \rightarrow} \theta_\beta \in \hat{\mathcal{A}}^2$$

satisfies $\{\int P_R, \int P_R\} = 0$. (Here $\rightarrow$ means: replace λ by ∂ and move all ∂ ’s to the right.) Equivalently, set $\Theta_R := \int(P_R - P)$. Then Θ_R is a Maurer–Cartan element of the dg Lie algebra $(\hat{\mathcal{F}}_{cl}, [-, -], d_P := [P, -])$, i.e. $d_P \Theta_R + \frac{1}{2}[\Theta_R, \Theta_R] = 0$.

Example ($X = \mathfrak{g}^*$). In the affine Kac–Moody Lie conformal algebra case, let $\mathcal{V} := \mathcal{V}_k(\mathfrak{g})$. We identify $\text{Spec } \mathcal{V}$ with $\mathfrak{g}_{k, \infty}^*$, where k indicates the level of the λ -bracket. In particular, [55, Corollary 5.6] shows that if $(\mathfrak{g}, (\cdot, \cdot))$ is a quadratic Lie algebra and $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a Manin triple with respect to $(\cdot, \cdot)$, then the projection maps give a classical R -matrix for $v_k(\mathfrak{g})$. This can be viewed as a chiral counterpart of the classical Manin triple, compatible with the viewpoint from generalized complex geometry [66].

Remark 4. Another point of view on Theorem 3.3.2 is that the classical R -matrix induces a *BRST charge* on the complex $\hat{\mathcal{F}}_{cl}^\bullet$: $Q_R := \int P_R$ satisfies $Q_R^2 = [Q_R, Q_R] = 0$, so the derivation $d_R := [Q_R, -]$ on $\hat{\mathcal{F}}_{cl}^\bullet$ squares to zero.

Motivated by the above formalism and by the works of Ikeda–Xu [72, 71], Voronov [108], and Gualtieri [66], we reformulate and generalize the notion of a classical R -matrix as a degree-two function on a shifted manifold and on its jet space, which naturally connects to the Maurer-Cartan equation. This approach extends the definition of classical r -matrices to Lie conformal algebras [87]. Related aspects of this shifted symplectic picture of classical r -matrices and Lie conformal algebras are currently being developed further in joint work with Noriaki Ikeda.

This also motivates our definition of an R -matrix for classical finite $\mathcal{W}$ -algebras in Section 3.2. Using this notion, we apply the Adler-Kostant-Symes scheme to construct an integrable hierarchy on $\mathcal{W}_{\text{fin}}^{\text{cl}}(\mathfrak{sl}_3, f_{\text{sub}})$ and present several explicit examples of R -matrices. Motivated by the work of Belavin-Drinfel'd [18], we describe the constant R -deformations of the subregular classical finite $\mathcal{W}$ -algebra $\mathcal{W}_{\text{fin}}^{\text{cl}}(\mathfrak{sl}_3, f_{\text{sub}})$ in Section 3.3.5, and we also obtain an integrable system whose solutions are of hyperelliptic type. Still in Section 3.2, as a natural consequence of the QP -chiralization picture, we also define the classical R -matrix for affine $\mathcal{W}$ -algebras; see Definition 3.3.8.

This thesis is organized as follows. In Chapter 1, we recall the definitions of vertex algebras and Poisson vertex algebras. In Chapter 2, we review the connection between Poisson vertex algebras and integrable hierarchies. In Chapter 2, we collect the results from the first paper [55], where we define the classical R -matrix of Lie conformal algebras and its classical (modified) Yang-Baxter equation. We also construct several integrable hierarchies using an approach analogous to the Adler-Kostant-Symes scheme. Chapter 3 is based on the forthcoming paper “Shifted symplectic structure and Poisson vertex algebras”. In particular, we chiralize the results of Kontsevich and Drinfel'd, respectively. Moreover, as a consequence of Theorem 3.2.6, we show that our definition of the classical R -matrix for Lie conformal algebras is a natural chiralization in the sense of Drinfel'd-Kosmann-Schwarzbach.

Throughout this thesis, we consider a reduced smooth scheme X of finite-type over the complex numbers $\mathbb{C}$.

CHAPTER 1

Notation and Preliminaries

1. Preliminaries I: Vertex algebras

1.1. Vertex algebras and Lie conformal algebras. In this part we recall the definitions of vertex algebras. All concepts can be found in [59], [74], and [81].

A vertex algebra is a vector space V equipped with a vector $|0\rangle$ (called the vacuum vector), a linear map

$$Y(-, z) : V \longrightarrow \text{End } V[[z^{\pm}]], \quad a \longmapsto Y(a, z) = a(z) := \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1},$$

and a linear operator $T \in \text{End} V$. These data are subject to the following axioms:

- (vacuum axiom) $Y(|0\rangle, z) = Id_V$. For all $a \in V$, $Y(a, z)|0\rangle \in V[[z]]$, $Y(a, z)|0\rangle|_{z=0} = a$.
- (translation axiom) For any $a \in V$, $[T, Y(a, z)] = \partial_z Y(a, z)$, $T|0\rangle = 0$.
- (locality axiom) For all $a, b \in V$, $Y(a, z)$, $Y(b, z)$ are locally mutual, i.e., there exists N_{ab} in $\mathbb{Z}_{>0}$ such that $(z - w)^{N_{ab}}[Y(a, z), Y(b, w)] = 0$.

Example 1.1.1. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra with normalized invariant bilinear form $(\cdot | \cdot)$. The Kac-Moody affinization is defined as $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$. The commutator on $\widehat{\mathfrak{g}}$ is given by formula: $[xt^n, yt^m] = [x, y]t^{n+m} + n(x | y)\delta_{n+m,0}K$, $[\widehat{\mathfrak{g}}, K] = 0$. Let k be a complex number. The universal affine vertex algebra associated with $\mathfrak{g}$ at a level k is defined as $V^k(\mathfrak{g}) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} \mathbb{C}_k$, where $\mathbb{C}_k$ is the one-dimensional representation of $\mathfrak{g}[t] \oplus \mathbb{C}K$ on which $\mathfrak{g}[t]$ acts trivially and K acts as a multiplication by k . Then $V^k(\mathfrak{g})$ has a natural vertex algebra structure [58]. We shall call it the universal affine vertex algebra associated with $\mathfrak{g}$ at level k . It is customary to denote the vacuum vector in this case by v_k .

The center $\mathfrak{z}(V)$ of a vertex algebra V is a subspace defined by

$$\mathfrak{z}(V) = \{b \in V \mid a_{(n)}b = 0 \ \forall a \in V, \ n \geq 0\}.$$

Equivalently, b in $\mathfrak{z}(V)$ if and only if $[Y(a, z), Y(b, w)] = 0$ for all $a \in V$.

It is well-known that for the affine vertex algebra $V^k(\mathfrak{g})$, $\mathfrak{z}(V^k(\mathfrak{g}))$ equals $\mathbb{C}|0\rangle$, unless $k = -h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. Following [69, 57], the center of the affine vertex algebra $V^{-h^\vee}(\mathfrak{g})$ is termed the Feigin-Frenkel center, denoted by $\mathfrak{z}(\widehat{\mathfrak{g}})$. In addition, the elements residing in the Feigin-Frenkel center are called the Segal-Sugawara operators. For the affine vertex algebra $V^{-2}(\mathfrak{sl}_2)$, let $S = e_{-1}f_{-1} + f_{-1}e_{-1} + \frac{1}{2}h_{-1}h_{-1}$, then, $\mathfrak{z}(\widehat{\mathfrak{sl}}_2) = \mathbb{C}[T^r S \mid r \geq 0]$.

Recall the following definition; see [74].

Definition 1.1.2. A Lie conformal algebra (LCA) is a $\mathbb{C}[\partial]$ -module L endowed with a $\mathbb{C}$ -bilinear λ -bracket $[\cdot_\lambda \cdot] : L \otimes L \rightarrow L[\lambda]$ such that the following properties hold

$$\begin{aligned} (\text{Sesquilinearity}) \quad & [\partial a_\lambda b] = -\lambda[a_\lambda b], \quad [a_\lambda \partial(b)] = (\lambda + \partial)[a_\lambda b], \\ (\text{Skewsymmetry}) \quad & [b_\lambda a] = -[a_{\lambda+\partial} b], \\ (\text{Jacobi identity}) \quad & [a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + [b_\mu [a_\lambda c]] \text{ for all } a, b, c \in L. \end{aligned}$$

For any vertex algebra $(V, T, |0\rangle, Y(? , z))$, it is a Lie conformal algebra by “forgetting the negative products”, i.e., $(V, T, Y^-(? , z))$ is a LCA, where for all $a \in V$, we define $Y^-(a, z) := \sum_{n \geq 0} a_{(n)} z^{-n-1}$ and λ -bracket defined as $[a_\lambda b] := \sum_{n \geq 0} \frac{a_{(n)} b}{n!} \lambda^n$.

Example 1.1.3. Consider the affine vertex algebra defined in Example 1.1.1. We denote the vector space spanned by the vacuum vector $v_k \in V^k(\mathfrak{g})$ and $xt^n \cdot v_k$, $n \leq -1$ as $v_k(\mathfrak{g})$. Then $(v_k(\mathfrak{g}), Y^-(\cdot, z), T)$ has Lie conformal algebra structure. We will refer to the Lie conformal algebra $v_k(\mathfrak{g})$ as the affine Lie conformal algebra of Lie algebra $\mathfrak{g}$ at level k . As $\mathfrak{g} = \mathfrak{sl}_2$, let f, h, e be the standard generators of $\mathfrak{sl}_2$. The subspace of $v_k(\mathfrak{sl}_2)$ spanned by $v_k, h_{-1} \cdot v_k, e_{-1} \cdot v_k$ forms a Lie conformal subalgebra of $v_k(\mathfrak{sl}_2)$.

Let $(L_1, [\cdot_\lambda \cdot]_1, T_1)$ and $(L_2, [\cdot_\lambda \cdot]_2, T_2)$ be Lie conformal algebras. A linear map $\varphi : L_1 \longrightarrow L_2$ is called a Lie conformal algebra homomorphism, if

$$[T, \varphi] = 0, \quad \varphi([u_\lambda v]_1) = [\varphi(u)_\lambda \varphi(v)]_2, \quad \forall u, v \in L_1.$$

According to [81, Proposition 3.12.1], if $(L_1, [\cdot_\lambda \cdot]_1, T_1)$ and $(L_2, [\cdot_\lambda \cdot]_2, T_2)$ are Lie conformal algebras, then $(L_1 \oplus L_2, [\cdot_\lambda \cdot]_1 \oplus [\cdot_\lambda \cdot]_2, T_1 \oplus T_2)$ is also a Lie conformal algebra.

Following [59], let L be a Lie conformal algebra and denote $L(D^\times) := L \otimes \mathbb{C}((t))$. Consider the operator $\partial := T \otimes 1 + \text{Id} \otimes \partial_t$ acting on $L(D^\times)$. Then, $\text{Lie}(L) := L(D^\times)/\text{Im}(\partial)$ forms a Lie algebra, called the local Lie algebra of L . For all $a \in L$ we denote the image of $a \otimes t^n$ in $\text{Lie}(L)$ by $a_{[n]}$. The commutator on $\text{Lie}(L)$ is given by:

$$(1) \quad [a_{[m]}, b_{[k]}] = \sum_{n \geq 0} \binom{m}{n} (a_{(n)} b)_{[m+k-n]}.$$

Let $\text{Lie}(L)_+$ be the completion of the span of all $a_{[n]}$, $n \geq 0$. Then, $\text{Lie}(L)_+$ is a Lie subalgebra of $\text{Lie}(L)$ with respect to the Lie bracket defined by (1). Take the affine Lie

conformal algebra $v_k(\mathfrak{g})$ as in Example 1.1.3. Then, $\text{Lie}(v_k(\mathfrak{g}))$ is isomorphic to $\widehat{\mathfrak{g}}$ as Lie algebras.

By [59, Lemma 16.1.7], given a Lie conformal algebra L , we can construct a vertex algebra. Consider a Lie conformal algebra L . Let $U(L)$ be the universal enveloping algebra of $\text{Lie}(L)$ and $U(L)_+$ be the universal enveloping algebra of $\text{Lie}(L)_+$. Define $\text{Vac}(L) := U(L) \otimes_{U(L)_+} \mathbb{C}$, where $\mathbb{C}$ is the trivial one-dimensional representation of $\text{Lie}(L)_+$. The vacuum vector is $1 \otimes 1$. For all $a \in L$, the field is defined as $Y(a_{[-1]}|0\rangle, z) = \sum_{n \in \mathbb{Z}} a_{[n]} z^{-n-1}$, where $|0\rangle := 1 \otimes 1$. For example, let the Lie conformal algebra L be $v_k(\mathfrak{g})$. Then $\text{Vac}(L) \cong V^k(\mathfrak{g})$ as vertex algebras.

1.2. Poisson vertex algebras. Now, we define Poisson vertex algebras, which can be viewed as the classical limit of a vertex algebra.

A Poisson vertex algebra is a quintuple $(\mathcal{V}, \partial, 1, \cdot, \{\cdot_\lambda \cdot\})$ such that $(\mathcal{V}, \partial, 1, \cdot)$ is a commutative associative unital differential algebra and $(\mathcal{V}, \partial, \{\cdot_\lambda \cdot\})$ is a Lie conformal algebra and such that the λ -bracket $\{\cdot_\lambda \cdot\}$ and the associative product “ $\cdot$ ” satisfies

- (1) (left Leibniz rule) $\{a_\lambda bc\} = \{a_\lambda b\}c + b\{a_\lambda c\},$
- (2) (right Leibniz rule) $\{ab_\lambda c\} = \{a_{\lambda+\partial}c\}_{\rightarrow} b + \{b_{\lambda+\partial}c\}_{\rightarrow} a,$

where the right arrow and λ -bracket are understood as the following. If $\{a_\lambda b\} = \sum_{n \in \mathbb{Z}_+} \frac{\lambda^n}{n!} a_{(n)} b$, when a right arrow appears, that means that $\lambda + \partial$ has to be moved to the right i.e., $\{a_{\lambda+\partial} b\}_{\rightarrow} = \sum_{n \in \mathbb{Z}_+} \frac{a_{(n)} b}{n!} (\lambda + \partial)^n$. If no right arrow appears, the $\lambda + \partial$ moved to the left, as in $\{a_{-\lambda-\partial} b\} = \sum_{n \in \mathbb{Z}_+} \frac{(-\lambda-\partial)^n}{n!} a_{(n)} b$. The right Leibniz rule above can be obtained from the left Leibniz rule and the skewsymmetry, see [12].

Remark 1.1.4. Consider a Lie conformal algebra $(L, \{\cdot_\lambda \cdot\}, \partial)$, let $\mathcal{V} := S^\bullet(L)$ i.e., the symmetric algebra of the vector space L . If we extend the λ -bracket by the left and right Leibniz rule, then the symmetric algebra $\mathcal{V}$ has a Poisson vertex algebra structure. According to Proposition 3.8 in the paper [84], we observe that for an affine Lie conformal algebra $v_k(\mathfrak{g})$, the quotient space $\text{Sym}'(v_k(\mathfrak{g})) := \text{Sym}(v_k(\mathfrak{g})) / (1 - v_k)$ is also a Poisson vertex algebra. In this paper, we will call it the modified Poisson vertex algebra obtained from $v_k(\mathfrak{g})$.

One obtains the Poisson vertex algebra by taking Li's filtration. We recap the relevant details here.

Let V be a vertex algebra and

$$F^p V := \{a_{(-n_1-1)}^1 \cdots a_{(-n_r-1)}^r b \mid a^i \in V, b \in V, n_i \in \mathbb{Z}_{\geq 0}, n_1 + n_2 + \cdots + n_r \geq p\}.$$

Then,

$$\begin{aligned} V &= F^0V \supset F^1V \supset \cdots, \bigcap_p F^pV = 0, \\ TF^pV &\subset F^{p+1}V, \\ a_{(n)}F^qV &\subset F^{p+q-n-1}V \text{ for } a \in F^pV, n \in \mathbb{Z}, \\ a_{(n)}F^qV &\subset F^{p+q-n}V \text{ for } a \in F^pV, n \geq 0, \end{aligned}$$

and we set $F^pV = V$, for all $p < 0$. The filtration $\{F^pV\}$ is called the Li filtration of V .

According to [84, Proposition 4.2], the associated graded vector space $\text{gr}^F V := \bigoplus_p F^pV/F^{p+1}$ has a Poisson vertex algebra structure, as stated in the following theorem.

For a vertex algebra V , the associated graded vector space of V has a Poisson vertex algebra structure defined by

$$(2) \quad \sigma_p(a) \cdot \sigma_q(b) := \sigma_{p+q}(a_{(-1)}b),$$

$$(3) \quad \partial\sigma_p(a) := \sigma_{p+1}(Ta),$$

$$(4) \quad \sigma_p(a)_{(n)}\sigma_q(b) := \sigma_{p+q-n}(a_{(n)}b),$$

for all $a \in F^pV$, $b \in F^qV$, $n \geq 0$. Here $\sigma_p : F^pV \rightarrow \text{gr}^F V = F^pV/F^{p+1}V$ denotes the canonical quotient map.

Let X be a finite type affine scheme i.e., $X = \text{Spec } R$, where R is a finitely generated unital, associative, commutative $\mathbb{C}$ -algebra. The m -jet scheme of X denoted by J_mX is characterized by the following property: for every unital commutative, associative algebra A ,

$$\text{Hom}_{\text{Sch}}(\text{Spec } A, J_mX) \cong \text{Hom}_{\text{Sch}}(\text{Spec } A[z]/(z^{m+1}), X).$$

If $m > n$, we have a family of projection maps $\pi_{mn} : J_mX \rightarrow J_nX$. This family of projection maps yields a projective system $\{\pi_{mn}, J_mX\}_{m \geq n}$ of schemes. We define the infinite jet scheme $J_\infty X$ of X as $J_\infty X := \varprojlim J_mX$. Choose the presentation of algebra R as $R = \mathbb{C}[x^1, x^2, \dots, x^r]/\langle f_1, f_2, \dots, f_s \rangle$. Then, J_mX can be realized as follows. Define variables $x_{(-i)}^j$, $i = 1, 2, \dots, m+1$ and derivation T on the ring $\mathbb{C}[x_{(-i)}^j \mid i = 1, 2, \dots, m+1, j = 1, 2, \dots, r]$. The T acts as

$$Tx_{(-i)}^j = \begin{cases} ix_{(-i-1)}^j & \text{for } i \leq m, \\ 0 & \text{for } i = m+1, \end{cases}.$$

Identify x^j and $x_{(-1)}^j$ here, and let

$$J_mR = \mathbb{C} \left[x_{(-i)}^j \mid 1 \leq i \leq m+1, 1 \leq j \leq r \right] / \langle T^j f_i \mid 1 \leq i \leq s, 0 \leq j \leq m+1 \rangle.$$

Then, $J_m X \cong \text{Spec} J_m R$ as affine schemes. The infinite jet scheme of X is realized by taking $m \rightarrow \infty$,

$$J_\infty R = \mathbb{C} \left[x_{(-i)}^j \mid i \geq 1, j \geq 1 \right] / \langle T^j f_i \mid 1 \leq i \leq s, j \geq 0 \rangle,$$

and $J_\infty X = \text{Spec} J_\infty R$.

Given a Poisson algebra, there is a natural way to construct a Poisson vertex algebra.

THEOREM 1.1.5 ([5, Proposition 2.3.1]). *Let R be a Poisson algebra with Poisson bracket $\{\cdot, \cdot\}$. Then $J_\infty R$ has a unique Poisson vertex algebra structure called the Poisson vertex algebra at level 0, such that*

$$u_{(n)}v = \begin{cases} \{u, v\} & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases}$$

$\forall u, v \in R \subset J_\infty R$. Let $\mathfrak{g}$ be a Lie algebra and consider the Poisson Lie conformal algebra structure on $\mathbb{C}[J_\infty \mathfrak{g}^*]$. Then by taking the Li's filtration, for all $k \in \mathbb{C}$ $\text{gr}^F V^k(\mathfrak{g}) \cong \mathbb{C}[J_\infty \mathfrak{g}^*]$ as Poisson vertex algebras, where $\mathbb{C}[J_\infty \mathfrak{g}^*]$ is equipped with the Poisson vertex algebra structure at level 0. In particular, at the critical level $k = -h^\vee$, we have $\text{gr}^F \mathfrak{z}(\widehat{\mathfrak{g}}) \subset \text{gr}^F V^k(\mathfrak{g}) \cong \mathbb{C}[J_\infty \mathfrak{g}^*]$ as Poisson vertex algebras.

