

On the Generalized Sampling Theorem

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Summary

Being based on the generalized sampling theorem along the line of consideration of the reciprocity of integral transforms, presented previously by the authors¹⁾, some of the series-expansions of functions are given as its special examples.

The generalization of the sampling theorem and the reconstruction of a band-limited function from its sampled values and derivatives were made by Kohlenberg²⁾, Fogel³⁾, Jagerman and Fogel⁴⁾, Bond and Cahn⁵⁾, and Linden and Abramson⁶⁾. The sampling theorem was also generalized by Balakrishnan⁷⁾ to the case of a continuous-parameter stochastic process. On the other hand, it was pointed out that the sampling intervals need not be uniformly distributed⁸⁾.

In the previous paper¹⁾, the authors presented a generalized sampling theorem along the line of consideration of the reciprocity of integral transforms and gave some of its examples, which include Someya-Shannon's sampling theorem^{9) 10) 11)} as a special case. There the sampling intervals were not uniformly distributed. Recently the authors were informed that the sampling theorem was also generalized by Isomichi¹²⁾ in connection with the generalized frequency domain. Some of his results agreed with those given by us¹⁾. Here in the present paper, the authors wish to discuss the series-expansion of functions in an orthogonal set of functions, and also to give some of the new sampling formulae.

Let $f(t)$ belong to L_p , and let the following reciprocity relations hold :

$$F(\lambda) = \mathcal{L}_\lambda \cdot f \equiv \int_A K(\lambda, t) f(t) dt, \quad \text{for } \lambda \in B \quad (1)$$

and

$$f(t) = \mathcal{L}_t^{-1} \cdot F \equiv \int_B \tilde{K}(t, \lambda) F(\lambda) d\lambda, \quad \text{for } t \in A \quad (2)$$

From (1) and (2), we obtain at once :

$$\int_A K(\lambda, t) \tilde{K}(t, \sigma) dt = \delta(\lambda - \sigma), \quad (3)$$

and

$$\int_B \tilde{K}(t, \lambda) K(\lambda, \tau) d\lambda = \delta(t - \tau), \quad (4)$$

with delta function $\delta(t)$.

Further we assume that

$$F(\lambda) = 0 \quad \text{for } \lambda \notin D \subseteq B \quad (5)$$

Accordingly, we obtain from (2) and (5)

$$f(t) = \mathcal{L}_t^{-1} \cdot F = \int_D \tilde{K}(t, \lambda) F(\lambda) d\lambda. \quad (6)$$

Let $F(\lambda)$ be expanded in a complete orthogonal set of functions :

$$\left\{ \phi_n(\lambda) ; \int_D \phi_m(\lambda) \phi_n(\lambda) d\lambda = \gamma_m \cdot \delta_{m,n} \quad (m, n = \text{integers}) \right\}, \quad (7)$$

in the domain $D \subseteq B$, i.e.

$$F(\lambda) = \sum_n a_n \phi_n(\lambda). \quad \text{for } \lambda \in D \subseteq B \quad (8)$$

The expression (8), being multiplied by $\phi_m(\lambda)$ and integrated over D , gives :

$$\int_D F(\lambda) \phi_m(\lambda) d\lambda = \sum_n a_n \cdot \int_D \phi_m(\lambda) \phi_n(\lambda) d\lambda = \sum_n a_n \cdot \gamma_m \cdot \delta_{m,n} = \gamma_m \cdot a_m. \quad (9)$$

From (6), (7), (8), and (9), we obtain :

$$f(t) = \mathcal{L}_t^{-1} \cdot F = \sum_n a_n \cdot \mathcal{L}_t^{-1} \cdot \phi_n$$

$$= \sum_n \left(\frac{1}{\gamma_n} \int_D F(\lambda) \phi_n(\lambda) d\lambda \right) \cdot \int_D \tilde{K}(t, \lambda) \phi_n(\lambda) d\lambda. \quad (10)$$

If the functions $f(t)$, $\{\phi_n(\lambda)\}$, and the integral kernels $K(\lambda, t)$ and $\tilde{K}(t, \lambda)$, are given, then we can construct a series-expansion of $f(t)$ by means of (10).

If we can take

$$\phi_n(\lambda) = \tilde{K}(\lambda_n, \lambda), \quad \text{for } \lambda \in D \quad (11)$$

with constants λ_n , i.e. if the kernel $\tilde{K}(\lambda_n, \lambda)$ can be put equal to $\phi_n(\lambda)$, then the expression (10) is simplified into:

$$\begin{aligned} f(t) &= \sum_n \left(\frac{1}{\gamma_n} \int_D F(\lambda) \tilde{K}(\lambda_n, \lambda) d\lambda \right) \cdot \int_D \tilde{K}(t, \lambda) \tilde{K}(\lambda_n, \lambda) d\lambda \\ &= \sum_n \frac{1}{\gamma_n} f(\lambda_n) \cdot \int_D \tilde{K}(t, \lambda) \tilde{K}(\lambda_n, \lambda) d\lambda, \end{aligned} \quad (12)$$

by means of (6). The expression (12) gives a generalized sampling theorem¹⁾. The points at the variable t :

$$t = \lambda_n, \quad (n = \text{integers}) \quad (13)$$

are called sampling points, and the function $g_n(t)$, defined by

$$g_n(t) \equiv \mathcal{L}_t^{-1} \cdot \phi_n = \int_D \tilde{K}(t, \lambda) \phi_n(\lambda) d\lambda = \int_D \tilde{K}(t, \lambda) \tilde{K}(\lambda_n, \lambda) d\lambda, \quad (n = \text{integers}) \quad (14)$$

is the sampling function.

