

Birational geometry of moduli spaces of parabolic connections

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2021/10/27 @城崎代数幾何学シンポジウム

Based on the preprint

T.M., Birational geometry of moduli spaces of rank 2 logarithmic connections, arXiv: 2105.06892

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Introduction (1)

- An **regular singular parabolic connection** (E, ∇, l_*) over a smooth complex projective curve C with marked points $\mathbf{t} = (t_1, \dots, t_n)$ is the pair of a parabolic bundle (E, l_*) and a logarithmic connection ∇ compatible with the parabolic structure l_* .

- moduli space of parabolic connections**

$$\mathcal{M}_{(C, \mathbf{t})}^{\alpha}(\nu, r, d) := \left\{ (E, \nabla, l_*) \left| \begin{array}{l} \alpha\text{-stable } \nu\text{-parabolic connections of rank } r \text{ and degree } d \\ \text{over } (C, \mathbf{t}). \end{array} \right. \right\} / \sim$$

- constructed by Arinkin-Lysenko and Inaba-Iwasaki-Saito.
- motivated by the search for the differential equation determined by the isomonodromic deformation (e.g. Painlevé equation) and the geometric Langlands correspondence.

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Introduction (2)

Theorem (Loray-Saito, 2015)

In the case $C = \mathbb{P}^1$, assume n, d, ν, α satisfy appropriate conditions. Then the rational map

$$\text{App} \times \text{Bun} : \mathcal{M}_{(C, \mathbf{t})}^{\alpha}(\nu, 2, d) \cdots \rightarrow |\mathcal{O}_C(n-3)| \times \mathcal{P}_{(C, \mathbf{t})}^{\alpha}(2, d)$$

is birational. In particular, $\mathcal{M}^{\alpha}(\nu, 2, d)$ is a **rational variety**.

Problem Is $\text{App} \times \text{Bun}$ birational for any rank and any genus?

In the case of fixing the determinant and the trace connection;

- $r = 2, g = 0$: Loray-Saito
- $r = 2, g = 1$: Fassarella-Loray-Muniz (arXiv: 2008.11767)
- $r = 2, g \geq 2$: M ← Today's talk
- $r \geq 3$: in progress

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Moduli of parabolic connections with fixed determinant

- determinant map

$$\det : \mathcal{M}_{(C, \mathbf{t})}^{\alpha}(\nu, r, d) \longrightarrow \mathcal{M}_{(C, \mathbf{t})}(\text{tr } \nu, 1, d)$$

$$(E, \nabla, l_*) \longmapsto (\det E, \text{tr } \nabla)$$

- moduli space of parabolic connections with fixed determinant**

$$\mathcal{M}_{(C, \mathbf{t})}^{\alpha}(\nu, r, (L, \nabla_L))$$

$$:= \{(E, \nabla, l_*) \in \mathcal{M}_{(C, \mathbf{t})}^{\alpha}(\nu, r, d) \mid (\det E, \text{tr } \nabla) \simeq (L, \nabla_L)\}$$

- (Inaba, 2013)

When

$$g = 0, r \geq 2, rn - 2(r+1) > 0 \text{ or } g = 1, n \geq 2 \text{ or } g \geq 2, \geq 1,$$

$\mathcal{M}_{(C, \mathbf{t})}^{\alpha}(\nu, r, (L, \nabla_L))$ is an **irreducible smooth quasi-projective variety** of dimension

$$2r^2(g-1) + rn(n-1) + 2 - 2g.$$

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Apparent map

- An **apparent singularity** is a singular point of a differential equation with rational coefficients such that there is a basis of regular solutions in a neighborhood of the point.

a differential equation $\longrightarrow$ the set of apparent singularities

a connection $\longrightarrow$ the apparent singularity divisor

- (Saito-Szabó)
Suppose $d = r(g-1) + 1$. Then we define the **apparent map**

$$\text{App} : \mathcal{M}_{(C,t)}^\alpha(\nu, r, d) \cdots \rightarrow S^N(C).$$

- fixed determinant case

$$\text{App} : \mathcal{M}_{(C,t)}^\alpha(\nu, 2, (L, \nabla_L)) \cdots \rightarrow |L \otimes \Omega_C^1(D)|.$$