Remark 1.1.6. Using the λ -bracket language, the level-0 Poisson vertex algebra on $\mathbb{C}[J_\infty \mathfrak{g}^*]$ can be described as follows. Let $L := v_0(\mathfrak{g})$ be the affine Lie conformal algebra at level 0 (see Example 1.1.3). For all $u, v \in \mathfrak{g}$, the λ -bracket is $\{u_\lambda v\} = [u, v]$. Extend this to $S^\bullet(v_0(\mathfrak{g}))$ and consider the modified Poisson vertex algebra $S^{\bullet'}(v_0(\mathfrak{g}))$ (see Remark 1.1.4). By the Master Formula (5) and Theorem 1.2.1, we obtain an isomorphism of Poisson vertex algebras

$$\text{gr}^F V^{-h^\vee}(\mathfrak{g}) \cong \text{Sym}'(v_0(\mathfrak{g})).$$

2. Preliminaries II: Hamiltonian structure and Poisson vertex algebra

In the preceding section, we showed that a Poisson vertex algebra $\mathcal{V}$ is a differential algebra with a λ -bracket $\{\cdot, \cdot\}$. To delve into discussions about Hamiltonian PDEs, we have to extend the λ -bracket to a larger space. We first define this enlarged space.

We now recall the definition of an algebra of differential functions from [12]. An algebra of differential functions $\mathcal{V}$ in the variables $\{u^i\}_{i \in I}$ is a differential algebra (with derivation ∂) containing the algebra of differential polynomials

$$\mathcal{P}_\ell := \mathbb{C}[u^{i,n} \mid i \in I, n \in \mathbb{Z}_+], \quad u^{i,n} := \partial^n u^i, \quad |I| = \ell,$$

as a differential subalgebra. Moreover, $\mathcal{V}$ is endowed with linear maps

$$\frac{\partial}{\partial u^{i,n}} : \mathcal{V} \rightarrow \mathcal{V}, \quad i \in I, n \in \mathbb{Z}_+,$$

which are commuting derivations extending the usual partial derivatives on $\mathcal{P}_\ell$. The derivation ∂ of $\mathcal{P}_\ell$ extends to $\mathcal{V}$ and satisfies:

- (1) for every $f \in \mathcal{V}$, we have $\frac{\partial f}{\partial u^{i,n}} = 0$ for all but finitely many pairs $(i, n) \in I \times \mathbb{Z}_+$;
- (2) $\left[\frac{\partial}{\partial u^{i,n}}, \partial \right] = \frac{\partial}{\partial u^{i,n-1}}$, with the convention that $\frac{\partial}{\partial u^{i,-1}} = 0$ (equivalently, the right-hand side is 0 when $n = 0$).

For example, $\mathcal{P}_\ell$ itself is an example of the algebra of differential functions.

2.1. Hamiltonian PDE and PVA. Now, we recall the connections between Poisson vertex algebra and integrable hierarchies. We first recall a result in the [12]. Let $\mathcal{V}$ be a PVA. Let $\overline{\mathcal{V}} := \mathcal{V}/\partial\mathcal{V}$ and let $\int : \mathcal{V} \rightarrow \overline{\mathcal{V}}$ be the quotient map. Then we have the following well-defined brackets:

- (1) $\overline{\mathcal{V}} \times \overline{\mathcal{V}} \rightarrow \overline{\mathcal{V}}$, $\{\int a, \int b\} := \int \{a_\lambda b\}_{\lambda=0}$,
- (2) $\overline{\mathcal{V}} \times \mathcal{V} \rightarrow \mathcal{V}$, $\{\int a, b\} := \{a_\lambda b\}_{\lambda=0}$.

Moreover, (1) defines a Lie algebra on $\overline{\mathcal{V}}$, and (2) defines a representation of the Lie algebra $\overline{\mathcal{V}}$ on $\mathcal{V}$ (Here we first set $\lambda = 0$ and then take the quotient map).

The following theorem extends the λ -bracket on a Poisson vertex algebra to an algebra of differential functions $\mathcal{V}$ that contains it.

THEOREM 1.2.1 ([12, Theorem 1.15]). *Let $\mathcal{V}$ be an algebra of differential functions in the variables $\{u^i\}_{i \in I}$. For each pair $i, j \in I$, choose $\{u_\lambda^i u^j\} = H_{ji}(\lambda) \in \mathcal{V}[\lambda]$, then*

- (1) *The Master formula:*

$$(5) \quad \{f_\lambda g\} = \sum_{i,j \in I, p, q \in \mathbb{Z}_+} \frac{\partial g}{\partial u^{j,q}} (\partial + \lambda)^q \{u_{\partial+\lambda}^i u^j\} \rightarrow (-\partial - \lambda)^p \frac{\partial f}{\partial u^{i,p}},$$

defines a λ -bracket on $\mathcal{V}$, which satisfies sesquilinearity, the left and right Leibniz rules, and extends the given λ -bracket on the variables u_i 's. Conversely, any λ -bracket on the algebra of differential polynomials satisfying these properties is given by the Master Formula.

- (2) *This λ -bracket is skewsymmetric if skew-symmetry holds for every pair of variables: $\{u_\lambda^i u^j\} = -\{u_\lambda^j u^i\}$ for all $i, j \in I$.*
- (3) *If this λ -bracket is skewsymmetric, then it satisfies the Jacobi identity, provided the Jacobi identity holds for every triple of variables: $\{u_\lambda^i \{u_\mu^j u^k\}\} - \{u_\mu^j \{u_\lambda^i u^k\}\} = \{\{u_\lambda^i u^k\}_{\lambda+\mu} u^j\}$ for all $i, j, k \in I$.*

Given a PVA $\mathcal{V}$ and a local functional $\int h \in \overline{\mathcal{V}}$, the associated Hamiltonian PDE is $\frac{du}{dt} = \{\int h, u\}$. The local functional $\int h$ is called the Hamiltonian of this equation. One views this formation as a generalization of the traditional definition in integrable systems, see [95]. Let the PVA $\mathcal{V}$ be an algebra of differential functions in the variables $\{u_i\}_{i \in I}$ and the λ -bracket is given by the Master Formula (5), then

- (1) Hamiltonian PDE: $\frac{du}{dt} = \{\int h, u\} = H \frac{\delta}{\delta u} \int h$,

$$(2) \text{ Poisson bracket on } \mathcal{V}: \{\int f, \int g\} = \int \frac{\delta g}{\delta u} \cdot H \frac{\delta f}{\delta u},$$

where H is a matrix differential operator $H(\partial) \in \text{Mat}_{\ell \times \ell} \mathcal{V}[\partial]$ with entries given by $H_{ij}(\partial) := \{u_{\partial+\lambda}^j u^i\}_{\rightarrow|\lambda=0}$. Here, the differential operator $H(\partial)$ is the Poisson structure associated with the Poisson vertex algebra $\mathcal{V}$; see [12].

On the basis of the above discussions, we now re-formulate Liouville integrability of a field-theoretic system in a more rigorous way. For a Hamiltonian system $u_t(x, t) = \{\int h, u\}$, we say this system is Liouville integrable if $\int h$ is contained in an infinite-dimensional Poisson commutative subalgebra in $\bar{\mathcal{V}}$ with respect to $\{\cdot, \cdot\}$.

We give an example here. Consider the Poisson vertex algebra $\mathcal{V} := \mathbb{C}[u, u', u'', \dots]$ equipped with a λ -bracket $\{u_\lambda u\} = \lambda$. Choosing $h = \int \frac{1}{2} u_x^2 + u^3 dx$, and we recover the KdV equation $\frac{du}{dt} = 6uu_x + u_{xxx}$. The integrability of the KdV equation can be argued by its bi-Hamiltonian structure, which was first discovered by Faddeev and Zakharov, see [54].

3. Preliminaries III: Graded Geometry

In this section, we recall some known results about supermanifold and superalgebra. The main references are

3.1. Graded algebra. We recall the graded manifold and the graded algebra here. The main references are [30, 31, 97].

Definition 1.3.1. A $\mathbb{Z}$ graded vector space V is a direct sum $V = \bigoplus_{i \in \mathbb{Z}} V_i$ of complex number $\mathbb{C}$. The V_i 's are called the component of V of degree i and the degree of a homogeneous element $a \in V$ is denoted by $|a|$.

For graded vector spaces V, W the tensor product of them is a graded vector space whose degree component r is given by $(V \otimes W)_r := \bigoplus_{p+q=r} V_p \otimes W_q$. We can also define the exterior algebra and the symmetric algebra of a graded vector space V as follows.

Definition 1.3.2. The symmetric and exterior algebra of a graded vector space V are defined respectively as

$$S^\bullet(V) := T(V)/I_s, \quad \bigwedge^\bullet(V) := T(V)/I_\wedge,$$

where $T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$ is the tensor algebra of V and I_s (resp. $I_\wedge$) is the two sided ideal generated by the elements of the form $a \otimes b - (-1)^{|a||b|} b \otimes a$ (resp. $a \otimes b + (-1)^{|a||b|} b \otimes a$). The images of $V^{\otimes n}$ in $S^\bullet V$ and $\bigwedge^\bullet V$ are denoted by $S^n V$ and $\bigwedge^n V$, respectively.

Let V be a graded vector space over $\mathbb{C}$. For an integer n we write $V[n]$ for the degree-shift of V ; explicitly $V[n] := \bigoplus_{i \in \mathbb{Z}} (V[n])_i$ where $(V[n])_i := V_{n+i}$.

Definition 1.3.3. A graded Lie algebra (GLA) of degree n is a graded vector space A endowed with a graded Lie bracket on $A[n]$ such that a bracket can be seen as a degree Lie bracket on A i.e., as a bilinear operation

$$(6) \quad \{-, -\} : A \times A \rightarrow A[-n],$$

such that graded anti-symmetric and graded Jacobi identity holds:

$$(7) \quad \{a, b\} = -(-1)^{(|a|+n)(|b|+n)} \{b, a\},$$

$$(8) \quad \{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{(|a|+n)(|b|+n)} \{b, \{a, c\}\},$$

for all homogeneous $a, b, c \in A$.

In particular, a Lie superalgebra $\mathfrak{g}$ is a special case of a graded Lie algebra whose bracket has degree 0 [73]. Note that there exists a canonical décalage isomorphism $S^n V[1] \cong \bigwedge^n V$.

Definition 1.3.4. A graded Poisson algebra of degree n or n -Poisson algebra is a triple $(A, \circ, \{-, -\})$ consisting of a graded vector space $A = \bigoplus_{i \in \mathbb{Z}} A_i$ endowed with a degree-zero graded commutative product and a Lie bracket of degree $-n$. For homogeneous, $a, b, c \in A$ the bracket satisfies the graded Leibniz rule $\{a, b \circ c\} = \{a, b\} \circ c + (-1)^{|b|(|a|+n)} b \circ \{a, c\}$. When $n = 1$, we call such an algebra a Gerstenhaber algebra and the bracket $\{-, -\}$ is called Nijenhuis-Schouten bracket. Also, if $n = 0$, it is the usual graded Poisson algebra.

For the $\mathbb{Z}_2$ -graded case, one simply says even or odd Poisson algebra. In this situation, an even Poisson algebra coincides with a Poisson superalgebra.

Example 1.3.5. Let $\mathfrak{g}$ be a graded Lie algebra of degree 0. Put $A := S^\bullet \mathfrak{g}[n]$ whose multiplication is the one naturally induced from multiplication induced from the tensor algebra $T^\bullet \mathfrak{g}[n]$. The Poisson bracket is specified as follows. On the linear component $S^1 \mathfrak{g}[n]$ it is the suspension of the bracket on $\mathfrak{g}$; for homogeneous elements $a, b, c \in A$ and for $k > 1$, we extend it by the graded Leibniz rule

$$\{a, b \circ c\} = \{a, b\} \circ c + (-1)^{|b|(|a|+n)} b \circ \{a, c\}.$$

Hence $S^\bullet \mathfrak{g}[n]$ is an n -Poisson algebra.

Example 1.3.6. Consider Lie superalgebra $\mathfrak{g} = \mathfrak{osp}(1|2)$, then $S^\bullet \mathfrak{g}[1]$ has Gerstenhaber algebra structure.

By shifting the degree of a Gerstenhaber algebra $(V^\bullet, \{-, -\}, \cdot)$, we can reformulate its definition as that of a graded Lie algebra equipped with a compatible Leibniz rule (see [80]). In Section 2, we follow the conventions of [51], which are standard in the theory of integrable systems.

Remark 1.3.7. By shifting the degree of vector space, for a Gerstenhaber algebra $(V^\bullet, \{-, -\}, \cdot)$, the graded skew-symmetry and graded Leibniz rule can be written as $[b, a] =$

$(-1)^{|a||b|}[a, b]$ and $[c, a \cdot b] = [c, a] \cdot b + (-1)^{|c|(|a|+1)}a \cdot [c, b]$, see [51, 64] and [80, Section 3.3.2].

Remark 1.3.8. This shows that the Gerstenhaber algebra is a kind of generalization of graded Poisson algebra.

In particular, this gives a way to characterize Poisson structure for a finite-dimensional variety, see Section 1.1.

3.2. Graded manifold. In this section, we recall the graded manifold/scheme here. The main references are [97, 30, 31], for their connection to vertex algebra and Poisson vertex algebra, see [4, 86].

Definition 1.3.9. A $\mathbb{Z}$ -graded scheme is a pair $(X, \mathcal{O}_X = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X^n)$ such that $(X, \mathcal{O}_X^0)$ is a scheme and $\mathcal{O}_X$ is a sheaf of $\mathbb{Z}$ -graded $\mathcal{O}_X^0$ -algebras (in particular, restriction maps preserve degrees). There exists an affine open cover $X = \bigcup U_i$ with $U_i = \text{Spec } A_i$ and isomorphisms of graded $\mathcal{O}_{U_i}$ -algebras

$$\mathcal{O}_X|_{U_i} \cong \widetilde{B^{(i)}},$$

where each $B^{(i)} = \bigoplus_{n \in \mathbb{Z}} B^{(i),n}$ is a finitely generated $\mathbb{Z}$ -graded, graded-commutative A_i -algebra (i.e. $ab = (-1)^{mn}ba$ for homogeneous $a \in B^{(i),m}$, $b \in B^{(i),n}$), with $B^{(i),0} \cong A_i$ (hence $\mathcal{O}_X^0|_{U_i} \cong \mathcal{O}_{U_i}$), and each $\mathcal{O}_X^n$ is a quasi-coherent $\mathcal{O}_X^0$ -module.

Example 1.3.10. Let $R^{n|m} = \mathbb{C}[x^1, x^2, \dots, x^n, \theta_1, \theta_2, \dots, \theta_n]$ be a polynomial where $x^i, i = 1, 2, \dots, n$ are even variables and $\theta_i, i = 1, 2, \dots, n$ are odd variables, then $\mathbb{A}^{n|m} = (\text{Spec}(R^{n|m}), R^{n|m})$ is a quasi-compact affine superscheme of dimension $n|m$.

Example 1.3.11. A basic example of a graded (and hence super) manifold is the shifted tangent bundle $T[1]X$. Let X be a finite-dimensional smooth manifold with local coordinates $x^i, i = 1, \dots, n$. On $T[1]X$ we use local coordinates (x^i, θ^i) , where x^i have degree 0 and θ^i have degree 1 (hence are odd). On an overlapping chart with coordinates $\tilde{x}^i = \tilde{x}^i(x)$, the gluing rules are

$$\tilde{x}^i = \tilde{x}^i(x), \quad \tilde{\theta}^i = \frac{\partial \tilde{x}^i}{\partial x^j} \theta^j,$$

so that the θ^i transform as the basis dx^i of one-forms. Consequently the algebra of global functions on $T[1]X$ is naturally identified with the graded commutative algebra of differential forms $\Omega^\bullet(X)$, where a k -form has degree k .

Similar construction works for cotangent bundle.

Example 1.3.12. Similarly, consider the shifted cotangent bundle $Y := T^*[1]X$. On a smooth manifold X with local coordinates (x^μ) , we use coordinates (x^μ, θ_μ) on $T^*[1]X$, where the x^μ have degree 0 and the θ_μ have degree 1. On an overlapping chart with

coordinates $\tilde{x}^\mu = \tilde{x}^\mu(x)$, the gluing is given by the degree-preserving maps

$$\tilde{x}^\mu = \tilde{x}^\mu(x), \quad \tilde{\theta}_\mu = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} \theta_\nu.$$

The coordinate functions generate a graded-commutative algebra $\Gamma(Y, \mathcal{O}_Y)$, whose degree- k part identifies with the space $\Gamma(X, \wedge^k T_X)$ of k -vector fields on X . In particular,

$$\Gamma(T^*[1]X, \mathcal{O}_{T^*[1]X}) \cong \Gamma(X, \wedge^\bullet T_X).$$

Example 1.3.13. [105, Letter 7] Consider the graded cotangent bundle $T^*[2](T^*[1]X)$ and let $Z := T^*[2](T^*[1]X)$. Let $(x^\mu, \theta^\mu, \psi_\mu, p_\mu)$ be local coordinates of degree 0, 1, 1, 2, respectively. On an overlapping chart with coordinates $\tilde{x}^\mu = \tilde{x}^\mu(x)$, the gluing between patches is given by the degree-preserving maps

$$\begin{aligned} \tilde{x}^\mu &= \tilde{x}^\mu(x), & \tilde{\theta}^\mu &= \frac{\partial \tilde{x}^\mu}{\partial x^\nu} \theta^\nu, & \tilde{\psi}_\mu &= \frac{\partial x^\nu}{\partial \tilde{x}^\mu} \psi_\nu, \\ \tilde{p}_\mu &= \frac{\partial x^\nu}{\partial \tilde{x}^\mu} p_\nu + \frac{\partial^2 x^\nu}{\partial \tilde{x}^\gamma \partial \tilde{x}^\mu} \frac{\partial \tilde{x}^\gamma}{\partial x^\sigma} \psi_\nu \theta^\sigma. \end{aligned}$$

The coordinate functions generate a graded-commutative algebra $\Gamma(Z, \mathcal{O}_Z)$. In degree 0, this algebra coincides with $\Gamma(X, \mathcal{O}_X)$, and in degree 1 with $\Gamma(X, T_X \oplus T_X^*)$. This graded manifold is a basic example in the study of Courant algebroids and is closely related to the chiral de Rham complex; see for instance [99].

CHAPTER 2

Generalized Adler-Kostant-Symes scheme of Integrability via vertex algebra

This chapter is based on the paper [55]. We will define the classical R -matrix of LCA and give the statement of it. On the otherhand, we give the statement of generalized AKS scheme of integrability and examples.

1. Modified Yang-Baxter equations of Lie conformal algebra

Definition 2.1.1. Let $(L, [\cdot_\lambda \cdot], T)$ be a Lie conformal algebra. An R -matrix for the Lie conformal algebra is a linear map R_L in $\text{End}L$ that satisfies the following conditions:

1. $(L, [\cdot_\lambda \cdot]_{R_L}, T)$ is a Lie conformal algebra where $[\cdot_\lambda \cdot]_{R_L}$ is defined by

$$[u_\lambda v]_{R_L} = [R_L(u)_\lambda v] + [u_\lambda R_L(v)] \text{ for all } u, v \in L.$$

2. R_L satisfies

$$(9) \quad [R_L, T] = 0.$$

We denote the resulting Lie conformal algebra as L_{R_L} and refer to it as the R_L -twisted Lie conformal algebra.

Remark 2.1.2. Note that the R -matrix defined in Definition 2.1.1 is the conformal analogue of the definition of the R -matrix for a Lie algebra, as introduced in [104]. Liberati introduced a different notation for the r -matrix of Lie conformal algebras in [87]. It may be interesting to compare Liberati's R -matrix with the author's.