If the set of functions $\{\tilde{K}(\lambda_n, \lambda); n = \text{integers}\}$ is not necessarily orthogonal, but complete and linearly independent, we can find normalized biorthogonal set $\{\psi_m(\lambda); m = \text{integers}\}$, in such a way that

$$\int_D \tilde{K}(\lambda_m, \lambda) \psi_n(\lambda) d\lambda = \delta_{m,n}, \quad (15)$$

is to be satisfied. From the completeness of the set $\{\tilde{K}(\lambda_m, \lambda); m = \text{integers}\}$,

we have:

$$\sum_n \tilde{K}(\lambda_n, \lambda) \psi_n(\sigma) = \delta(\lambda - \sigma). \quad (16)$$

Isomichi¹²⁾ expanded $F(\lambda)$ in $\{\psi_n(\lambda); n = \text{integers}\}$, and obtained :

$$F(\lambda) = \sum_n b_n \psi_n(\lambda), \quad \text{for } \lambda \in D \subseteq B \quad (17)$$

$$b_n = \int_D \tilde{K}(\lambda_n, \lambda) F(\lambda) d\lambda. \quad (18)$$

From (6) and (18), we obtain at once :

$$b_n = f(\lambda_n) = \int_D \tilde{K}(\lambda_n, \lambda) F(\lambda) d\lambda. \quad (19)$$

Operating $\mathcal{L}_t^{-1}$ on (17), we obtain, by means of (6) :

$$\begin{aligned} f(t) &= \mathcal{L}_t^{-1} \cdot F = \sum_n b_n \cdot \int_D \tilde{K}(t, \lambda) \psi_n(\lambda) d\lambda \\ &= \sum_n f(\lambda_n) \cdot \int_D \tilde{K}(t, \lambda) \psi_n(\lambda) d\lambda. \end{aligned} \quad (20)$$

The expression (18) or (19) becomes to :

$$\begin{aligned} f(\lambda_n) &= \int_D d\lambda \cdot \tilde{K}(\lambda_n, \lambda) \cdot \int_A dt \cdot K(\lambda, t) f(t) \\ &= \int_A dt \cdot f(t) \cdot \int_D d\lambda \cdot \tilde{K}(\lambda_n, \lambda) K(\lambda, t), \end{aligned} \quad (21)$$

by means of (1). The expression (20) is the generalized sampling theorem obtained by Isomichi¹²⁾, and the expression

$$g_n(t) = \int_D \tilde{K}(t, \lambda) \psi_n(\lambda) d\lambda, \quad (n = \text{integers}) \quad (22)$$

is the sampling function, with sampling points $t = \lambda_n$ ($n = \text{integers}$) and with

$$g_n(\lambda_m) = \int_D \tilde{K}(\lambda_m, \lambda) \psi_n(\lambda) d\lambda = \delta_{m,n}.$$

The idea of the generalized sampling theorem was also given by one of the present authors, Takizawa, in his book¹³⁾ of information theory.

If we take $\{\tilde{K}(\lambda_n, \lambda)\}$ to be orthogonal, then we have at once

$$\{\psi_n(\lambda)\} = \{\phi_n(\lambda)\} = \{\tilde{K}(\lambda_n, \lambda)\}, \quad (23)$$

as was given in (11), the expression (20) reduces to the sampling theorem (12), the dual vector $\{\psi_n(\lambda)\}$ for $\{\tilde{K}(\lambda_n, \lambda)\}$ reducing to $\{\tilde{K}(\lambda_n, \lambda)\}$ itself.

The corresponding expressions of $F(t)$ for (20) and (21) are as follows :

$$F(t) = \sum_n F(\mu_n) \cdot \int_E K(t, \lambda) \psi_n(\lambda) d\lambda, \quad (24)$$

and

$$F(\mu_n) = \int_B dt \cdot F(t) \cdot \int_E d\lambda \cdot K(\mu_n, \lambda) \tilde{K}(\lambda, t), \quad (25)$$

with the sampling function :

$$p_n(t) = \int_E K(t, \lambda) \tilde{\psi}_n(\lambda) d\lambda, \quad (26)$$

and the sampling points $t = \mu_n$ ($n = \text{integers}$), under the condition that for the complete set $\{K(\mu_n, t); n = \text{integers}\}$ we have a set of biorthogonal functions $\{\tilde{\psi}_n(\lambda); n = \text{integers}\}$ in the domain $E \subseteq A$, i.e.

$$\int_E K(\mu_m, t) \tilde{\psi}_n(t) dt = \delta_{m,n} \quad \text{for } t \in E \subseteq A \quad (27)$$

From the completeness of the set $\{K(\mu_m, \lambda); m = \text{integers}\}$, it follows that the expression :

$$\sum_n K(\mu_n, \lambda) \tilde{\psi}_n(\sigma) = \delta(\lambda - \sigma), \quad (28)$$

holds.