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Birational structure of moduli spaces (1)

- $\mathcal{P}^\alpha(L)$: moduli space of rank 2 α -semistable parabolic bundles with determinant L over (C, t)
- $\mathcal{M}^\alpha(\nu, (L, \nabla_L)) = \mathcal{M}_{(C,t)}^\alpha(\nu, 2, (L, \nabla_L))$
- Bundle map**

$$\begin{aligned} \text{Bun} : \mathcal{M}^\alpha(\nu, (L, \nabla_L)) \cdots &\rightarrow \mathcal{P}^\alpha(L) \\ (E, \nabla, l_*) &\longmapsto (E, l_*) \end{aligned}$$

Theorem (M)

Assume $g \geq 1$, $d = 2g - 1$, $\sum_{i=1}^n \nu_0^{(i)} \neq 0$ and $\sum_{i=1}^n (\alpha_2^{(i)} - \alpha_1^{(i)}) < 1$. Then the rational map

$$\text{App} \times \text{Bun} : \mathcal{M}^\alpha(\nu, (L, \nabla_L)) \cdots \rightarrow |L \otimes \Omega_C^1(D)| \times \mathcal{P}^\alpha(L)$$

is birational. In particular, $\mathcal{M}^\alpha(\nu, (L, \nabla_L))$ is a **rational variety**.

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Birational structure of moduli spaces (2)

- $V_0 \subset \mathcal{P}^\alpha(L)$: the distinguished open subset

- There is an open immersion

$$V_0 \hookrightarrow \mathbb{P} \text{Ext}^1((L, D), (\mathcal{O}_C, \emptyset)) \simeq |L \otimes \Omega_C^1(D)|^*.$$

- Σ : incidence variety

$$\Sigma \subset |L \otimes \Omega_C^1(D)| \times |L \otimes \Omega_C^1(D)|^*$$

- $\mathcal{M}^0 \subset \mathcal{M}^\alpha(\nu, (L, \nabla_L))$: a nonempty open subset

$$\mathcal{M}^0 := \{(E, \nabla, l_*) \in \mathcal{M}^\alpha(\nu, (L, \nabla_L)) \mid (E, l_*) \in V_0\}$$

The map

$$\text{App} \times \text{Bun} : \mathcal{M}^0 \longrightarrow (|L \otimes \Omega_C^1(D)| \times V_0) \setminus \Sigma$$

is an **isomorphism**.

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Birational structure of moduli spaces (3)

Proposition

Assume $g \geq 1$, $d = 2g - 1$, $\sum_{i=1}^n \nu_0^{(i)} = 0$ and $\sum_{i=1}^n (\alpha_2^{(i)} - \alpha_1^{(i)}) < 1$. Then $\mathcal{M}^\alpha(\nu, (L, \nabla_L))$ is birational to $T^*\mathcal{P}^\alpha(L)$. In particular, $\mathcal{M}^\alpha(\nu, (L, \nabla_L))$ is a **rational variety**.

$$T_{(E,l_*)}^* \mathcal{P}^\alpha(L) \simeq \left\{ \begin{array}{l} \text{parabolic Higgs fields over} \\ (E, l_*) \text{ such that } \text{tr} = 0 \end{array} \right\}$$

$$\text{Bun}^{-1}((E, l_*)) = \nabla_0 + \left\{ \begin{array}{l} \text{parabolic Higgs fields over} \\ (E, l_*) \text{ such that } \text{tr} = 0 \end{array} \right\}$$

$$\begin{array}{ccc} \mathcal{M}^0 & \xrightarrow{\sim} & T^*V_0 \\ \text{Bun} \searrow & & \swarrow \text{projection} \\ & V_0 & \end{array} \quad \begin{array}{ccc} \mathcal{M}^0 & \xrightarrow{\text{App} \times \text{Bun}} & \Sigma \\ \wr \searrow & & \swarrow \wr \\ T^*V_0 & \xrightarrow{\text{projectivization}} & \mathbb{P}T^*V_0 \end{array}$$

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