THEOREM 2.1.3. *Suppose $R_L \in \text{End}L$ satisfies (9). The following condition is sufficient for R_L to be an R -matrix: there exists $\epsilon \in \mathbb{C}$ such that*

$$(10) \quad [R_L(u)_\lambda R_L(v)] - R_L([u_\lambda v]_{R_L}) + \epsilon [u_\lambda v] = 0, \quad \forall u, v \in L.$$

In particular, when $\epsilon = 0$, equation (10) becomes

$$(11) \quad [R_L(u)_\lambda R_L(v)] = R_L([u_\lambda v]_{R_L}), \quad \forall u, v \in L,$$

which we call the constant classical Yang-Baxter equation (CCYBE) for L . When $\epsilon = 1$, equation (10) becomes

$$(12) \quad [R_L(u)_\lambda R_L(v)] - R_L([u_\lambda v]_{R_L}) = -[u_\lambda v], \quad \forall u, v \in L,$$

which we call the modified Yang-Baxter equation (mYBE) for L .

PROOF. The proof of sesquilinearity and skew-symmetry are straightforward. Hence, it suffices to verify the Jacobi identity for $(L, [\cdot]_{\lambda}, \partial)$, given that R_L satisfies (10). For all $a, b, c \in L$, we have:

$$(13) \quad [a_{\lambda}[b_{\mu}c]_{R_L}]_{R_L} = [R_L(a)_{\lambda}[b_{\mu}c]_{R_L}] + [a_{\lambda}[R_L(b)_{\mu}R_L(c)]] + \epsilon[a_{\lambda}[b_{\mu}c]]$$

$$(14) \quad [b_{\mu}[a_{\lambda}c]_{R_L}]_{R_L} = [R_L(b)_{\mu}[a_{\lambda}c]_{R_L}] + [b_{\mu}[R_L(a)_{\lambda}R_L(c)]] + \epsilon[b_{\mu}[a_{\lambda}c]]$$

Thus, by the Jacobi identity of $(L, [\cdot]_{\lambda}, \partial)$ and equation (10), we obtain:

$$\begin{aligned} & [a_{\lambda}[b_{\mu}c]_{R_L}]_{R_L} - [b_{\mu}[a_{\lambda}c]_{R_L}]_{R_L} \\ &= [R_L(a)_{\lambda}[b_{\mu}c]_{R_L}] + [a_{\lambda}[R_L(b)_{\mu}R_L(c)]] + \epsilon[a_{\lambda}[b_{\mu}c]] - [R_L(b)_{\mu}[a_{\lambda}c]_{R_L}] - [b_{\mu}[R_L(a)_{\lambda}R_L(c)]] - \epsilon[b_{\mu}[a_{\lambda}c]] \\ &= [[R_L(a)_{\lambda}R_L(b)]_{\lambda+\mu}c] + [[R_L(a)_{\lambda}b]_{\lambda+\mu}R_L(c)] + [[a_{\lambda}R_L(b)]_{\lambda+\mu}R_L(c)] + \epsilon[[a_{\lambda}b]_{\lambda+\mu}c] \\ &= [R_L([a_{\lambda}b]_{R_L})_{\lambda+\mu}c] - \epsilon[[a_{\lambda}b]_{\lambda+\mu}c] + [[a_{\lambda}b]_{R_L}R_L(c)]_{\lambda+\mu} + \epsilon[[a_{\lambda}b]_{\lambda+\mu}c] \\ &= [R_L([a_{\lambda}b]_{R_L})_{\lambda+\mu}c] + [[a_{\lambda}R_L(b)]_{\lambda+\mu}R_L(c)] \\ &= [[a_{\lambda}b]_{R_L}R_L(c)]_{\lambda+\mu}. \end{aligned}$$

This completes the proof. $\square$

2. Factorization theorems

2.1. Factorization theorem. In this section, we prove the following theorem.

FACTORIZATION THEOREM . Let L be a $\mathbb{C}[\partial]$ -module, $R_L : L \rightarrow L$ a homomorphism of $\mathbb{C}[\partial]$ -modules. The following conditions are equivalent. (i) R_L satisfies the modified Yang-Baxter equation (mYBE). (ii) There exist conformal subalgebras $L_{\pm} \subset L$, conformal ideals $\mathcal{L}_{\pm} \subset L_{\pm}$, a Lie conformal isomorphism $\theta : L_+/\mathcal{L}_+ \rightarrow L_-/\mathcal{L}_-$ and a Lie conformal algebra isomorphism

$$\Phi : \tilde{L} \cong L, \quad (u_+, u_-) \mapsto u_+ - u_-,$$

such that

$$(15) \quad (R_L \circ \Phi)(u_+, u_-) = u_+ + u_-,$$

where $\tilde{L}$ is defined as

$$\tilde{L} := \{(u_+, u_-) \in L_+ \oplus L_- \mid \theta(\overline{u_+}) = \overline{u_-}\},$$

which is a Lie conformal subalgebra of $L_+ \oplus L_-$. Moreover, $L_{\pm} = \text{Im}(R_L \pm 1)$ and $\mathcal{L}_{\pm} = \text{Ker}(R_L \mp 1)$.

PROOF. Let us show (i) implies (ii). We first show that if $R_L \in \text{End}(L)$ satisfies the modified Yang-Baxter equation, then defining

$$(16) \quad L_{\pm} := (R_L \pm 1)(L), \quad \mathcal{L}_{\pm} := \text{Ker}(R_L \mp 1),$$

we have:

- (a) $(L_{\pm}, [\cdot_{\lambda} \cdot], T)$ are Lie conformal subalgebras of $(L, [\cdot_{\lambda} \cdot], T)$.
- (b) The subspaces $\mathcal{L}_{\pm}$ are Lie conformal algebra ideals of $(L, [\cdot_{\lambda} \cdot]_{R_L}, T)$ and $L_{\pm} \cong L/\mathcal{L}_{\mp}$ as Lie conformal algebras.
- (c) The subspaces $\mathcal{L}_{\pm}$ are Lie conformal algebra ideals of $(L_{\pm}, [\cdot_{\lambda} \cdot], T)$.

For (a), equation (12) implies for all $u, v \in L$, we have $(R_L \pm 1)([u_{\lambda} v]) = [(R_L \pm 1)(u)_{\lambda} (R_L \pm 1)(v)]$. This shows that $(L_{\pm}, [\cdot_{\lambda} \cdot], T)$ are Lie conformal subalgebras of $(L, [\cdot_{\lambda} \cdot], T)$. Moreover, this also implies that $(R_L \pm 1)$ are Lie conformal algebra homomorphisms from $(L, [\cdot_{\lambda} \cdot], T)$ to $(L_{\pm}, [\cdot_{\lambda} \cdot], T)$. Thus, (b) follows: $\mathcal{L}_{\pm}$ are Lie conformal algebra ideals of $(L, [\cdot_{\lambda} \cdot]_{R_L}, T)$, and we have the isomorphism $L_{\pm} \cong L/\mathcal{L}_{\mp}$ as Lie conformal algebras.

Now, we show (c). First, we verify that $\mathcal{L}_{\pm} \subset L_{\pm}$. This follows from the fact that for any $u \in \mathcal{L}_{\pm}$, we have $(R_L \mp 1)u = 0$. Hence, $R_L(u) = \pm u$ and $\pm u = \frac{1}{2}(R_L \pm 1)(u) \in L_{\pm}$. Next, we show that for all $v \in L$, the identity $(R_L \mp 1)[(R_L + 1)u_{\lambda} v] = 0$ for all u in $\mathcal{L}_{\pm}$ holds. This follows from $(R_L \mp 1)[(R_L \pm 1)v_{\lambda} u] = 0$, which is a consequence of the modified Yang-Baxter equation (12) and $R_L(u) = \pm u$ for all u in $\mathcal{L}_{\pm}$. The inclusion $T\mathcal{L}_{\pm} \subset \mathcal{L}_{\pm}$ follows from $[T, R_L] = 0$, (c) holds.

Now, let $\overline{\cdot} : L_{\pm} \longrightarrow L_{\pm}/\mathcal{L}_{\pm}$ be the quotient map. For any $u, v \in L_{\pm}$, we define

$$\overline{T}u := \overline{T}u, \quad \overline{\{u_{\lambda} v\}} := \{\overline{u}_{\lambda} \overline{v}\} := \sum_{n \geq 0} \frac{\overline{u^{(n)}v}}{n!} \lambda^n.$$

From (c), we see that $(L_{\pm}/\mathcal{L}_{\pm}, \overline{T}, \overline{\{\cdot_{\lambda} \cdot\}})$ form Lie conformal algebras. We claim that the map $\theta(\overline{(R_L + 1)u}) := \overline{(R_L - 1)u}$ for all u in L is a Lie conformal algebra isomorphism from $(L_{+}/\mathcal{L}_{+}, \overline{\{\cdot_{\lambda} \cdot\}}, \overline{T})$ to $(L_{-}/\mathcal{L}_{-}, \overline{\{\cdot_{\lambda} \cdot\}}, \overline{T})$. The well-definedness follows directly from (16). For injectivity, note that the surjective linear map $L_{+} \rightarrow L_{-}/\mathcal{L}_{-}$ given by $(R_L + 1)u \mapsto \overline{(R_L - 1)u}$ has kernel given by elements of the form $a = (R_L + 1)u \in \mathcal{L}_{+}$ such that $(R_L - 1)u \in \mathcal{L}_{-}$. This is equivalent to $(R_L - 1)a = 0$, which means that the kernel of θ is precisely $\mathcal{L}_{+}$. Hence, this map induces a bijection $L_{+}/\mathcal{L}_{+} \simeq L_{-}/\mathcal{L}_{-}$. Next, we verify that $[\overline{T}, \theta] = 0$. Since $[T, R_L] = 0$ and by the definition of θ , we have

$$\theta(\overline{T(R_L + 1)u}) = \theta(\overline{T(R_L + 1)u}) = \overline{T}\theta(\overline{(R_L + 1)u}),$$

$\forall u \in L$. By repeatedly applying the mYBE (12),

$$\theta(\overline{[(R_L + 1)u_{\lambda} (R_L + 1)v]}) = [\theta(\overline{(R_L + 1)u})]_{\lambda} \theta(\overline{(R_L + 1)v})]$$

Therefore, we conclude θ is an isomorphism of conformal algebras. We now prove that Φ is a Lie conformal algebra isomorphism. To show that Φ is a Lie conformal algebra isomorphism, we define the map $\Psi : L \rightarrow \tilde{L}$, $u \rightarrow \frac{1}{2}((R_L + 1)u, (R_L - 1)u)$ for all $u \in L$. By definition of Φ , Ψ , we have $\Phi \circ \Psi = \text{Id}_L$. We now show that $\Psi \circ \Phi = \text{Id}_{\tilde{L}}$. Suppose $(u_{+}, u_{-}) \in \tilde{L}$. By construction there exists $a, b \in L$ such that $u_{+} = (R_L + 1)a$, $u_{-} = (R_L - 1)b$. Since $\theta(\overline{u_{+}}) = \overline{u_{-}}$, it follows that $(R_L - 1)b = (R_L - 1)a - l$ for some $l \in \mathcal{L}_{-}$. By direct computation, this identity implies $(R_L \pm 1)(u_{+} - u_{-}) = 2u_{\pm}$. Therefore, $\Psi \circ \Phi(u_{+}, u_{-}) =$

(u_+, u_-) as required. Next, note that equation (12) implies that for all $u, v \in L$, we have $(R_L \pm 1)([u_\lambda v]) = [(R_L \pm 1)(u)_\lambda (R_L \pm 1)(v)]$. Thus, Ψ is a Lie conformal algebra morphism. Now, we prove (ii) implies (i). Suppose $u, v \in L$. Since Φ is a Lie conformal algebra isomorphism, there exist $u_\pm, v_\pm \in L_\pm$ such that $u = u_+ - u_-$, $v = v_+ - v_-$. Then, we obtain

$$(17) \quad R_L(u) = R_L \circ \Phi((u_+, u_-)) = u_+ + u_-.$$

Therefore, it follows that $[u_\lambda v]_{R_L} = 2([u_{+\lambda} v_+] - [u_{-\lambda} v_-])$. Thus, we derive

$$(18) \quad \begin{aligned} R_L([u_\lambda v]_{R_L}) &= 2R_L([u_{+\lambda} v_+] - [u_{-\lambda} v_-]) \\ &= 2R_L \circ \Phi([u_{+\lambda} v_+], [u_{-\lambda} v_-]) \\ &= 2([u_{+\lambda} v_+] + [u_{-\lambda} v_-]). \end{aligned}$$

Now, we observe that

$$[R_L(u)_\lambda R_L(v)] - R_L([u_\lambda v]_{R_L}) = [u_{+\lambda} v_+] - [u_{+\lambda} v_-] + [u_{-\lambda} v_+] - [u_{-\lambda} v_-] = -[u_\lambda v],$$

since $u = u_+ - u_-$, and $v = v_+ - v_-$. This confirms that R_L satisfies the modified Yang-Baxter equation. Next, we prove the equalities $\text{Im}(R_L \pm 1) = L_\pm$ and $\text{Ker}(R_L \mp 1) = \mathcal{L}_\pm$. Since the two cases are symmetric, it is sufficient to show that $\text{Im}(R_L + 1) = L_+$ and $\text{Ker}(R_L - 1) = \mathcal{L}_+$. Let $x \in L$. Then there exists $(u_+, u_-) \in \tilde{L}$ such that $\Phi((u_+, u_-)) = x$. Applying R_L to both sides, we obtain

$$R_L(x) = R_L \circ \Phi((u_+, u_-)) = u_+ + u_-.$$

Hence, $(R_L + 1)(x) = R_L(x) + x = 2u_+$. Thus, we conclude that $\text{Im}(R_L + 1) \subseteq L_+$. To show the reverse inclusion, $L_+ \subseteq \text{Im}(R_L + 1)$, let $x \in L_+$. Since Φ is an isomorphism, there exists $(u_+, u_-) \in \tilde{L}$ such that $x = \Phi((u_+, u_-)) \in L$. By the equation (15), we have $u_+ = \frac{1}{2}(R_L + 1)x \in L_+$. Thus, $u_+ \in \text{Im}(R_L + 1)$, which completes the proof. We now show that $\mathcal{L}_+ = \text{Ker}(R_L - 1)$. Let $x \in \mathcal{L}_+$. Then $x \in L_+$. By arguments above, $x \in \text{Im}(R_L + 1)$. Suppose $x = (R_L + 1)(y)$ for some $y \in L$. Since Φ is an isomorphism, there exists $u_\pm \in L_\pm$ such that $y = \Phi(u_+, u_-)$. We have

$$\begin{aligned} (R_L - 1)x &= (R_L - 1) \circ (R_L + 1)y \\ &= (R_L - 1) \circ (R_L + 1) \circ \Phi((u_+, u_-)) \\ &= 0. \end{aligned}$$

Thus, $x \in \text{Ker}(R_L - 1)$, proving the inclusion $\mathcal{L}_+ \subseteq \text{Ker}(R_L - 1)$. Conversely, suppose $x \in \text{Ker}(R_L - 1)$, i.e., $R_L(x) = x$. Since Φ is an isomorphism, there exists $(u_+, u_-) \in \tilde{L}$ such that $x = \Phi((u_+, u_-))$. By the equation (15),

$$0 = \Phi((u_+, u_-)) - R_L \circ \Phi((u_+, u_-)) = -2u_-.$$

Since Φ is an isomorphism, $x = \Phi((u_+, 0))$. It follows from the definition of θ that $\theta(\bar{x}) = \bar{0}$. Since θ is an isomorphism, $x \in \mathcal{L}_+$. We conclude that $\text{Ker}(R_L - 1) = \mathcal{L}_+$.

Thus, $\text{Ker}(R_L - 1) = \mathcal{L}_+$. A similarly argument shows that $L_- = \text{Im}(R_L - 1)$ and $\mathcal{L}_- = \text{Ker}(R_L + 1)$. $\square$

Corollary 2.2.1. Let L be a Lie conformal algebra and suppose $L = L_1 \oplus L_2$ as $\mathbb{C}[\partial]$ -modules, where L_1 and L_2 are Lie conformal subalgebras of L . Define the operator $R_L \in \text{End}L$, by $R_L(u) = u_1 - u_2$ for $u = u_1 + u_2$ where $u_1, u_2 \in L$.

PROOF. Set $L_+ = \mathcal{L}_+ = L_1$ and $L_- = \mathcal{L}_- = L_2$ in the Factorization Theorem. Then, $L_+/\mathcal{L}_+ \cong L_-/\mathcal{L}_- \cong \{0\}$. Thus, R_L defined in the above satisfies the mYBE (12). $\square$

2.2. Comparison of the classical R -matrices of Lie algebras and LCA. In this subsection, we explain the connection between the classical R -matrix of Lie algebras and the classical R -matrix of Lie conformal algebras. For convenience, we adopt the λ -bracket notation. In particular, we introduce the following definition.

Definition 2.2.2. Let L be a Lie conformal algebra with λ -bracket $\{\cdot_\lambda \cdot\}$, and R_L be a classical R -matrix of Lie conformal algebra L . According to the above discussions, the R -matrix R_L gives another λ -bracket structure on the vector space L . We denote this new λ -bracket by $\{\cdot_\lambda \cdot\}_{R_L}$ and refer to it as the λ -bracket twisted by the R -matrix R_L .

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra of dimension ℓ and let $u^1, u^2, \dots, u^\ell$ be its basis. Consider the affine Lie conformal algebra $v_0(\mathfrak{g})$ at level 0 equipped with the λ -bracket $\{\cdot_\lambda \cdot\}$ and its $\mathbb{C}$ -vector subspace $v_0(\mathfrak{g})_0 = \text{Span}_{\mathbb{C}}\{u^i \mid i = 1, 2, \dots, \dim \mathfrak{g}\}$. Restrict $\{\cdot_\lambda \cdot\}$ to $v_0(\mathfrak{g})_0$. Then in this case we have for all $a, b \in v_0(\mathfrak{g})_0$, $\{a_\lambda b\} = [a, b]$. This defines a Lie algebra structure on $v_0(\mathfrak{g})_0$. Moreover, $(v_0(\mathfrak{g})_0, \{\cdot_\lambda \cdot\}_{res}) \cong (\mathfrak{g}, [\cdot, \cdot])$ as Lie algebras. Suppose $R_L \in \text{End}(v_0(\mathfrak{g}))$ and $R_L(v_0(\mathfrak{g})_0) \subseteq v_0(\mathfrak{g})_0$. Then if R_L is a solution of the classical or modified Yang-Baxter equations of Lie conformal algebra $v_0(\mathfrak{g})$, it also satisfies the classical or modified Yang-Baxter equation of Lie algebra $(v_0(\mathfrak{g})_0, [\cdot, \cdot]_{res})$. Hence, the classical R -matrix of Lie conformal algebra may viewed as a generalization of classical R -matrix of Lie algebra.

2.3. Examples of the classical R -matrix. In this section, we present two examples of the classical R -matrix for the $v_k(\mathfrak{g})$ obtained from the universal affine vertex algebras $V^k(\mathfrak{g})$. Specifically, we calculate λ -brackets $\{\cdot_\lambda \cdot\}_{R_L}$ on the generators for $v_k(\mathfrak{g})_{R_L}$ obtained from the Borel and Iwasawa decompositions of $\mathfrak{g}$. Corollary 2.2.1 leads us to an effective way of constructing the classical R -matrix for $v_k(\mathfrak{g})$.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra admitting a decomposition $\mathfrak{g} \cong \mathfrak{a} \oplus \mathfrak{b}$ i.e., $\mathfrak{g} \cong \mathfrak{a} \oplus \mathfrak{b}$ as vector space and $\mathfrak{a}, \mathfrak{b}$ are Lie subalgebras of $\mathfrak{g}$. Also, suppose $\mathfrak{b}$ is an isotropic subalgebra of $\mathfrak{g}$. Consider the affine Lie conformal algebra $(v_k(\mathfrak{g}), [\cdot_\lambda \cdot], T)$. As a vector space, $v_k(\mathfrak{g})$ can be decomposed as $v_k(\mathfrak{g}) = v_k(\mathfrak{a}) \oplus v'_k(\mathfrak{b})$, where $v'_k(\mathfrak{b}) := \text{Span}_{\mathbb{C}}\{xt^n \cdot v_k \mid x \in \mathfrak{b}, n \leq -1\}$. Furthermore, $(v'_k(\mathfrak{b}), [\cdot_\lambda \cdot], T)$ forms a Lie conformal subalgebra of $v_k(\mathfrak{g})$. This motivates the following theorem.

THEOREM 2.2.3. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra with an ad -invariant non-degenerate, bilinear form $(\cdot | \cdot)$. Suppose $\mathfrak{g}$ has a Lie algebra decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$, where $\mathfrak{b}$ is an isotropic subalgebra of $\mathfrak{g}$ w.r.t. $(\cdot | \cdot)$. Let $R_L := \frac{1}{2}(P_+ - P_-)$, where P_+ , P_- are the projection operators from $v_k(\mathfrak{g})$ to $v_k(\mathfrak{a})$ and $v'_k(\mathfrak{b})$, respectively. Then R_L is a classical R -matrix of Lie conformal algebra $v_k(\mathfrak{g})$. Following Lie bialgebra's terminology, we refer to the R -matrix obtained in this way as the factorizable R -matrix of the Lie conformal algebra.*

Remark 2.2.4. In the case of Theorem 2.2.3, we have $L_+ = v_k(\mathfrak{a})$, $L_- = v_k(\mathfrak{b})$, $\mathcal{L}_\pm = \{0\}$, and Φ is a map from $\tilde{L}$ to L that sends (u_+, u_-) to $u_+ - u_-$.

Remark 2.2.5. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra with a Lie algebra decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$, consider the affine Lie conformal algebra $v_0(\mathfrak{g})$. Then, $v_0(\mathfrak{g})$ admits a Lie conformal algebra decomposition $v_0(\mathfrak{g}) \cong v_0(\mathfrak{a}) \oplus v'_0(\mathfrak{b})$, where both $v_0(\mathfrak{a})$, $v'_0(\mathfrak{b})$ are Lie conformal subalgebras of $v_0(\mathfrak{g})$. It follows from Factorization Theorem, and above discussion that for any Lie algebra decomposition $\mathfrak{g} \cong \mathfrak{a} \oplus \mathfrak{b}$, there exists an associated factorizable R -matrix for the level 0 Lie conformal algebra of $\mathfrak{g}$. In this case, we simply say that R_L is an R -matrix associated with Lie algebra decomposition $\mathfrak{a} \oplus \mathfrak{b}$.

Recall a finite-dimensional Manin triple, which is a triple of finite-dimensional Lie algebras $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$, where $\mathfrak{g}$ has an invariant bilinear form $(\cdot | \cdot)$ such that

- i. $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ where $\mathfrak{g}_\pm$ are Lie subalgebras of $\mathfrak{g}$.
- ii. $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are isotropic subspaces of $\mathfrak{g}$ with respect to the bilinear form $(\cdot | \cdot)$.

This immediately leads to the following corollary, which is similar to Corollary 2.2.5 but applies at nonzero levels.

Corollary 2.2.6. Let $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ be a finite-dimensional Manin triple equipped with an invariant bilinear form $(\cdot | \cdot)$. Then the affine Lie conformal algebra $v_k(\mathfrak{g})$ admits a $\mathbb{C}[\partial]$ -module factorization: $v_k(\mathfrak{g}) \cong v_k(\mathfrak{g}_+) \oplus v'_k(\mathfrak{g}_-)$. Furthermore, $v_k(\mathfrak{g})$ has a factorizable R -matrix $R_L := \frac{1}{2}(P_+ - P_-)$, where P_+ , P_- are the projection operators from $v_k(\mathfrak{g})$ to $v_k(\mathfrak{g}_+)$ and $v'_k(\mathfrak{g}_-)$, respectively.

Definition 2.2.7. Let R_L be an R -matrix of Lie conformal algebra $v_k(\mathfrak{g})$ obtained by Theorem 2.2.3. Then, we say the R_L is an R -matrix of $v_k(\mathfrak{g})$ associated with the Lie algebra decomposition $\mathfrak{a} \oplus \mathfrak{b}$.

Now, we adopt the following notations. Let $\mathfrak{g}$ be a semi-simple Lie algebra with a bilinear, non-degenerate, ad -invariant form $(\cdot, \cdot)$ and $\mathfrak{h}$ be its Cartan subalgebra. The root space decomposition of $\mathfrak{g}$ given by

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha, \quad \mathfrak{g}_\alpha := \{y \in \mathfrak{g} : [x, y] = \alpha(x)y \text{ for all } x \in \mathfrak{h}\},$$

where Δ denotes the set of roots of $\mathfrak{g}$. Let $\Pi := \{\alpha_1, \alpha_2, \dots, \alpha_r\}$ be the set of simple roots of $\mathfrak{g}$, where $r = \dim \mathfrak{h}$. Also, we let Δ_+ , Δ_- be the sets of positive roots, and

negative roots of $\mathfrak{g}$, respectively. Then the triangle decomposition of $\mathfrak{g}$ can be written as $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ = \mathfrak{n}_- \oplus \mathfrak{b}$, where $\mathfrak{n}_\pm = \bigoplus_{\alpha \in \Delta_\pm} \mathfrak{g}_\alpha$ and $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}_+$ is called the Borel subalgebra of $\mathfrak{g}$. Since $\dim \mathfrak{g}_\alpha = 1$, one chooses a non-zero element e^α in $\mathfrak{g}_\alpha$ for all α in Δ , and $h^i \in \mathfrak{h}$, $i = 1, 2, \dots, r$, as generators of the Lie algebra $\mathfrak{g}$.

2.4. Example: Borel decomposition. Consider the Borel decomposition of a finite-dimensional Lie algebra $\mathfrak{g}$ rank r , $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}_-$ with $\dim \mathfrak{n}_\pm = n$ and the R -matrix R_L associated with it. The corresponding Lie conformal algebra factorization is $v_k(\mathfrak{g}) = v_k(\mathfrak{b}) \oplus v'_k(\mathfrak{n}_-)$, where

$$v_k(\mathfrak{b}) = \text{Span}_{\mathbb{C}}\{v_k, T^p h_{-1}^i \cdot v_k, T^p e_{-1}^{\alpha_j} \cdot v_k \mid i = 1, 2, \dots, r, j = 1, 2, \dots, n, p \geq 1\},$$

and

$$v'_k(\mathfrak{n}_-) = \text{Span}_{\mathbb{C}}\{T^p e_{-1}^{-\alpha_i} \cdot v_k \mid i = 1, 2, \dots, n, p \geq 1\}.$$

For convenience, we may also denote $e_{-1}^{-\alpha_i} \cdot v_k$, $h_{-1}^j \cdot v_k$, $e_{-1}^{\alpha_i} \cdot v_k$ by $e^{-\alpha_i}$, h^j , e^{α_i} , $\forall i = 1, 2, \dots, n$, $j = 1, 2, \dots, r$. Let $L := v_k(\mathfrak{g})_{R_L}$. The commutators of the local Lie algebra $\text{Lie}(L)$ are computed as follows. Using the identities,

$$R_L(e_{-1}^{-\alpha_i} \cdot v_k) = -\frac{1}{2} f_{-1}^i \cdot v_k, \quad R_L(h_{-1}^j \cdot v_k) = \frac{1}{2} h_{-1}^j \cdot v_k, \quad R_L(e_{-1}^{\alpha_i} \cdot v_k) = \frac{1}{2} e_{-1}^{\alpha_i} \cdot v_k,$$

for all $i = 1, 2, \dots, n$, $j = 1, 2, \dots, r$. Hence, the operators $Y_{R_L}^-(\cdot, z)$ read: for all $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, r$,

$$Y_{R_L}^-(e_{-1}^{-\alpha_i} \cdot v_k, z) = \sum_{n \geq 0} \left(-\frac{1}{2} e_{(n)}^{-\alpha_i} + e_{(n)}^{-\alpha_i} R_L \right) z^{-n-1}, \quad Y_{R_L}^-(h_{-1}^j \cdot v_k, z) = \sum_{n \geq 0} \left(\frac{1}{2} h_{(n)}^j + h_{(n)}^j R_L \right) z^{-n-1},$$

$$Y_{R_L}^-(e_{-1}^{\alpha_i} \cdot v_k, z) = \sum_{n \geq 0} \left(\frac{1}{2} e_{(n)}^{\alpha_i} + e_{(n)}^{\alpha_i} R_L \right) z^{-n-1}.$$

By formula (1), we have for all $i, j = 1, 2, \dots, n$, $p, q = 1, 2, \dots, r$, and $m, k \in \mathbb{Z}$,

$$\begin{aligned} [e_{[m]}^{-\alpha_i}, e_{[k]}^{-\alpha_j}]_{R_L} &= [e_{[m]}^{-\alpha_i}, e_{[k]}^{-\alpha_j}], \quad [e_{[m]}^{-\alpha_i}, h_{[k]}^j]_{R_L} = 0, \quad [e_{[m]}^{-\alpha_i}, e_{[k]}^{\alpha_j}]_{R_L} = 0, \\ [h_{[m]}^p, h_{[k]}^q]_{R_L} &= [h_{[m]}^p, h_{[k]}^q], \quad [h_{[m]}^p, e_{[k]}^{\alpha_j}]_{R_L} = [h_{[m]}^p, e_{[k]}^{\alpha_j}], \\ [e_{[m]}^{\alpha_i}, e_{[k]}^{\alpha_j}]_{R_L} &= [e_{[m]}^{\alpha_i}, e_{[k]}^{\alpha_j}]_{[m+k]} + m(e^{\alpha_i}, e^{\alpha_j}) K \delta_{m+k, 0}. \end{aligned}$$

The calculations in Example 2.4 show the following statement.

THEOREM 2.2.8. *For a semi-simple Lie algebra $\mathfrak{g}$ with triangle decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, let R_L be the R -matrix associated with the Borel decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{b}$. Denote the universal enveloping vertex algebra of the R_L -twisted Lie conformal algebra*

$v_k(\mathfrak{g})_{R_L}$ by $V^k(\mathfrak{g})_{R_L}$. Then, generators of fields in $V^k(\mathfrak{g})_{R_L}$ have the following OPEs,

$$\begin{aligned} e_{R_L}^{-\alpha_i}(z)e_{R_L}^{-\alpha_j}(w) &\sim -\frac{[e^{-\alpha_i}, e^{-\alpha_j}](w)}{z-w} - \frac{(e^{\alpha_i}, e^{\alpha_j})K}{(z-w)^2}, \quad e_{R_L}^{-\alpha_i}(z)h_{R_L}^p(w) \sim 0, \quad e_{R_L}^{-\alpha_i}(z)e_{R_L}^{\alpha_j}(w) \sim 0, \\ h_{R_L}^p(z)h_{R_L}^q(w) &\sim \frac{(h^p, h^q)K}{(z-w)^2}, \quad h_{R_L}^p(z)e_{R_L}^{\alpha_j}(w) \sim \frac{\alpha_j(h^p)e^{\alpha_j}(w)}{z-w} + \frac{(h^p, e^{\alpha_j})K}{(z-w)^2}, \\ e_{R_L}^{\alpha_i}(z)e_{R_L}^{\alpha_j}(w) &\sim \frac{[e^{\alpha_i}, e^{\alpha_j}](w)}{z-w} + \frac{(e^{\alpha_i}, e^{\alpha_j})K}{(z-w)^2}. \end{aligned}$$

If we use the λ -bracket,

$$\begin{aligned} [e^{-\alpha_i}{}_{\lambda}e^{-\alpha_j}]_{R_L} &= [e^{-\alpha_i}, e^{-\alpha_j}] - (e^{\alpha_i}, e^{\alpha_j})\lambda, \quad [e^{-\alpha_i}{}_{\lambda}h^p]_{R_L} = 0, \quad [e^{-\alpha_i}{}_{\lambda}e^{\alpha_j}]_{R_L} = 0, \\ [h^p{}_{\lambda}h^q]_{R_L} &= (h^p, h^q)K\lambda, \quad [h^p{}_{\lambda}e^{\alpha_j}]_{R_L} = \alpha_j(h^p)e^{\alpha_j} + (h^p, e^{\alpha_j})K\lambda, \\ [e^{\alpha_i}{}_{\lambda}e^{\alpha_j}]_{R_L} &= [e^{\alpha_i}, e^{\alpha_j}] + (e^{\alpha_i}, e^{\alpha_j})K\lambda, \end{aligned}$$

for all $i, j = 1, 2, \dots, n$ and $p, q = 1, 2, \dots, r$.

Similarly to the case of Borel decomposition 2.4.

Example 2.2.9 (Iwasawa decomposition). Let $\mathfrak{g}_0$ be a real semisimple Lie algebra and let θ be a Cartan involution. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition. Choose a maximal abelian subspace $\mathfrak{a}_0 \subset \mathfrak{p}_0$ and let $\Sigma \subset \mathfrak{a}_0^*$ be the restricted root system. Fix a positive system Σ^+ and set

$$(\mathfrak{g}_0)_{\lambda} := \{X \in \mathfrak{g}_0 \mid [H, X] = \lambda(H)X \ \forall H \in \mathfrak{a}_0\}, \quad \mathfrak{n}_0 := \bigoplus_{\lambda \in \Sigma^+} (\mathfrak{g}_0)_{\lambda}, \quad \mathfrak{b}_0 := \mathfrak{a}_0 \oplus \mathfrak{n}_0.$$

Then $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0 = \mathfrak{k}_0 \oplus \mathfrak{b}_0$ as a vector space, where $\mathfrak{a}_0$ is abelian, $\mathfrak{n}_0$ is nilpotent, and $\mathfrak{b}_0$ is a solvable Lie subalgebra. Let $L = v_0(\mathfrak{g}_0)$ and define $R_L = \frac{1}{2}(P_{\mathfrak{k}_0} - P_{\mathfrak{b}_0})$ using the projections associated with $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{b}_0$.

In this $\mathfrak{sl}_n(\mathbb{C})$ case, we get $\mathfrak{k}_0 \cong \mathfrak{so}_n(\mathbb{C})$. Let A_{ij} , H_i , N_{ij} , be the basis of $\mathfrak{k}_0$, $\mathfrak{a}_0$, $\mathfrak{n}_0$, respectively. Apply Theorem 2.2.3, we get an R -matrix, which gives the following relation

$$R_L(A_{ij}) = \frac{1}{2}A_{ij}, \quad R_L(H_i) = -\frac{1}{2}H_i, \quad R_L(N_{ij}) = -\frac{1}{2}N_{ij}.$$

Therefore, take the universal enveloping vertex algebra we find the following OPEs.

$$\begin{aligned} A_{ij}^{R_L}(z)A_{kl}^{R_L}(w) &\sim \frac{\delta_{jk}A_{il}(w) - \delta_{jl}A_{ik}(w) - \delta_{ik}A_{jl}(w) + \delta_{il}A_{jk}(w)}{z-w}, \\ A_{ij}^{R_L}(z)H_p^{R_L}(w) &\sim 0, \quad A_{ij}^{R_L}(z)N_{kl}^{R_L}(w) \sim 0, \\ H_p^{R_L}(z)H_q^{R_L}(w) &\sim 0, \\ H_p^{R_L}(z)N_{ij}^{R_L}(w) &\sim -\frac{(\delta_{i,p} - \delta_{i,p+1} - \delta_{j,p} + \delta_{j,p+1})N_{ij}(w)}{z-w}, \\ N_{ij}^{R_L}(z)N_{kl}^{R_L}(w) &\sim -\frac{\delta_{jk}N_{il}(w) - \delta_{li}N_{kj}(w)}{z-w}. \end{aligned}$$

In particular, in the $\mathfrak{sl}_2$ case (so $A := A_{12}$, $H := H_1$, $N := N_{12}$),

$$A^{R_L}(z)A^{R_L}(w) \sim 0, \quad H^{R_L}(z)H^{R_L}(w) \sim 0, \quad A^{R_L}(z)H^{R_L}(w) \sim 0, \quad A^{R_L}(z)N^{R_L}(w) \sim 0,$$

and

$$H^{R_L}(z)N^{R_L}(w) \sim -\frac{2N^{R_L}(w)}{z-w}, \quad N^{R_L}(z)N^{R_L}(w) \sim 0.$$

3. Generalized AKS Scheme

In this section, we present and prove the generalized AKS scheme. To aid our discussion, we begin by fixing some notation here. Throughout this section and the next, we consider the Poisson vertex algebra constructed as follows. Let L be a Lie conformal algebra with λ -bracket $\{\cdot_\lambda \cdot\}$, and suppose L is free as a $\mathbb{C}[\partial]$ -module. Let the Poisson vertex algebra $\mathcal{V} := S^\bullet(L)$ be equipped with the λ -bracket $\{\cdot_\lambda \cdot\}$ and extended via the Master formula (5). Let R_L be an R -matrix of L , and define the λ -bracket $\{\cdot_\lambda \cdot\}_{R_L}$ as the λ -bracket twisted by R_L . Extend $\{\cdot_\lambda \cdot\}_{R_L}$ to $\mathcal{V}$ via the Master formula and denote the resulting Poisson vertex algebra by $\mathcal{V}_{R_L}$. In this setup, we say that the Poisson vertex algebra $\mathcal{V}$ has λ -bracket structures $\{\cdot, \cdot\}$ and $\{\cdot, \cdot\}_{R_L}$.

By the definition of an R -matrix, we have the following lemma.

Lemma 2.3.1. *Consider a Lie conformal algebra L , and let R_L be an R -matrix of L . For all elements $a, b \in L$, the identity $\{a_\lambda b\}_{R_L} = \{R_L(a)_\lambda b\} + \{a_\lambda R_L(b)\}$ holds.*

The following remark is useful for proving the involution theorem and its proof is straightforward.

Remark 2.3.2. Let $I := \{1, 2, \dots, n\}$ be an index set. For all $k, \ell \in I$, $r \in \mathbb{Z}_+$,

$$\{u_{\lambda+\partial}^{k,r} u^\ell\}_{\rightarrow} = \{u_{\lambda+\partial}^k u^\ell\}_{\rightarrow} (-\lambda - \partial)^r, \quad \{u_{\lambda+\partial}^\ell u^{k,r}\}_{\rightarrow} = (\partial + \lambda)^r \{u_{\lambda+\partial}^\ell u^k\}_{\rightarrow}.$$

INVOLUTION THEOREM. Let $\mathcal{V}$ be a Poisson vertex algebra with λ -brackets $\{\cdot_\lambda \cdot\}$ and $\{\cdot_\lambda \cdot\}_{R_L}$. In addition, suppose $\mathcal{V}$ is an algebra of differential functions in the variables $\{u_i\}_{i \in I}$, where $I = \{1, 2, \dots, n\}$ is an index set. Then $\forall f, g \in \mathcal{V}$, identity

$$(19) \quad \{f_\lambda g\}_{R_L} = \sum_{p \in \mathbb{Z}_+} \{R_L(u^i)_\lambda g\} (-\lambda - \partial)^p \frac{\partial f}{\partial u^{i,p}} + \sum_{q \in \mathbb{Z}_+} \frac{\partial g}{\partial u^{j,q}} (\partial + \lambda)^q \{f_\lambda R_L(u^j)\},$$

holds. Let $\mathfrak{z}(\mathcal{V})$ be the λ -bracket center of $\mathcal{V}$ with respect to the λ -bracket $\{\cdot_\lambda \cdot\}$, then for all $f, g \in \mathfrak{z}(\mathcal{V})$, $\{f_\lambda g\}_{R_L} = 0$.

PROOF OF INVOLUTION THEOREM. In this proof, we use the Einstein convention for the indices in I . By Lemma 2.3.1, we have for all $f, g \in \mathcal{V}$, $\{f_\lambda g\}_{R_L} = (*) + (**)$, where

$$(*) = \sum_{p, q \in \mathbb{Z}_+} \frac{\partial g}{\partial u^{j,q}} (\partial + \lambda)^q \left\{ R_L(u^i)_{\lambda+\partial} u^j \right\}_{\rightarrow} (-\partial - \lambda)^p \frac{\partial f}{\partial u^{i,p}},$$

and

$$(**) = \sum_{p, q \in \mathbb{Z}_+} \frac{\partial g}{\partial u^{j,q}} (\partial + \lambda)^q \left\{ u_{\lambda+\partial}^i R_L(u^j) \right\}_{\rightarrow} (-\partial - \lambda)^p \frac{\partial f}{\partial u^{i,p}}.$$

To prove (19), it suffices to verify that

$$(*) = \sum_{p \in \mathbb{Z}_+} \{R_L(u^i)_\lambda g\}(-\lambda - \partial)^p \frac{\partial f}{\partial u^{i,p}}, \text{ and } (**) = \sum_{q \in \mathbb{Z}_+} \frac{\partial g}{\partial u^{j,q}}(\partial + \lambda)^q \{f_\lambda R_L(u^j)\}.$$

Suppose $R_L(u^i) = \sum_{r \in \mathbb{Z}_+} c_{k,r}^i u^{k,r}$. Then, by Remark 2.3.2,

$$(*) = \sum_{p,q \in \mathbb{Z}_+} \sum_{r \in \mathbb{Z}_+} c_{k,r}^i \frac{\partial g}{\partial u^{j,q}}(\partial + \lambda)^q \{u_{\lambda+\partial}^k u^j\}_\rightarrow (-\lambda - \partial)^r (-\partial - \lambda)^p \frac{\partial f}{\partial u^{i,p}}.$$

By the Master formula,

$$\sum_{p \in \mathbb{Z}_+} \{R_L(u^i)_\lambda g\}(-\lambda - \partial)^p \frac{\partial f}{\partial u^{i,p}} = \sum_{p,q \in \mathbb{Z}_+} \sum_{r \in \mathbb{Z}_+} c_{k,r}^i \frac{\partial g}{\partial u^{j,q}}(\partial + \lambda)^q \{u_{\lambda+\partial}^k u^j\}_\rightarrow (-\lambda - \partial)^r (-\partial - \lambda)^p \frac{\partial f}{\partial u^{i,p}} = (*)$$

Similarly, one can show that $(**) = \sum_{q \in \mathbb{Z}_+} \frac{\partial g}{\partial u^{j,q}}(\partial + \lambda)^q \{f_\lambda R_L(u^j)\}$. Consequently,

$$\{f_\lambda g\}_{R_L} = (*) + (**) = \sum_{p \in \mathbb{Z}_+} \{R_L(u^i)_\lambda g\}(-\lambda - \partial)^p \frac{\partial f}{\partial u^{i,p}} + \sum_{q \in \mathbb{Z}_+} \frac{\partial g}{\partial u^{j,q}}(\partial + \lambda)^q \{f_\lambda R_L(u^j)\}.$$

In particular, for $f, g \in \text{gr}^F(\mathfrak{z}(\widehat{\mathfrak{g}}))$, it follows that $\{f_\lambda g\}_{R_L} = 0$. $\square$

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra equipped with a non-degenerate, ad-invariant bilinear form $(\cdot, \cdot)$. Consider the affine Kac–Moody vertex algebra $V^k(\mathfrak{g})$ of $\mathfrak{g}$ at the critical level $k = -h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. By Theorem 1.1.5 and Remark 1.1.6, we have

$$\text{gr}^F \mathfrak{z}(\widehat{\mathfrak{g}}) \subset \text{gr}^F V^k(\mathfrak{g}) \cong \mathbb{C}[J_\infty \mathfrak{g}^*] \cong S^{\bullet'}(v_0(\mathfrak{g})).$$

Hence, in this case, we have the following integrability scheme.

THEOREM 2.3.3 (Generalized AKS theorem). *Let $\mathfrak{g}$ be a simple Lie algebra with a non-degenerate, ad-invariant bilinear form $(\cdot, \cdot)$. Consider the affine Kac–Moody vertex algebra $V^k(\mathfrak{g})$ of $\mathfrak{g}$ at the critical level $k = -h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. Let $\mathcal{V} := \text{gr}^F V^k(\mathfrak{g})$ and let R_L be a classical R-matrix of $v_0(\mathfrak{g})$. Then the Poisson vertex algebra $\mathcal{V}$ possesses two λ -bracket structures $\{\cdot, \cdot\}$ and $\{\cdot, \cdot\}_{R_L}$. Let $h_n \in \text{gr}^F \mathfrak{z}(\widehat{\mathfrak{g}})$, where $n \in \mathbb{Z}_+$. Then, the sequence of functionals $\int h_n \in \mathcal{V}/\partial \mathcal{V}$, $n \in \mathbb{Z}$ satisfies $\{\int h_n, \int h_m\}_{R_L} = 0$. This gives rise to a Hamiltonian system $\frac{du^i}{dt_n} := \{\int h_n, u^i\}$, where $i = 1, 2, \dots, \dim \mathfrak{g}$ and $n \in \mathbb{Z}_+$.*

PROOF. This scheme follows from the involution theorem, Theorem 1.1.5, and Remark 1.1.6. $\square$

3.1. Classical and generalized AKS scheme. We show that in a special case, the generalized AKS scheme for the affine Lie conformal algebra reduces to the classical AKS scheme for Lie algebras. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra with a non-degenerate, ad-invariant bilinear form $(\cdot, \cdot)$. Suppose $\mathfrak{g}$ has a basis u^i , $i = 1, 2, \dots, \ell$, where $\ell = \dim \mathfrak{g}$. Now, consider the affine Lie conformal algebra $v_0(\mathfrak{g})$ with λ -bracket $\{\cdot, \cdot\}$. Using the notations from Section 2.2, the λ -bracket on $v_0(\mathfrak{g})$ naturally endows the vector space

$v_0(\mathfrak{g})_0$ a Lie algebra structure denoted by $\{\cdot, \cdot\}_{res}$. Then, $(v_0(\mathfrak{g})_0, \{\cdot, \cdot\}_{res}) \cong (\mathfrak{g}^*, \{\cdot, \cdot\})$ as Poisson varieties, where the right hand side is viewed as a Poisson variety equipped with the Kirillov-Kostant-Souriau (KKS) Poisson bracket. Thus, all results from the classical AKS scheme can be recovered through the generalized AKS scheme.

3.2. Applications. In this section, we provide several examples of our scheme. In particular, we find systems of PDEs, ODEs, and their Poisson structures associated with Poisson vertex algebras $\text{Sym}'(v_0(\mathfrak{sl}_2))$ and $\text{Sym}'(v_0(\mathfrak{sl}_3))$. For convenience, we introduce the following definitions.

Definition 2.3.4. Let $v_k(\mathfrak{g})$ be an affine Lie conformal algebra, let R_L be a classical R -matrix for $v_k(\mathfrak{g})$, and let $S^\bullet(v_k(\mathfrak{g}))$ denote the Poisson vertex algebra whose λ -bracket is the Master-Formula extension of the λ -bracket of $v_k(\mathfrak{g})$. As discussed in Section 3, $S^\bullet(v_k(\mathfrak{g}))$ carries two λ -brackets, namely $\{\cdot, \cdot\}$ and $\{\cdot, \cdot\}_{R_L}$. Set $\mathcal{V} := S^{\bullet'}(v_k(\mathfrak{g}))$; it inherits these two λ -brackets from $S^\bullet(v_k(\mathfrak{g}))$. We denote by $H_{R_L}(\partial)$ the Poisson structure on $\mathcal{V}$ corresponding to $\{\cdot, \cdot\}_{R_L}$, and we write $\mathcal{V}_{R_L}$ for $\mathcal{V}$ endowed with this bracket.

In the following, we consider the affine vertex algebras at level 0. Using the same notations as in Definition 2.3.4, consider an affine Lie conformal algebra $v_0(\mathfrak{g})$. Due to Remark 2.2.5, any Lie algebra factorization $\mathfrak{g} \cong \mathfrak{a} \oplus \mathfrak{b}$ gives an R -matrix. For simplicity, we will refer to the Poisson structure $H_{R_L}(\partial)$ on $\mathcal{V}_{R_L}$ as the Poisson structure associated with decomposition $\mathfrak{a} \oplus \mathfrak{b}$. Recall that in Examples 2.4 and 2.2.9, we have denoted the basis elements of $v_k(\mathfrak{g})$ by $e_{-1}^{-\alpha_i} \cdot v_k, h_{-1}^j \cdot v_k, e_{-1}^{\alpha_i} \cdot v_k$. In the following examples, we write f_i, h_j, e_i for the images of $f_i := e_{-1}^{-\alpha_i} \cdot v_k, h_j := h_{-1}^j \cdot v_k$, and $e_i := e_{-1}^{\alpha_i} \cdot v_k$ in the modified Poisson vertex algebra $S^{\bullet'}(v_0(\mathfrak{g}))$, respectively.

Example 2.3.5. By the data we computed in Example 2.4, the Poisson structure associated with the Borel decomposition of $\mathfrak{sl}_2$ is $H_{R_L}(\partial) =$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2e \\ 0 & -2e & 0 \end{bmatrix}. \text{ Take } \int h_n = \int S^n,$$

then the first equation of our hierarchy is $\frac{df}{dt_1} = 0, \frac{dh}{dt_1} = 4ef, \frac{de}{dt_1} = -2eh$. This system admits a special solution: $f = 1, h = \frac{1}{t_1+C}, e = -\frac{1}{4}(\frac{1}{t_1+C})^2$, where C is an integration constant.

Example 2.3.6. Use the Iwasawa decomposition for $\mathfrak{sl}_2(\mathbb{C})$ as we calculated in Example 2.2.9. Then $H_{ij}(\partial) = \{u_j \partial_{+\lambda} u_i\}_{\lambda=0}$ and $u_t = \{\int h, u\} = H(\partial) \frac{\delta}{\delta u} \int h$. The R -matrix gives the following Hamiltonian operator (in the ordered variables (f, h, e)):

$$H_{R_L}(\partial) = \begin{pmatrix} 0 & -2e & 0 \\ 2e & 0 & 2e \\ 0 & -2e & 0 \end{pmatrix}.$$

Here in the case of $\mathfrak{sl}_2(\mathbb{C})$ we set $A := f - e$, $H := h$, $N := e$. Let $C = \frac{1}{2}h^2 + 2fe$ (the quadratic Casimir polynomial) and set $\int h_n := \int C^n$ ($n \geq 1$). The first flow (t_1 -flow) is

$$f_{t_1} = -2he, \quad h_{t_1} = 4e(f + e), \quad e_{t_1} = -2he.$$

In particular $(f - e)_{t_1} = 0$, so write $c := f - e$ and put $\xi := f + e$, $\eta := h$. Then

$$\xi_{t_1} = -2\eta(\xi - c), \quad \eta_{t_1} = 2\xi(\xi - c), \quad \text{and} \quad (\xi^2 + \eta^2)_{t_1} = 0.$$

Thus $\xi^2 + \eta^2 = r^2$ is conserved and the system is solvable by quadrature.

Example 2.3.7. Take the same Poisson structure as in Example 2.3.5. Let $\int h_n := \int SS^{(n)}$, $n = 1, 2, \dots$, in particular, note that $n = 1$ gives a trivial flow. The second flow is,

$$(20) \quad \begin{aligned} \frac{df}{dt_2} &= 0, \quad \frac{dh}{dt_2} = -8fe(h'^2 + 4f'e' + 2ef'' + hh'' + 2fe''), \\ \frac{de}{dt_2} &= 4he(h'^2 + 4f'e' + 2ef'' + hh'' + 2fe''). \end{aligned}$$

This system may be viewed as infinitely many integrable systems parameterized by t_2 -independent function f . For example, let $f := \frac{1}{8}$, then our system is equivalent to system,

$$\frac{dh}{dt_2} = -e(h'^2 + hh'' + \frac{1}{4}e''), \quad \frac{de}{dt_2} = 4he(h'^2 + hh'' + \frac{1}{4}e'').$$

Take $f = 0$, then we may choose $h = \frac{1}{4}$, and our equations deduce to

$$(21) \quad \frac{de}{dt_2} = ee''.$$

We seek a traveling wave solution of (21) with form $f(\xi) = f(x + ct_2)$. Plugging ansatz $f(\xi)$ into (21), separating variables and integrating, we obtain that $c \ln f - A = \frac{df}{d\xi}$, where A is an integration constant. Setting $A := 0$, and integrating it once more, we obtain that $\xi + \xi_0 = \frac{1}{c} \text{Ei}(\ln f)$, where $\text{Ei}(x)$ is the exponential integral, i.e., $\text{Ei}(x) := -\int_{-x}^{\infty} \frac{e^{-t}}{t} dt$, see [65] and [13]. The function $\text{Ei}(x)$ is a higher transcendental function [13], see also [110]. Also, we seek a traveling solution for the system (20) in full generality. Note,

$$h'^2 + 4f'e' + 2ef'' + hh'' + 2fe'' = \frac{1}{2}\partial_x^2 h^2 + 2\partial_x^2 fe,$$

and $\frac{df}{dt_2} = 0$, hence we may suppose $f(x, t_2) = \frac{1}{4}C(x)$. Let $\xi(x, t_2) := h(x, t_2)$, $\eta(x, t_2) := e(x, t_2)$, then the system (20) is equivalent to

$$\frac{d\xi}{dt_2} = -4C(x)\eta\partial_x^2(\frac{1}{2}h^2 + \frac{1}{2}C(x)\eta), \quad \frac{d\eta}{dt_2} = 4\xi\eta\partial_x^2(\frac{1}{2}h^2 + \frac{1}{2}C(x)\eta).$$

Therefore, $\frac{d}{dt_2}(\frac{1}{2}\xi^2 + \frac{1}{2}C(x)\eta) = 0$. This means that quantity $\frac{1}{2}\xi^2 + \frac{1}{2}C(x)\eta$ is a first integral of system (23). Let $\frac{1}{2}B(x)^2 := \frac{1}{2}\xi^2 + \frac{1}{2}C(x)\eta$, then

$$(22) \quad \frac{d\xi}{dt_2} = -2C(x)(B(x)^2)_{xx}\eta = -2(B(x)^2)_{xx}B(x)^2 + 2(B(x)^2)_{xx}\xi^2,$$

$$(23) \quad \left(\frac{d\eta}{dt_2}\right)^2 = 4\xi^2\eta^2(B(x)^2)_{xx}^2 = 4\eta^2(B(x)^2)_{xx}^2(B(x)^2 - C(x)\eta).$$

From equation (23), we find

$$(24) \quad \frac{d\eta}{dt_2} = 2\eta(B(x)^2 - C(x)\eta)^{\frac{1}{2}}(B(x)^2)_{xx}.$$

Separating variables and integrating (24) once, we have $\int \frac{1}{2\eta\sqrt{B^2-C(x)\eta}}d\eta = \int (B(x)^2)_{xx}dt_2$.

This gives that $-\frac{1}{B} \tanh^{-1}\left[\frac{\sqrt{B^2-C(x)\eta}}{B(x)}\right] = (B(x)^2)_{xx}t_2 + c(x)$, where $c(x)$ is a function that only depends on x . Therefore, by rescaling the integration constant,

$$\eta(x, t_2) = \frac{B^2(x)}{C(x)} \operatorname{sech}^2[-B(B^2)_{xx}t_2 + c(x)] = \frac{B^2(x)}{C(x)} \operatorname{sech}^2\phi,$$

where $\phi = -B(B^2)_{xx}t_2 + c(x)$. Using formula $\frac{1}{2}B(x)^2 := \frac{1}{2}\xi^2 + \frac{1}{2}C(x)\eta$, we find $\xi(x, t_2) = B(x) \tanh \phi$. Choose $c(x) = \frac{\sqrt{c}}{2}x$, where c is a real number, $B(x) = \frac{9^{\frac{1}{3}}c^{\frac{1}{2}}}{2}(x - x_0)^{\frac{2}{3}}$, where x_0 is a constant, and $C(x) = \frac{c}{2B(x)^2}$, then

$$(25) \quad \eta(x, t_2) = \frac{c}{2} \operatorname{sech}^2 \frac{\sqrt{c}}{2}(x - ct_2)$$

is coincident with the KdV one soliton. Also, in this case, $\xi(x, t_2) = B(x) \tanh(\frac{\sqrt{c}}{2}(x - ct_2))$.

Rewrite equation (22) as

$$(26) \quad \frac{d\xi}{dt_2} = E(x)\xi^2 - F(x),$$

where $E(x) := 2(B(x)^2)_{xx}$, $F(x) := 2(B(x)^2)_{xx}B(x)^2$. We obtain a Ricatti equation whose coefficients depend on the parameter x . Hence, equation (26) can be linearized: let $\tau(t_2)$ be a function satisfying $\xi(t_2) = -\frac{1}{E(x)}\frac{d}{dt_2} \log \tau(t_2)$. Then, the equation (26) is equivalent to

$$(27) \quad \frac{d^2\tau}{dt_2^2} - P(x)^2\tau = 0,$$

where $P(x) = 2(B(x)^2)_{xx}B(x)$. Take two linearly independent solution of equation (27), say $\tau_1(x, t_2) = \exp(-c(x) + P(x)t_2)$ and $\tau_2(x, t_2) = \exp(c(x) - P(x)t_2)$, where $c(x)$ is an arbitrary function that only depends on variable x . We see all solutions of (27) have form $\tau(x, t_2) = c_1e^{-c(x)+P(x)t_2} + c_2e^{c(x)-P(x)t_2}$ for some $c_1, c_2 \in \mathbb{C}$. This means all solutions of (26) have form

$$\xi(x, t_2) = -\frac{1}{E(x)} \frac{c_1\tau_1'(t_2) + c_2\tau_2'(t_2)}{c_1\tau_1(t_2) + c_2\tau_2(t_2)},$$

where $\prime$'s are derivative with respect to the variable t_2 . Since this solution is completely determined by the ratio of c_1, c_2 , we may identify them with points on $\mathbb{CP}^1$. Note that by

the choice of $B(x)$, $E(x) = -\frac{1}{B(x)}$. Hence,

$$\xi(x, t_2) = -\frac{B(x)}{P(x)} \frac{c_1 P(x) \tau_1(t_2) - c_2 P(x) \tau_2(t_2)}{c_1 \tau_1(t_2) - c_2 \tau_2(t_2)} = B(x) \frac{-c_1 e^{-\zeta} + c_2 e^{\zeta}}{c_1 e^{-\zeta} + c_2 e^{\zeta}},$$

where $\zeta := c(x) - P(x)t_2$. Note that as $[c_1 : c_2] = [1 : 1]$, $\xi(x, t_2) = B(x) \tanh \zeta$. Therefore, by identity $\frac{1}{2}B(x)^2 := \frac{1}{2}\xi^2 + \frac{1}{2}C(x)\eta$, $\eta(x, t_2) = \frac{B(x)^2}{C(x)} \operatorname{sech}^2 \zeta$. Let $c \in \mathbb{R}$, choose $c(x) := \frac{\sqrt{c}}{2}x$, $B(x) := (-\frac{9}{16})^{\frac{1}{3}} c^{\frac{1}{2}} (x - x_0)^{\frac{2}{3}}$, where x_0 is a constant. We find $\eta(x, t_2) = \frac{c}{2} \operatorname{sech}^2(\frac{\sqrt{c}}{2}(x - ct_2))$. We see this solution is the same as equation (25).

Example 2.3.8. Take the same Poisson structure as in Example 2.3.6. Let $\int h_n := \int SS^{(n)}$, $n = 1, 2, \dots$, in particular, note that $n = 1$ gives a trivial system. The second system, i.e., $n = 2$, is

$$\begin{aligned} \frac{df}{dt_2} &= 2h(f - e)(h'^2 + 4f'e' + 2ef'' + hh'' + 2fe''), \\ \frac{dh}{dt_2} &= -4(f - e)(f + e)(h'^2 + 4f'e' + 2ef'' + hh'' + 2fe''), \\ \frac{de}{dt_2} &= 2h(f - e)(h'^2 + 4f'e' + 2ef'' + hh'' + 2fe''). \end{aligned}$$

Observing this system, we see the first and third equations guarantee that $f - e$ is a first integral. Therefore, we may put $f - e = C(x)$. Let

$$(28) \quad \xi(x, t_2) := h(x, t_2), \quad \eta(x, t_2) := f(x, t_2) + e(x, t_2),$$

and note $h'^2 + 4f'e' + 2ef'' + hh'' + 2fe'' = \partial_x^2(\frac{1}{2}h^2 + 2fe)$. In addition, Equation (28) implies $2fe = \frac{1}{2}\eta^2 - \frac{1}{2}C(x)^2$. Hence, our system is equivalent to

$$\frac{d\xi}{dt_2} = -4C(x)\eta\partial_x^2(\frac{1}{2}\xi^2 + \frac{1}{2}\eta^2 - \frac{1}{2}C(x)^2), \quad \frac{d\eta}{dt_2} = 4C(x)\xi\partial_x^2(\frac{1}{2}\xi^2 + \frac{1}{2}\eta^2 - \frac{1}{2}C(x)^2).$$

This implies that $\xi \frac{d\xi}{dt_2} = -\eta \frac{d\eta}{dt_2}$ and hence $\frac{d}{dt_2}(\frac{1}{2}\xi^2 + \frac{1}{2}\eta^2) = 0$. Namely, $\frac{1}{2}\xi^2 + \frac{1}{2}\eta^2$ is a first integral. Let $\frac{1}{2}\xi^2 + \frac{1}{2}\eta^2 = \frac{1}{2}A(x)^2$, then $\frac{d\xi}{dt_2} = -4C(x)P_{xx}\eta$, $\frac{d\eta}{dt_2} = 4C(x)P_{xx}\xi$, where $P(x) := \frac{1}{2}A(x)^2 - \frac{1}{2}C(x)^2$. Note that if $A(x) = C(x)$, we get trivial equation. Hence, we may suppose $A(x) \neq C(x)$. The identity $\frac{1}{2}\xi^2 + \frac{1}{2}\eta^2 = \frac{1}{2}A(x)^2$ gives

$$(29) \quad \frac{d\xi}{dt_2} = -4C(x)P_{xx}\sqrt{A(x)^2 - \xi^2}.$$

Use separation of variables and integrate (29) once. We find $\int \frac{d\xi}{\sqrt{A(x)^2 - \xi^2}} = -4C(x)P_{xx} \int dt_2$.

Therefore, $\tan^{-1}(\frac{\xi}{\sqrt{A(x)^2 - \xi^2}}) = -4C(x)P_{xx}(t_2 + \zeta_0(x))$, where $\zeta_0(x)$ does not depend on t_2 . Let $\psi(x, t_2) = -4C(x)P_{xx}(t_2 + \zeta_0)$, then

$$(30) \quad \xi(x, t_2) = \frac{A(x)\tan\psi}{\sqrt{1 + \tan^2\psi}} = \frac{A(x)\tan\psi}{\sec\psi} = A(x)\sin\psi.$$

Using the identity $\frac{1}{2}\xi^2 + \frac{1}{2}\eta^2 = \frac{1}{2}A(x)^2$, we find $\eta(x, t_2) = A(x)\cos\psi$. One writes solution (30) in terms of a traveling wave solution. To do this, suppose $A(x)$ and $-4C(x)P_{xx} = \alpha$ are constants with respect to both x, t_2 . Since $A(x)$ is a constant with respect to

the variable x , $-4C(x)P_{xx} = -4C(x)\frac{\partial^2}{\partial x^2}(\frac{1}{2}A^2 - \frac{1}{2}C(x)^2) = 2C(x)\frac{\partial^2 C(x)^2}{\partial x^2} = \alpha$. Choose $C(x) = \frac{(9\alpha)^{\frac{1}{3}}}{2}(x - x_0)^{\frac{2}{3}}$ and $\zeta_0(x) := \frac{x}{\alpha}$, then we have $\xi(x, t_2) = A \sin(x + \alpha t_2)$, $\eta(x, t_2) = A \cos(x + \alpha t_2)$.

Now, we present several Hamiltonian equations associated with the Lie algebra $\mathfrak{sl}_3$, derived from our integrability scheme.

Example 2.3.9. The Poisson structure associated with the Borel decomposition of $\mathfrak{sl}_3$ is

$$H_{RL}(\partial) = \begin{bmatrix} 0 & f_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -f_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2e_1 & -e_2 & e_3 \\ 0 & 0 & 0 & 0 & 0 & -e_1 & 2e_2 & e_3 \\ 0 & 0 & 0 & -2e_1 & e_1 & 0 & e_3 & 0 \\ 0 & 0 & 0 & e_2 & -2e_2 & -e_3 & 0 & 0 \\ 0 & 0 & 0 & -e_3 & -e_3 & 0 & 0 & 0 \end{bmatrix}.$$

Let S_1 be the quadratic Sugawara operator of $\mathfrak{sl}_2$ and $\int h_n = \int S_1^n$, then the first system is given by

$$\begin{aligned} \frac{df_1}{dt_1} &= -2f_3e_2, \quad \frac{df_2}{dt_1} = 2f_3e_1, \quad \frac{df_3}{dt_1} = 0, \quad \frac{dh_1}{dt_1} = -4f_1e_1 + 2f_2e_2 - 2f_3e_3, \\ \frac{dh_2}{dt_1} &= 2f_1e_1 - 2f_2e_2 - 2f_3e_3, \quad \frac{de_1}{dt_1} = 2h_1e_1 - 2f_2e_3, \\ \frac{de_2}{dt_1} &= \frac{2}{3}(-h_1e_2 + h_2e_2 + 3f_1e_3), \quad \frac{de_3}{dt_1} = 2(h_1 + h_2)e_3. \end{aligned}$$

Take the cubic Sugawara operator S_2 of $\mathfrak{sl}_3$ and $\int h_n = \int S_2^n$, the first system is a system of ODEs that takes the form

$$\begin{aligned} \frac{df_1}{dt_1} &= \frac{1}{3}f_3(2h_1e_2 + h_2e_2 - 3f_1e_3), \quad \frac{df_2}{dt_1} = f_3(h_1e_1 + f_2e_3), \quad \frac{df_3}{dt_1} = 0, \\ \frac{dh_1}{dt_1} &= \frac{1}{3}(-f_2(2h_1 + h_2)e_2 + f_3(-h_1 + h_2)e_3 - 3f_1(2h_1e_1 + f_2e_3)), \\ \frac{dh_2}{dt_1} &= \frac{1}{3}(f_2(2h_1 + h_2)e_2 + f_3(-h_1 + h_2)e_3 + 3f_1(h_1e_1 - h_1e_3)), \\ \frac{de_1}{dt_1} &= \frac{1}{3}(h_1^2e_1 + 6f_1e_1^2 - 3f_2e_1e_2 + f_2h_2e_3 + 3f_3e_1(-h_1 + 2h_2 + e_3) + 2h_1(h_2e_1 + f_2e_3)), \\ \frac{de_2}{dt_1} &= \frac{1}{9}(-h_1^2e_2 + e_2(-h_2^2 - 9f_1e_1 + 3f_2e_2) + 3f_3e_2(3h_1 - 3h_2 - 2e_3) + h_1(-4h_2e_2 + 9f_1e_3)), \\ \frac{de_3}{dt_1} &= \frac{1}{3}(h_1^2 - h_2^2 + 3f_3(h_1 + h_2) + 3f_1e_1 - 3f_2e_2)e_3. \end{aligned}$$

Example 2.3.10. By the computations in Example 2.2.9, the Poisson structure associated with the Iwasawa decomposition of $\mathfrak{sl}_3$ is:

$$H_{R_L}(\partial) = \begin{pmatrix} 0 & f_3 & e_2 - f_2 & -2e_1 & e_1 & 0 & e_3 & 0 \\ -f_3 & 0 & f_1 - e_1 & e_2 & -2e_2 & -e_3 & 0 & 0 \\ f_2 - e_2 & e_1 - f_1 & 0 & -e_3 & -e_3 & 0 & 0 & 0 \\ 2e_1 & -e_2 & e_3 & 0 & 0 & 2e_1 & -e_2 & e_3 \\ -e_1 & 2e_2 & e_3 & 0 & 0 & -e_1 & 2e_2 & e_3 \\ 0 & e_3 & 0 & -2e_1 & e_1 & 0 & e_3 & 0 \\ -e_3 & 0 & 0 & e_2 & -2e_2 & -e_3 & 0 & 0 \\ 0 & 0 & 0 & -e_3 & -e_3 & 0 & 0 & 0 \end{pmatrix}.$$

$$\frac{df_1}{dt_1} = 3e_1e_2^2 - e_1h_1^2 - 2e_1h_1h_2 + e_2e_3(h_1 - h_2) - e_2f_3(2h_1 + h_2) + 3e_3f_1f_3 - 3e_3f_2h_1,$$

$$\frac{df_2}{dt_1} = -3e_1^2e_2 - e_1e_3(h_1 - h_2) - e_1f_3(h_1 + 2h_2) + 2e_2h_1h_2 + e_2h_2^2 - 3e_3f_1h_2 - 3e_3f_2f_3,$$

$$\frac{df_3}{dt_1} = -3e_1e_2(h_1 + h_2) + e_1f_2(h_1 + 2h_2) + e_2f_1(2h_1 + h_2) - 3e_3f_1^2 + 3e_3f_2^2 - e_3h_1^2 + e_3h_2^2,$$

$$\begin{aligned} \frac{dh_1}{dt_1} &= 2e_1^2h_1 + 4e_1^2h_2 + 3e_1e_2e_3 + 3e_1e_2f_3 + 6e_1e_3f_2 + 2e_1f_1h_1 + 4e_1f_1h_2 \\ &\quad + 2e_2^2h_1 + e_2^2h_2 - 3e_2e_3f_1 + 2e_2f_2h_1 + e_2f_2h_2 + e_3^2(h_1 - h_2) \\ &\quad + 3e_3f_1f_2 + e_3f_3(h_1 - h_2), \end{aligned}$$

$$\begin{aligned} \frac{dh_2}{dt_1} &= -e_1^2h_1 - 2e_1^2h_2 + 3e_1e_2e_3 + 3e_1e_2f_3 - 3e_1e_3f_2 - e_1f_1h_1 - 2e_1f_1h_2 \\ &\quad - 4e_2^2h_1 - 2e_2^2h_2 + 6e_2e_3f_1 - 4e_2f_2h_1 - 2e_2f_2h_2 + e_3^2(h_1 - h_2) \\ &\quad + 3e_3f_1f_2 + e_3f_3(h_1 - h_2), \end{aligned}$$

$$\frac{de_1}{dt_1} = 3e_1e_2f_2 - e_1h_1^2 - 2e_1h_1h_2 - e_2e_3(2h_1 + h_2) + 3e_3^2f_1 - e_3f_2(2h_1 + h_2),$$

$$\frac{de_2}{dt_1} = -3e_1e_2f_1 - e_1e_3(h_1 + 2h_2) + 2e_2h_1h_2 + e_2h_2^2 - 3e_3^2f_2 - e_3f_1(h_1 + 2h_2),$$

$$\frac{de_3}{dt_1} = -3e_1e_3f_1 + 3e_2e_3f_2 - e_3h_1^2 + e_3h_2^2.$$

CHAPTER 3

Shifted symplectic structures and Poisson vertex algebras

In this chapter, we introduce the language of graded geometry in order to describe the classical R -matrix for Lie conformal algebras.

1. Graded conformal algebra

In this section, we introduce graded conformal algebras and chiral Gerstenhaber algebras. We give several examples drawn from geometry and algebra. The geometry of PVA/VOA can be found in [4]. The super case is studied in detail in [86].

1.1. Poisson vertex superalgebra and related geometry. For completeness we now define the graded Lie conformal algebra L . It may be interesting to compare this algebra with the dg vertex algebra introduced by Caradot, Jiang, and Lin (see [24]).

Definition 3.1.1. A graded Lie conformal algebra L is a left $\mathbb{Z}$ -graded $\mathbb{C}[\partial]$ -module $L = \bigoplus_{i \in \mathbb{Z}} L_i$, where ∂ is an endomorphism of L of degree 0 together with a $\mathbb{C}$ -bilinear operation

$$[-_\lambda -] : L \otimes L \rightarrow \mathbb{C}[\lambda] \otimes_{\mathbb{C}} L, \quad a \otimes b \mapsto [a_\lambda b] := \sum_{n \geq 0} \frac{\lambda^n}{n!} (a(n)b) = \sum_{n \geq 0} \lambda^{(n)} (a(n)b),$$

where λ is an even variable. Moreover, $(L, \partial, [-_\lambda -])$ satisfies the following axioms: for all $a, b, c \in L$,

1. Sesquilinearity: $[\partial(a)_\lambda b] = -\lambda[a_\lambda b]$, $[a_\lambda \partial(b)] = (\lambda + \partial)[a_\lambda b]$,
2. Skew-symmetry: $[b_\lambda a] = -(-1)^{|a||b|}[a_{-\lambda-\partial}b]$,
3. Jacobi identity: $[a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu}c] + (-1)^{|a||b|}[b_\mu [a_\lambda c]]$.

The coefficient $a(n)b$ is called the n th product of a and b . A graded LCA of degree n is a graded $\mathbb{C}[\partial]$ -module L endowed with a graded λ -bracket on $L[n]$.

Remark 3.1.2. Similar to the graded Lie algebra case, as a bilinear operation $\{-_\lambda -\} : L \otimes L \rightarrow L[-n][\lambda]$ satisfying graded skew-symmetry and graded Jacobi relations:

$$\begin{aligned} \{a_\lambda b\} &= -(-1)^{(|a|+n)(|b|+n)} \{b_{-\lambda-\partial}a\}, \\ \{a_\lambda \{b_\mu c\}\} - (-1)^{(|a|+n)(|b|+n)} \{b_\mu \{a_\lambda c\}\} &= \{\{a_\lambda b\}_{\lambda+\mu}c\}. \end{aligned}$$

Note the Lie conformal superalgebra is a special case of graded Lie conformal algebra. It may be interesting to compare Definition 3.1.1 with the graded vertex algebra discussed in [24].

Example 3.1.3. [56, Example 3.3] Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra. We can associate $\mathfrak{g}$ a Lie conformal superalgebra $\text{Cur}(\mathfrak{g}) = \mathbb{C}[\partial] \otimes \mathfrak{g}$ called the current conformal superalgebra associated to $\mathfrak{g}$ with λ -bracket $[a_\lambda b] = [a, b]$, $\forall a, b \in \mathfrak{g}$.

Here, we define a chiral Gerstenhaber algebra. It is interesting to compare this definition with the one given by Tamarkin in his ICM talk (see [106]), as well as with those in [39] and [9].

Definition 3.1.4. A chiral Gerstenhaber algebra is a quintuple $(\mathcal{V}^\bullet, 1, \cdot, \{-_\lambda-\}, \partial)$ such that $\mathcal{V}^\bullet$ is a $\mathbb{Z}$ -graded vector space, ∂ is a degree 0 endomorphism with respect to this $\mathbb{Z}$ -grading. In addition, product $\cdot$ is a degree 0 graded commutative product and the bilinear operator $\{-_\lambda-\}$ is a degree -1 λ -bracket. Moreover, this quintuple satisfies the following axioms.

1. $(\mathcal{V}^\bullet, 1, \partial, \cdot)$ is a commutative, associative differential algebra whose product has degree 0.
 2. $(\mathcal{V}^\bullet, \partial, \{-_\lambda-\})$ is a graded Lie conformal algebra of degree 1.
- $$(31) \quad \{a_\lambda \{b_\mu c\}\} = \{\{a_\lambda b\}_{\lambda+\mu} c\} + (-1)^{(|a|+1)(|b|+1)} \{b_\mu \{a_\lambda c\}\}.$$

3. $\{-_\lambda-\}$ and $\cdot$ satisfy the following shifted Leibniz rule: $\{a_\lambda bc\} = \{a_\lambda b\}c + (-1)^{(|a|+1)|b|} b \{a_\lambda c\}$.

We shall also call λ -bracket the chiral Nijenhuis-Schouten bracket.

Remark 3.1.5. By shifting the grading on $\mathcal{V}^\bullet$, the graded skew-symmetry and the odd Leibniz rule may be written as $\{b_\lambda a\} = (-1)^{|a||b|} \{a_{-\lambda-\partial} b\}$, $\{c_\lambda(a \cdot b)\} = \{c_\lambda a\} \cdot b + (-1)^{(|c|+1)|a|} a \cdot \{c_\lambda b\}$. See, e.g., [51, 64, 80].

In Section 2, we adopt this graded skew-symmetry and this odd Leibniz rule, both standard in the integrable systems literature.

Example 3.1.6. Let $L := \text{Cur}(\mathfrak{g})$ as in Example 3.1.3. Then $S^\bullet(L[1])$ is a chiral Gerstenhaber algebra. In particular, the odd Leibniz rule reads

$$\{a_\lambda(b \cdot c)\} = \{a_\lambda b\} \cdot c + (-1)^{(|a|+1)|b|} b \cdot \{a_\lambda c\}.$$

Similar to Lemma 8 in [75] and to the corresponding proposition in [12], we have the following proposition.

Proposition 3.1.7. Let $(\mathcal{V}^\bullet, \{-_\lambda -\}, \partial, \cdot)$ be a chiral Gerstenhaber algebra. Define $\hat{\mathcal{F}}_{cl}^\bullet := \mathcal{V}^\bullet / \partial \mathcal{V}^\bullet$ and let $\int : \mathcal{V}^\bullet \rightarrow \hat{\mathcal{F}}_{cl}^\bullet$ be the quotient map. Then the following brackets are well-defined:

- (1) $\hat{\mathcal{F}}_{cl}^\bullet \times \hat{\mathcal{F}}_{cl}^\bullet \rightarrow \hat{\mathcal{F}}_{cl}^\bullet, \quad \{\int a, \int b\} := \int \{a_\lambda b\}|_{\lambda=0},$
- (2) $\hat{\mathcal{F}}_{cl}^\bullet \times \mathcal{V}^\bullet \rightarrow \mathcal{V}^\bullet, \quad \{\int a, b\} := \{a_\lambda b\}|_{\lambda=0}.$

Moreover, (1) defines a graded Lie bracket on $\hat{\mathcal{F}}_{cl}^\bullet$, and (2) defines a representation of the graded Lie algebra $\hat{\mathcal{F}}_{cl}^\bullet$ on $\mathcal{V}^\bullet$ by derivations whose degree is shifted by 1; these derivations act on both the product and the λ -bracket of $\mathcal{V}^\bullet$ and commute with ∂ . Here, “degree shifted by 1” means that the bracket in (2) satisfies

$$\{\int a, bc\} = \{\int a, b\}c + (-1)^{|b|(|a|+1)} b \{\int a, c\}.$$

PROOF. First, by left and right *sesquilinearity*, the brackets in (1) and (2) are well-defined. For all $\int a, \int b, \int c \in \hat{\mathcal{F}}_{cl}^\bullet$, the graded skew-symmetry and the graded Jacobi identity of the chiral Gerstenhaber algebra imply that

$$\{\int a, \int b\} = -(-1)^{(|a|+1)(|b|+1)} \{\int b, \int a\},$$

and

$$\{\int a, \{\int b, \int c\}\} = \{\{\int a, \int b\}, \int c\} + (-1)^{(|a|+1)(|b|+1)} \{\int b, \{\int a, \int c\}\},$$

which shows that $\hat{\mathcal{F}}_{cl}^\bullet$, together with the first bracket, is a graded Lie algebra.

We now show that the bracket in (2) defines a representation of this graded Lie algebra. By the above, (1) defines a graded Lie algebra on $\hat{\mathcal{F}}_{cl}^\bullet$. Let

$$\rho : \hat{\mathcal{F}}_{cl}^\bullet \rightarrow \text{End}(\mathcal{V}^\bullet), \quad \rho(\int a)(b) := \{\int a, b\}.$$

By the shifted graded Jacobi identity (31), we obtain

$$\rho(\{\int a, \int b\}) = \rho(\int a) \circ \rho(\int b) - (-1)^{(|a|+1)(|b|+1)} \rho(\int b) \circ \rho(\int a),$$

and

$$\rho(\int a)(\{b_\mu c\}) = \{\rho(\int a)b_\mu c\} + (-1)^{(|a|+1)(|b|+1)} \{b_\mu \rho(\int a)c\},$$

for all $a, b, c \in \mathcal{V}^\bullet$. Also, the shifted graded Leibniz rule implies that, for all $a, b, c \in \mathcal{V}^\bullet$, $\{\int a, bc\} = \{\int a, b\}c + (-1)^{(|a|+1)|b|} b \{\int a, c\}$. Finally, by sesquilinearity of $\mathcal{V}^\bullet$, we conclude the commutativity of ∂ with $\{\int a, -\}$ for all $a \in \mathcal{V}^\bullet$. $\square$

2. Shifted 1-symplectic structure and Poisson vertex algebra

In this section, we present our main theorem. We begin by recalling a theorem of Arakawa [4] (generalized by Li [86]) on the Poisson vertex algebra structure on arc spaces. After that, we fix conventions for the geometric setup used later in this section.

THEOREM 3.2.1 ([4, 86]). *Let $X = \operatorname{Spec} R$ be an affine Poisson superscheme (so R is a Poisson superalgebra). Then the arc space $J_\infty X$ carries a unique Poisson vertex superalgebra structure on*

$$R_\infty := \mathcal{O}(X_\infty)$$

characterized by

$$(32) \quad \{a_\lambda b\} = \{a, b\}, \quad \text{for all } a, b \in R.$$

This implies the following proposition.

Proposition 3.2.2. *Let $X = \operatorname{Spec} R$ be an affine scheme, where R is a Gerstenhaber algebra. Then there exists a unique chiral Gerstenhaber algebra structure on*

$$R_\infty := \mathcal{O}(X_\infty)$$

characterized by

$$(33) \quad \{a_\lambda b\} = \{a, b\}, \quad \text{for all } a, b \in R.$$

PROOF. Since the translation operator ∂ is even, the axioms of a chiral Gerstenhaber algebra follow directly from those of a level-0 Poisson vertex algebra. $\square$

2.1. Conventions. Now, we fix conventions. We mainly follow the notation of [8, Chs. 1 and 4].

We first fix notation. Let $X = \operatorname{Spec} R$ be a smooth affine scheme. Denote its coordinate ring by $\mathcal{A}_0 := \Gamma(X, \mathcal{O}_X)$, and set $Y := T^*[1]X$ to be the degree-shifted cotangent bundle of X . Since X is smooth, for any $x \in X$ there exists an affine open neighbourhood $U \ni x$ and a local coordinate system $\{u^i, \partial_i\}_{i=1}^n$ on U in the sense of [70, Appendix A.5], i.e. $[\partial_i, \partial_j] = 0$ and $\partial_i(u^j) = \delta_{ij}$. In particular, $u = (u^1, \dots, u^n) : U \rightarrow \mathbb{A}^n$ is étale. Let $\theta_1, \dots, \theta_n$ be the parity-reversed fiber coordinates dual to $du^1, \dots, du^n$ (equivalently, corresponding to $\partial_1, \dots, \partial_n$).

Write $\hat{\mathcal{A}}_0 := \Gamma(Y, \mathcal{O}_Y)$ for the algebra of functions on Y . It has a natural decomposition

$$\hat{\mathcal{A}}_0 = \bigoplus_{p=0}^n \hat{\mathcal{A}}_0^p.$$

Étale locally, $\hat{\mathcal{A}}_0^p$ consists of functions of the form

$$P = \sum P^{\alpha_1 \dots \alpha_p} \theta_{\alpha_1} \theta_{\alpha_2} \dots \theta_{\alpha_p},$$

with the coefficients $P^{\alpha_1 \dots \alpha_p}$ totally skew-symmetric in the indices $(\alpha_1, \dots, \alpha_p)$.

Passing to jet schemes, fix an affine open neighbourhood $U \subset X$ equipped with an étale coordinate map $u = (u^1, \dots, u^n) : U \rightarrow \mathbb{A}^n$ as above. We set

$$\mathcal{A}_U := \Gamma(U, (\pi_{\infty,0})_* \mathcal{O}_{X_\infty}), \quad \hat{\mathcal{A}}_U := \Gamma(T^*[1]U, (\pi_{\infty,0})_* \mathcal{O}_{Y_\infty}).$$

We now give a geometric example of a chiral Gerstenhaber algebra.

Example 3.2.3. Let X be an affine scheme over $\mathbb{C}$, and set $Y := T^*[1]X$. We use the same notation as in Example 1.3.11. By Proposition 3.1.7, on an étale open subset $U \subset X$, the section $\hat{\mathcal{A}}_U$ carries a natural structure of a chiral Gerstenhaber algebra with λ -bracket

$$\{u^\alpha \lambda \theta_\beta\} = \delta^\alpha_\beta,$$

extended to the algebra by the (shifted) Leibniz rule.

Therefore, following the notation of Section 2.1, we obtain a chiral Gerstenhaber algebra

$$\mathcal{V}^\bullet := \hat{\mathcal{A}}_U$$

with λ -bracket as in Example 3.2.3. Let ∂ denote the translation operator on $\mathcal{V}^\bullet$. We set

$$\hat{\mathcal{F}}_{cl}^\bullet := \mathcal{V}^\bullet / \partial \mathcal{V}^\bullet$$

and call $\hat{\mathcal{F}}_{cl}^\bullet$ the space of (classical) local functionals. We write

$$\int : \mathcal{V}^\bullet \rightarrow \hat{\mathcal{F}}_{cl}^\bullet$$

for the quotient map, and for $P \in \mathcal{V}^\bullet$ we denote its class in $\hat{\mathcal{F}}_{cl}^\bullet$ by $\int P$. By Proposition 3.1.7, the brackets from Proposition 3.1.7 endow $\hat{\mathcal{F}}_{cl}^\bullet$ with the structure of a graded Lie algebra,

$$[-, -] : \hat{\mathcal{F}}_{cl}^\bullet \otimes \hat{\mathcal{F}}_{cl}^\bullet \rightarrow \hat{\mathcal{F}}_{cl}^\bullet,$$

and make $\mathcal{V}^\bullet$ a representation of $\hat{\mathcal{F}}_{cl}^\bullet$ by derivations.

In the following, we work on an étale neighbourhood $U = \mathbb{A}^n$ for some $n \in \mathbb{N}$, and we simply denote $\mathcal{A}_U$ and $\hat{\mathcal{A}}_U$ by $\mathcal{A}$ and $\hat{\mathcal{A}}$, respectively.

2.2. Normalizing operator and normalized expression. We recall the normalizing operator $\mathcal{N}$, first introduced by A. Barakat in her Ph.D. thesis [11, Section 2.6]; see also [90, Lemma 2.3.6]. For $P = \int \tilde{P} \in \hat{\mathcal{F}}_{cl}^p$ we set

$$\mathcal{N}(P) := \sum_\alpha \theta_\alpha \frac{\delta \tilde{P}}{\delta \theta_\alpha}.$$

One checks that for every $P = \int \tilde{P} \in \hat{\mathcal{F}}_{cl}^p$,

$$\int \mathcal{N}(P) = p P.$$

Thus $\mathcal{N}$ provides a canonical normalized representative of each class in $\hat{\mathcal{F}}_{cl}^p$.

In particular, let

$$P = \int \tilde{P} = \int \sum_{s,t} P_{st}^{\alpha\beta} \theta_\alpha^s \theta_\beta^t \in \hat{\mathcal{F}}_{cl}^2.$$

After applying $\mathcal{N}$ and integrating by parts we obtain a normalized density of the form

$$(34) \quad \tilde{P}_{\text{norm}} = \frac{1}{2} \sum_{\alpha,\beta,s} \theta_\alpha (\mathcal{P}_s^{\alpha\beta} \partial^s) \theta_\beta.$$

In the notation of integrable systems, this is encoded by the matrix differential operator

$$\mathcal{P}(\partial) = (\mathcal{P}^{\alpha\beta}(\partial))_{\alpha,\beta}, \quad \mathcal{P}^{\alpha\beta}(\partial) = \sum_s \mathcal{P}_s^{\alpha\beta} \partial^s.$$

In the sequel we will always work with such normalized representatives, obtained from differential polynomials $\tilde{P} \in \hat{\mathcal{A}}^2$.

As observed by Casati and Wang in [27, p. 1507] (in the difference setting), and as explained to the author by M. Casati (personal communication) and in lectures of Si-Qi Liu, the graded commutativity of the odd variables θ_α and of the trace/integral implies that only the skewsymmetric part of $H(\partial)$ contributes to the class P . We follow the notation of Section 2.1; the main theorem below gives an explicit description of the graded Lie algebra structure on $\hat{\mathcal{F}}_{cl}^\bullet$.

THEOREM 3.2.4. *For any $P \in \hat{\mathcal{A}}^p$ and $Q \in \hat{\mathcal{A}}^q$, the graded Lie algebra structure on $\hat{\mathcal{F}}_{cl}^\bullet$, induced by the shifted symplectic structure on $T^*[1]X$ and the shifted Leibniz rule (see Example 3.2.3), is given by*

$$\left\{ \int P, \int Q \right\} = \int \frac{\delta P}{\delta \theta_\sigma} \frac{\delta Q}{\delta u^\sigma} + (-1)^p \frac{\delta P}{\delta u^\sigma} \frac{\delta Q}{\delta \theta_\sigma}.$$

PROOF. In this proof, we adopt the following conventions: Greek indices are summed by the Einstein convention; m, n are summed explicitly. Also, for differential monomials

$$P = P_{s_1 \dots s_p}^{\alpha_1 \dots \alpha_p} \Theta_{\alpha_1 \dots \alpha_p}^{s_1 \dots s_p}, \quad Q = Q_{r_1 \dots r_q}^{\beta_1 \dots \beta_q} \Theta_{\beta_1 \dots \beta_q}^{r_1 \dots r_q},$$

where $\Theta_{\alpha_1 \dots \alpha_p}^{s_1 \dots s_p} = \theta_{\alpha_1}^{s_1} \theta_{\alpha_2}^{s_2} \dots \theta_{\alpha_p}^{s_p}$ and $\Theta_{\beta_1 \dots \beta_q}^{r_1 \dots r_q} = \theta_{\beta_1}^{r_1} \theta_{\beta_2}^{r_2} \dots \theta_{\beta_q}^{r_q}$. We have

$$\begin{aligned} \{P_\lambda Q\} &= P_{s_1 \dots s_p}^{\alpha_1 \dots \alpha_p} \left\{ \Theta_{\alpha_1 \dots \alpha_p}^{s_1 \dots s_p} \lambda Q_{r_1 \dots r_q}^{\beta_1 \dots \beta_q} \right\} \Theta_{\beta_1 \dots \beta_q}^{r_1 \dots r_q} \\ &\quad + (-1)^{pq} Q_{r_1 \dots r_q}^{\beta_1 \dots \beta_q} \left\{ P_{s_1 \dots s_p}^{\alpha_1 \dots \alpha_p} \lambda \Theta_{\beta_1 \dots \beta_q}^{r_1 \dots r_q} \right\} \Theta_{\alpha_1 \dots \alpha_p}^{s_1 \dots s_p}. \end{aligned}$$

Using the Master Formula and then setting $\lambda = 0$, we obtain

$$\begin{aligned} &P_{s_1 \dots s_p}^{\alpha_1 \dots \alpha_p} \left\{ \Theta_{\alpha_1 \dots \alpha_p}^{s_1 \dots s_p} \lambda Q_{r_1 \dots r_q}^{\beta_1 \dots \beta_q} \right\} \Theta_{\beta_1 \dots \beta_q}^{r_1 \dots r_q} \Big|_{\lambda=0} \\ &= P_{s_1 \dots s_p}^{\alpha_1 \dots \alpha_p} \sum_{m,n \geq 0} \frac{\partial Q_{r_1 \dots r_q}^{\beta_1 \dots \beta_q}}{\partial u^{\sigma,m}} \partial^n (-\partial)^m \left(\frac{\partial \Theta_{\alpha_1 \dots \alpha_p}^{s_1 \dots s_p}}{\partial \theta_\sigma^n} \right) \Theta_{\beta_1 \dots \beta_q}^{r_1 \dots r_q}. \end{aligned}$$

Similarly,

$$\begin{aligned} &(-1)^{pq} Q_{r_1 \dots r_q}^{\beta_1 \dots \beta_q} \left\{ P_{s_1 \dots s_p}^{\alpha_1 \dots \alpha_p} \lambda \Theta_{\beta_1 \dots \beta_q}^{r_1 \dots r_q} \right\} \Theta_{\alpha_1 \dots \alpha_p}^{s_1 \dots s_p} \Big|_{\lambda=0} \\ &= (-1)^{pq} Q_{r_1 \dots r_q}^{\beta_1 \dots \beta_q} \sum_{m,n \geq 0} \frac{\partial \Theta_{\beta_1 \dots \beta_q}^{r_1 \dots r_q}}{\partial \theta_\sigma^n} \partial^n (-\partial)^m \left(\frac{\partial P_{s_1 \dots s_p}^{\alpha_1 \dots \alpha_p}}{\partial u^{\sigma,m}} \right) \Theta_{\alpha_1 \dots \alpha_p}^{s_1 \dots s_p}. \end{aligned}$$

Therefore, passing to the quotient, we find

$$\int \{P_\lambda Q\} \Big|_{\lambda=0} = \int \frac{\delta P}{\delta \theta_\sigma} \frac{\delta Q}{\delta u^\sigma} + (-1)^p \frac{\delta P}{\delta u^\sigma} \frac{\delta Q}{\delta \theta_\sigma}.$$

Another way to prove it is to combine the uniqueness statement of [90, Theorem 2.4.1] with Proposition 3.1.7. $\square$

Remark 3.2.5. The same result extends to the setting of continuous Poisson vertex algebras (in the sense of De Sole–Kac–Valeri) by working in the inverse-limit differential algebra and using continuous λ -brackets; see [46]. See also [90] for related infinite-dimensional Hamiltonian/Jacobi structures. In addition, such Poisson–vertex–algebraic/bihamiltonian structures arise naturally in Gromov–Witten and cohomological field theories; compare, for example, [51, 23, 89].

THEOREM 3.2.6 (Correspondence Theorem). *Let X be a smooth, reduced, finitely generated $\mathbb{C}$ -scheme, and set $Y := T^*[1]X$. Then a degree-2 element $P \in \Gamma(Y_\infty, \mathcal{O}_{Y_\infty})_2$ defines a Poisson vertex algebra structure on X_∞ (i.e., it makes $\mathcal{O}_{X_\infty}$ a sheaf of PVAs) if and only if $[\int P, \int P] = 0$, where $[\cdot, \cdot]$ denotes the Lie bracket on local functionals induced by the chiral Gerstenhaber bracket; in this case, the λ -bracket corresponding to P is skew-symmetric.*

Conversely, any PVA λ -bracket on $\mathcal{O}_{X_\infty}$ determines a unique degree-2 local functional $\int P$ such that $[\int P, \int P] = 0$; equivalently, it determines P uniquely modulo total derivatives.

PROOF. Fix an affine open neighbourhood $U \subset X$ equipped with étale coordinates $u = (u^1, \dots, u^n) : U \rightarrow \mathbb{A}^n$ as in §4.1, and work with $\mathcal{A}_U$ and $\hat{\mathcal{A}}_U$ as above. For convenience, let $\mathcal{A} := \mathcal{A}_U$, $\hat{\mathcal{A}} := \hat{\mathcal{A}}_U$. Let $\hat{\mathcal{F}}_{cl} := \hat{\mathcal{A}}/\partial\hat{\mathcal{A}}$.

Let $P \in \Gamma(Y_\infty, \mathcal{O}_{Y_\infty})_2$. By the normalization in Section , we may represent the class $\int P \in \hat{\mathcal{F}}_{cl}$ in the form

$$\int P = \frac{1}{2} \int \sum_{\alpha, \beta} \theta_\alpha H^{\alpha\beta}(\partial) \theta_\beta$$

with $H(\partial)$ skew-adjoint. Define on generators

$$\{u^\alpha_\lambda u^\beta\} := H^{\beta\alpha}(\lambda),$$

and extend this λ -bracket to $\mathcal{A}$ by the Master Formula. Skew-adjointness of H implies skew-symmetry. By Theorem 3.2.4, the bracket $[\cdot, \cdot]$ on local functionals coincides with the variational Schouten bracket (cf. [90, Remark 2.4.6]). Under the normalization in Section , $\int P = \frac{1}{2} \int \theta H(\partial) \theta$, and together with [90, Remark 2.4.6.] $[\int P, \int P] = 0$ is equivalent to the corresponding Hamiltonian operator's Schouten bracket vanishes. On the other hand, by [12, Thm. 1.15(a)] the assignment $\{u^\alpha_\lambda u^\beta\} = H^{\beta\alpha}(\lambda)$ extends by the Master Formula, and by [12, Prop. 1.16(b), Def. 1.17] it defines a PVA structure if and only if H is Hamiltonian; finally, by [12, Def. 4.22, Cor. 4.23] this is equivalent to the Schouten bracket of H vanishes $[H, H]_{SB} = 0$. Conversely, any PVA λ -bracket on $\mathcal{O}_{X_\infty}(U)$ is determined by $\{u^\alpha_\lambda u^\beta\}$. By [12, Thm. 1.15(a), (1.47)], this is equivalent to a matrix $H(\lambda)$ with entries $H^{\beta\alpha}(\lambda) := \{u^\alpha_\lambda u^\beta\}$. Write $H^{\beta\alpha}(\lambda) = \sum_{s \geq 0} H_s^{\beta\alpha} \lambda^s$ and set $H^{\beta\alpha}(\partial) = \sum_{s \geq 0} H_s^{\beta\alpha} \partial^s$. As described in Section , this determines the normalized density $\tilde{P}_{\text{norm}}$ in (34), hence determines a unique class $\int P = \frac{1}{2} \int \theta H \theta$ modulo total derivatives.

Since the PVA Jacobi is equivalent to the vanishing of the corresponding Schouten bracket, the same argument as in the first part of the proof implies that $[\int P, \int P] = 0$. $\square$

Proposition 3.2.7. Let $P \in \hat{\mathcal{F}}_{cl}^0$ be in canonical form

$$P = \frac{1}{2} \int \sum_s \theta_\alpha \mathcal{P}_s^{\alpha\beta} \theta_\beta^s.$$

For $F, G \in \hat{\mathcal{F}}_{cl}^1$ define the derived bracket by

$$\{F, G\}_P := -\{\{P, F\}, G\}.$$

Then

$$\{F, G\}_P = \int \frac{\delta G}{\delta u^\beta} \mathcal{P}^{\beta\alpha}(\partial) \frac{\delta F}{\delta u^\alpha}.$$

Moreover, the corresponding PVA structure is given by the λ -bracket $\mathcal{P}^{\alpha\beta}(\lambda)$.

PROOF. For any $H \in \hat{\mathcal{F}}_{cl}^0$ and $F \in \hat{\mathcal{F}}_{cl}^1$ we have

$$\{H, F\} = \int X^\alpha \theta_\alpha \in \hat{\mathcal{F}}_{cl}^1.$$

By the definition of the normalising operator,

$$X^\alpha = -\mathcal{P}^{\alpha\beta}(\partial) \frac{\delta F}{\delta u^\beta}.$$

Hence

$$\begin{aligned} \{F, G\}_P &= -\{\{P, F\}, G\} = -\left\{\int X^\alpha \theta_\alpha, G\right\} \\ &= -\int \frac{\delta G}{\delta u^\alpha} X^\alpha = -\int \frac{\delta G}{\delta u^\alpha} \left(-\mathcal{P}^{\alpha\beta}(\partial) \frac{\delta F}{\delta u^\beta}\right) \\ &= \int \frac{\delta G}{\delta u^\alpha} \mathcal{P}^{\alpha\beta}(\partial) \frac{\delta F}{\delta u^\beta}. \end{aligned}$$

Since $\mathcal{P}$ is skew-adjoint, $\mathcal{P}^{\alpha\beta}(\partial) = -(\mathcal{P}^{\beta\alpha}(\partial))^*$, integration by parts gives

$$\int \frac{\delta G}{\delta u^\alpha} \mathcal{P}^{\alpha\beta}(\partial) \frac{\delta F}{\delta u^\beta} = \int \frac{\delta G}{\delta u^\beta} \mathcal{P}^{\beta\alpha}(\partial) \frac{\delta F}{\delta u^\alpha},$$

as claimed. $\square$

We now illustrate how the above shifted symplectic formalism reproduces some standard Poisson vertex algebras.

Example 3.2.8 (Heisenberg PVA). Consider the Heisenberg Poisson vertex algebra. Geometrically, we take $X = \mathbb{A}^1$ with coordinate u and its arc space X_∞ , so that

$$\mathcal{A} := \mathbb{C}[J_\infty X] = \mathbb{C}[u^{(n)} \mid n \geq 0],$$

and the PVA structure is given by

$$\{u_\lambda u\} = \lambda.$$

In the shifted cotangent picture, we consider the odd cotangent bundle $Y = T^*[1]X$ with odd fiber coordinate θ , and its arc space Y_∞ with jet coordinates $\theta^{(n)}$. The above λ -bracket corresponds to the section

$$P = \theta \theta^{(1)} \in \Gamma(Y_\infty, \mathcal{O}_{Y_\infty})_2,$$

viewed as a degree 2 function on the odd symplectic supermanifold Y_∞ . A direct computation gives

$$[P_\lambda P]|_{\lambda=0} = 0,$$

so P defines a Hamiltonian operator on X_∞ . Equivalently, the associated Hamiltonian operator is the constant operator $\mathcal{P}(\partial) = \partial$, and the derived bracket recovers the Heisenberg PVA above.

Example 3.2.9 (Virasoro-Magri PVA). Let $X = \mathbb{A}^1$ with coordinate u , and consider its arc space X_∞ and odd cotangent bundle $Y = T^*[1]X$ as above. On $\mathcal{A} = \mathbb{C}[J_\infty X]$ we consider the Virasoro-Magri PVA structure

$$\{u_\lambda u\} = u' + 2u \lambda + c \lambda^3,$$

where $c \in \mathbb{C}$ is the central charge. The corresponding Hamiltonian operator is

$$\mathcal{P}(\partial) = u' + 2u \partial + c \partial^3.$$

Using Theorem 3.2.6, it corresponds to a degree-2 section

$$P = 2u \theta \theta^{(1)} + c \theta \theta^{(3)}$$

in $\Gamma(Y_\infty, \mathcal{O}_{Y_\infty})$. Using the definition of the λ -bracket, one computes

$$[P_\lambda P]|_{\lambda=0} = \partial(\theta \theta^{(1)} \theta^{(2)}) \in \partial \hat{\mathcal{A}}.$$

Therefore, P defines a Hamiltonian operator on X_∞ , and the associated derived bracket induces the Virasoro-Magri PVA structure above.

3. Applications of shifted symplectic structures in the theory of PVA

The main goal of this section is to present several applications of Theorem 3.2.6. In Section 3.1 we give several examples of sheaves of Poisson vertex algebras on different algebraic schemes; in particular, we obtain a classification of certain PVA sheaves on some algebraic curves. In Section 3.2 we give a Maurer-Cartan description of the classical R -matrix of Lie conformal algebras, as introduced by the author in [55].

3.1. Sheaves of PVAs. In this section we apply Theorem 3.2.6 to classify global scalar PVA structures on the arc space X_∞ for $X = \mathbb{CP}^1$.

We restrict our attention to λ -brackets of order at most 3, since this class contains the Dubrovin-Novikov hydrodynamic-type Hamiltonian operators [49] and it is the natural conformal (CFT-type) case in the sense of De Sole-Kac (see, e.g., [38]).

Example 3.3.1. Let $X = \mathbb{CP}^1$ with the standard affine cover U_i ($i = 1, 2$). We use a local coordinate u on U_1 and v on U_2 , related on $U_1 \cap U_2$ by

$$v = \frac{1}{u}.$$

We restrict to scalar PVA structures corresponding to differential operators of order at most 3. The skew-symmetry of the λ -bracket implies that on U_1 any such bracket must be of the form

$$(35) \quad \{u_\lambda u\} = \frac{1}{2}(f_1)' + f_1 \lambda + \frac{c}{12}\lambda^3, \quad f_1 \in \mathcal{A}_{U_1}, \quad c \in \mathbb{C}.$$

The coordinate change $u \mapsto v = \frac{1}{u}$ induces a transformation on the fiber coordinates; with the convention $\theta_u = \partial/\partial u$, $\theta_v = \partial/\partial v$, we have on $U_1 \cap U_2$

$$\theta_u = -\frac{1}{u^2} \theta_v.$$

The λ -bracket (35) corresponds to a Hamiltonian function

$$P = f_1 \theta \theta^{(1)} + \frac{c}{12} \theta \theta^{(3)} \in \Gamma(T^*[1]U_1, \pi_* \mathcal{O}_{Y_\infty})_2.$$

Requiring P to glue across $U_1 \cap U_2$ forces f_1 to be a linear combination of the following differential monomials:

$$(36) \quad 1, u, u^2, u^{(1)}, uu^{(1)}, u^3, u^{(2)}, uu^{(2)}, u^2u^{(1)}, u^4, (u^{(1)})^2, u^{(3)}.$$

Hence we obtain the following classification.

(a) $c \neq 0$. In this case, up to adding total derivatives and rescaling u , the only global scalar PVA structure on X_∞ of the form (35) is, in normal form,

$$\{u_\lambda u\} = u' + 2u \lambda + \frac{c}{12}\lambda^3,$$

i.e. the Virasoro-Magri λ -bracket with central charge c .

(b) $c = 0$. In this case, all global scalar PVA structures on X_∞ of the form (35) are given by

$$\{u_\lambda u\} = q'(u) u' + 2q(u) \lambda,$$

where, on an affine chart, $q(u)$ is a polynomial of degree at most 4, equivalently

$$q \in H^0(X, \mathcal{O}_X(4)).$$

3.2. Classical R-matrix. Recall in the paper [55], the first author introduced the classical R -matrix of Lie conformal algebra. Theorem 3.2.6 implies the following corollary.

THEOREM 3.3.2. *Let $(L, \partial, [\cdot, \cdot])$ be a Lie conformal algebra such that L is a free $\mathbb{C}[\partial]$ -module of finite rank ℓ , and fix a $\mathbb{C}[\partial]$ -basis $\{u^\alpha\}_{\alpha=1}^\ell$ of L . Let $\mathcal{V} := S^\bullet(L)$ be the free commutative differential algebra generated by L , endowed with the Poisson vertex algebra structure obtained by extending the λ -bracket via the Master Formula. Assume $\mathcal{V} \cong \mathcal{O}(X_\infty)$ for the arc space X_∞ of some smooth reduced scheme X of finite type over*

$\mathbb{C}$, and set $Y := T^*[1]X$. Suppose $R \in \text{End}_{\mathbb{C}}(L)$ satisfies $[R, \partial] = 0$. Then the (modified) classical Yang–Baxter equation for R is equivalent to the statement that

$$P_R = \theta_\alpha \{ u^\alpha \partial u^\beta \}_{R, \rightarrow} \theta_\beta \in \hat{\mathcal{A}}^2$$

satisfies $\{\int P_R, \int P_R\} = 0$. (Here $\rightarrow$ means: replace λ by ∂ and move all ∂ 's to the right.) Equivalently, set $\Theta_R := \int(P_R - P)$. Then Θ_R is a Maurer–Cartan element of the dg Lie algebra $(\hat{\mathcal{F}}_{cl}, [-, -], d_P := [P, -])$, i.e. $d_P \Theta_R + \frac{1}{2}[\Theta_R, \Theta_R] = 0$.

Example 3.3.3. In the affine Kac–Moody Lie conformal algebra case, let $\mathcal{V} := \mathcal{V}_k(\mathfrak{g})$. We identify $\text{Spec } \mathcal{V}$ with $\mathfrak{g}_{k, \infty}^*$, where k indicates the level of the λ -bracket. In particular, [55, Corollary 5.6] shows that if $(\mathfrak{g}, (\cdot, \cdot))$ is a quadratic Lie algebra and $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a Manin triple with respect to $(\cdot, \cdot)$, then the projection maps give a classical R -matrix for $v_k(\mathfrak{g})$. This can be viewed as a chiral counterpart of the classical fact that a Manin triple gives a projection-type classical R -matrix.

Motivated by the above discussion, we now define the notion of a classical R -matrix for a classical finite $\mathcal{W}$ -algebra.

Definition 3.3.4. Let $\mathcal{W}_{\text{fin}}^{\text{cl}}(\mathfrak{g}, f_{\text{sub}})$ be the subregular classical finite $\mathcal{W}$ -algebra. As a Poisson variety we have

$$\text{Spec } \mathcal{W}_{\text{fin}}^{\text{cl}}(\mathfrak{g}, f_{\text{sub}}) \cong \mathbb{C}[\mathcal{S}_f],$$

equipped with the Poisson bivector π_f . Set $X := \mathcal{S}_f$ and $Y := T^*[1]X$. We say that an element $\Pi_R \in \Gamma(Y, \mathcal{O}_Y)_2$ is a *classical R -matrix* of $\mathcal{W}_{\text{fin}}^{\text{cl}}(\mathfrak{g}, f_{\text{sub}})$ if and only if $d_{\pi_f} \Theta_R + \frac{1}{2}[\Theta_R, \Theta_R] = 0$, where $\Theta_R := \pi_f - \Pi_R$ and $[\cdot, \cdot]$ is the Schouten–Nijenhuis bracket on polyvector fields.

Example 3.3.5. Let $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathcal{W}_{\text{fin}}^{\text{cl}}(\mathfrak{sl}_3, f_{\text{sub}})$ be the subregular finite classical $\mathcal{W}$ -algebra with ordered basis u_1, u_2, u_3 and Poisson structure

$$H = \begin{bmatrix} 0 & -2u_1 & u_2^2 \\ 2u_1 & 0 & -2u_3 \\ -u_2^2 & 2u_3 & 0 \end{bmatrix}.$$

Let $X = \mathcal{S}_f$ be the corresponding Slodowy slice and $Y = T^*[1]X$. Set $V = \text{Span}_{\mathbb{C}}\{u_1, u_2, u_3\}$ and take an endomorphism $R \in \text{End}_{\mathbb{C}}(V)$, $R(u_i) = \sum_{j=1}^3 R_{ij} u_j$, $R_{ij} \in \mathbb{C}$. It induces a degree-2 function

$$\Pi_R = \sum_{i,j=1}^3 (\{R(u_i), u_j\} + \{u_i, R(u_j)\}) \theta_i \theta_j \in \Gamma(Y, \mathcal{O}_Y)_2,$$

where θ_i are the odd coordinates corresponding to u_i . The Jacobi identity for Π_R is equivalent to the polynomial relations

$$\begin{aligned} f_1 &:= R_{11}R_{21} + R_{21}R_{22} + R_{23}R_{31} = 0, \\ f_2 &:= R_{12}R_{31} + R_{22}R_{32} + R_{32}R_{33} = 0, \\ f_3 &:= R_{11}^2 + R_{12}R_{21} - R_{23}R_{32} - R_{33}^2 = 0, \\ f_4 &:= R_{13}R_{21} + R_{22}R_{23} + R_{23}R_{33} = 0, \\ f_5 &:= R_{11}R_{12} + R_{12}R_{22} + R_{13}R_{32} = 0. \end{aligned}$$

We call this the *constant R -deformation* of $\mathcal{W}_{\text{fin}}^{\text{cl}}(\mathfrak{sl}_3, f_{\text{sub}})$. In algebraic-geometric terms, put $\mathbb{C}[R] = \mathbb{C}[R_{ij} \mid 1 \leq i, j \leq 3]$ so that $\text{Spec } \mathbb{C}[R] \simeq \mathbb{A}^9$ and let $I = \langle f_1, \dots, f_5 \rangle \subset \mathbb{C}[R]$. Then $\mathfrak{R} := V(I) \subset \mathbb{A}^9$ parametrizes all constant 3×3 matrices $R = (R_{ij})$ such that Π_R is Poisson; equivalently, R is a classical R -matrix for $\mathcal{W}_{\text{fin}}^{\text{cl}}(\mathfrak{sl}_3, f_{\text{sub}})$ if and only if $(R_{ij}) \in \mathfrak{R}$. More general (non-constant) R -deformations are governed by the Maurer-Cartan equation of Definition 3.3.4; the variety $\mathfrak{R}$ describes the locus of constant R -operators in this moduli problem.

Like the Belavin-Drinfel'd classification of constant solutions of the classical Yang-Baxter equation [18], we single out three natural families of constant endomorphisms R .

Type I (Cartan type). Assume R is diagonal. Then R defines a constant R -deformation if and only if $R = \text{diag}(a, b, \pm a)$, $a, b \in \mathbb{C}$.

Type II (upper triangular type). Assume R is upper triangular, i.e. $R_{21} = R_{31} = R_{32} = 0$. Then the Jacobi identity for Π_R is equivalent to

$$R_{11}^2 = R_{33}^2, \quad R_{12}(R_{11} + R_{22}) = 0, \quad R_{23}(R_{22} + R_{33}) = 0.$$

Type III (mixed type). Assume $R_{12} = R_{21} = R_{23} = R_{32} = 0$. Then R defines a constant R -deformation if and only if $R_{11} = \pm R_{33}$.

We observe that the Adler-Kostant-Symes scheme remains valid under constant R -deformations.

Let $Z_{\text{Poi}}(\mathcal{W}_{\text{fin}}^{\text{cl}}(\mathfrak{g}, f_{\text{sub}}))$ denote the Poisson center of $\mathcal{W}_{\text{fin}}^{\text{cl}}(\mathfrak{g}, f_{\text{sub}})$, and write $\mathcal{W}_{\text{fin}}^{\text{cl}}(\mathfrak{g}, f_{\text{sub}})_R$ for the R -deformed classical finite $\mathcal{W}$ -algebra. It is known that $Z_{\text{Poi}}(\mathcal{W}_{\text{cl}}^{\text{fin}}(\mathfrak{g}, f)) \cong \mathbb{C}[\mathfrak{g}^*]^G$ via the restriction map $\mathbb{C}[\mathfrak{g}^*]^G \rightarrow \mathbb{C}[S_f] \cong \mathcal{W}_{\text{cl}}^{\text{fin}}(\mathfrak{g}, f)$ (hence $Z_{\text{Poi}}(\mathcal{W}_{\text{cl}}^{\text{fin}}(\mathfrak{g}, f)) \cong \mathbb{C}[\mathfrak{g}]^G$ after identifying $\mathfrak{g} \simeq \mathfrak{g}^*$ by $(\cdot|\cdot)$). Then for any $f, g \in Z_{\text{Poi}}(\mathcal{W}_{\text{fin}}^{\text{cl}}(\mathfrak{g}, f_{\text{sub}}))$ we have $\{f, g\}_R = 0$, where $\{\cdot, \cdot\}_R$ denotes the R -deformed Poisson bracket. Consequently, the elements of $Z_{\text{Poi}}(\mathcal{W}_{\text{fin}}^{\text{cl}}(\mathfrak{g}, f_{\text{sub}}))$ provide a family of mutually commuting Hamiltonians, and by restricting to suitable symplectic leaves we obtain integrable systems. The proof is entirely parallel to that of [55, Theorem 6.3]. We now present a three-dimensional integrable system obtained from the mixed-type constant R -matrix of Type III. In this case the center is given by

$$S = \frac{1}{6}u_2^3 + u_1u_3.$$

The corresponding Type III deformation induces the system

$$(37) \quad \begin{aligned} \dot{u}_1 &= u_2^2(\Delta u_1 + R_{13}u_3), \\ \dot{u}_2 &= 2(R_{31}u_1^2 - R_{13}u_3^2), \\ \dot{u}_3 &= -u_2^2(R_{31}u_1 + \Delta u_3), \end{aligned}$$

where $\Delta := R_{33} - R_{22}$.

The Type III R -matrix also produces a compatible Poisson pencil. More precisely, let H denote the original Poisson structure on $\mathcal{S}_f$ and let H^{mix} be the Poisson structure obtained from the mixed-type R -matrix. Then, for any $\epsilon \in \mathbb{C}$, the combination $H + \epsilon H^{\text{mix}}$ is again a Poisson structure on $\mathcal{S}_f$.

We now choose the parameters R_{ij} so as to obtain an elliptic-type ODE. For convenience we impose $\Delta = 0$, assume $R_{13} \neq 0$ and $R_{31} \neq 0$, and rescale the time variable so that $R_{13} = R_{31} = 1$. Then (37) reduces to

$$\dot{u}_1 = u_3u_2^2, \quad \dot{u}_2 = 2(u_1^2 - u_3^2), \quad \dot{u}_3 = -u_1u_2^2.$$

This system admits an additional integral of motion $J = u_1^2 + u_3^2$. In particular, it is a three-dimensional Hamiltonian system with two independent integrals and hyperelliptic-type solutions. To write it as a hyperelliptic ODE, we use polar coordinates in the (u_1, u_3) -plane: let $u_1 = r \cos \theta$, $u_3 = r \sin \theta$ so that $J = r^2$. Then one finds $\dot{u}_2^2 = P(u_2)$, where

$$P(u_2) = 4r^4 - 16\left(S - \frac{1}{6}u_2^3\right)^2 = \frac{4}{9}(u_2^3 - 3(2S - r^2))(3(2S + r^2) - u_2^3).$$

Thus the reduced equation defines a genus-2 hyperelliptic curve, and the solutions $u_2(t)$ are expressed in terms of hyperelliptic (genus 2) theta functions.

We summarize the above example as follows.

Proposition 3.3.6. With notation as above, the following three families of constant endomorphisms $R \in \text{End}_{\mathbb{C}}(V)$ define constant R -deformations of $\mathcal{W}_{\text{fin}}^{\text{cl}}(\mathfrak{sl}_3, f_{\text{sub}})$.

Type I (Cartan type). Assume R is diagonal. Then R defines a constant R -deformation if and only if

$$R = \text{diag}(a, b, \pm a), \quad a, b \in \mathbb{C}.$$

Type II (upper triangular type). Assume R is upper triangular, i.e. $R_{21} = R_{31} = R_{32} = 0$. Then the Jacobi identity for Π_R is equivalent to

$$R_{11}^2 = R_{33}^2, \quad R_{12}(R_{11} + R_{22}) = 0, \quad R_{23}(R_{22} + R_{33}) = 0.$$

Type III (mixed type). Assume $R_{12} = R_{21} = R_{23} = R_{32} = 0$. Then R defines a constant R -deformation if and only if $R_{11} = \pm R_{33}$.

In particular, any Type III R -matrix produces a compatible Poisson pencil.

THEOREM 3.3.7. *In the setting of the subregular finite classical $\mathcal{W}$ -algebra $\mathcal{W}_{\text{fin}}^{\text{cl}}(\mathfrak{sl}_3, f_{\text{sub}})$ with Poisson structure H as above, consider a constant R -matrix of Type III and impose*

$\Delta = R_{33} - R_{22} = 0$. Assume $R_{13} \neq 0$ and $R_{31} \neq 0$, and rescale the time variable so that $R_{13} = R_{31} = 1$. Then the corresponding R -deformation of the Hamiltonian flow generated by

$$S = \frac{1}{6}u_2^3 + u_1u_3$$

is given by the three-dimensional system

$$(38) \quad \dot{u}_1 = u_3u_2^2, \quad \dot{u}_2 = 2(u_1^2 - u_3^2), \quad \dot{u}_3 = -u_1u_2^2.$$

This system has another integral of motion, $J = u_1^2 + u_3^2$. Moreover, the component $u_2(t)$ can be expressed in terms of hyperelliptic functions.

Using the ideas explored above, we obtain a natural chiralization of classical R -matrices for (affine) classical $\mathcal{W}$ -algebras.

Definition 3.3.8. Let $X = \mathcal{S}_f$ and $Y := T^*[1]\mathcal{S}_f$. We fix an identification of Poisson vertex algebras $\mathcal{W}^{\text{cl}}(\mathfrak{g}, f) \cong \mathcal{O}(X_\infty)$ so that

$$\text{Spec } \mathcal{W}^{\text{cl}}(\mathfrak{g}, f) \cong X_\infty.$$

By Theorem 3.2.6, the corresponding PVA structure on X_∞ is encoded by a degree-2 element

$$P \in \Gamma(Y, \pi_*\mathcal{O}_{Y_\infty})_2$$

satisfying $[P, P] = 0$, where $[-, -]$ is the Schouten bracket from Theorem 3.2.6. We say that an element

$$\Pi_R \in \Gamma(Y, \pi_*\mathcal{O}_{Y_\infty})_2$$

is a *classical R -matrix* of $\mathcal{W}^{\text{cl}}(\mathfrak{g}, f)$ if and only if it satisfies the Maurer-Cartan equation

$$d_P\Theta_R + \frac{1}{2}[\Theta_R, \Theta_R] = 0, \quad d_P := [P, -],$$

where P is the Hamiltonian structure of the classical affine $\mathcal{W}$ -algebra and $\Theta_R := P - \Pi_R$.

In this formalism, a classical R -matrix for $\mathcal{W}^{\text{cl}}(\mathfrak{g}, f)$ may be viewed as a QP -chiralization of the classical R -matrix on the finite classical $\mathcal{W}$ -algebra $\mathcal{W}^{\text{fin}}(\mathfrak{g}, f)$ defined in Definition 3.3.4.

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