Now, let us take sampling functions $\{g_n(t); n = \text{integers}\}$, $\{h_n(t); n = \text{integers}\}$, $\{p_n(t); n = \text{integers}\}$, and $\{q_n(t); n = \text{integers}\}$, defined as follows :

$$g_n(t) \equiv \int_D \tilde{K}(t, \lambda) \psi_n(\lambda) d\lambda, \quad (29)$$

$$h_n(t) \equiv \int_D K(\lambda, t) \phi_n(\lambda) d\lambda = \int_D K(\lambda, t) \tilde{K}(\lambda_n, \lambda) d\lambda, \quad (30)$$

$$p_n(t) \equiv \int_E K(t, \lambda) \tilde{\psi}_n(\lambda) d\lambda, \quad (31)$$

and

$$g_n(t) \equiv \int_E \tilde{K}(\lambda, t) K(\mu_n, \lambda) d\lambda. \quad (32)$$

From (3) and (4), we have

$$\int_A K(\sigma, t) g_n(t) dt = \int_D \psi_n(\lambda) \cdot \delta(\sigma - \lambda) d\lambda = \begin{cases} \psi_n(\sigma), & \text{for } \sigma \in D \\ 0, & \text{for } \sigma \notin D \end{cases} \quad (33)$$

$$\int_A \tilde{K}(t, \sigma) h_n(t) dt = \int_D \phi_n(\lambda) \cdot \delta(\sigma - \lambda) d\lambda = \begin{cases} \tilde{K}(\lambda_n, \sigma), & \text{for } \sigma \in D \\ 0, & \text{for } \sigma \notin D \end{cases} \quad (34)$$

$$\int_B \tilde{K}(\sigma, t) p_n(t) dt = \int_E \tilde{\psi}_n(\lambda) \cdot \delta(\sigma - \lambda) d\lambda = \begin{cases} \tilde{\psi}_n(\sigma), & \text{for } \sigma \in E \\ 0, & \text{for } \sigma \notin E \end{cases} \quad (35)$$

and

$$\int_B K(t, \sigma) g_n(t) dt = \int_E K(\mu_n, \lambda) \cdot \delta(\sigma - \lambda) d\lambda = \begin{cases} K(\mu_n, \sigma), & \text{for } \sigma \in E \\ 0, & \text{for } \sigma \notin E \end{cases} \quad (36)$$

By means of (3) and (15), we obtain

$$\begin{aligned} \int_A g_m(t) h_n(t) dt &= \int_D d\lambda \cdot \int_D d\sigma \cdot \psi_m(\lambda) \phi_n(\sigma) \cdot \delta(\sigma - \lambda) = \\ &= \int_D \psi_m(\lambda) \phi_n(\lambda) d\lambda = \int_D \tilde{K}(\lambda_n, \lambda) \psi_m(\lambda) d\lambda = \delta_{m,n}, \end{aligned} \quad (37)$$

for any integers m and n . This expression (37) shows that any function of $\{g_n(t); n = \text{integers}\}$ and any function of $\{h_n(t); n = \text{integers}\}$ are mutually orthogonal in the domain D . Similarly, we obtain, from (4) and (27):

$$\int_B p_m(t) g_n(t) dt = \int_E \tilde{\psi}_m(\lambda) K(\mu_n, \lambda) d\lambda = \delta_{m,n}. \quad (38)$$

We shall return to the expressions (20), (21), (24), and (25).

They are written as follows:

$$f(t) = \sum_n f(\lambda_n) g_n(t), \quad (39)$$

$$f(\lambda_n) = \int_A f(t) h_n(t) dt, \quad (40)$$

$$F(t) = \sum_n F(\mu_n) p_n(t), \quad (41)$$

and

$$F(\mu_n) = \int_B F(t) g_n(t) dt, \quad (42)$$

in terms of (29)~(32).

Let $\xi(t)$ be any function which can be expanded in $\{g_n(t); n = \text{integers}\}$, i.e.

$$\xi(t) = \sum_n c_n g_n(t), \quad (43)$$

with constants c_n , then the integral operator :

$$\mathcal{J} \equiv \int_A d\tau \cdot \sum_n g_n(t) h_n(\tau) = \int_A d\tau \cdot \delta(t-\tau), \quad (44)$$

applied to (43), is the identical operator, in the sense that

$$\mathcal{J} \cdot \xi(\tau) = \xi(t). \quad (45)$$

The proof of (45) is quite obvious, if we put (43) into (45) and refer (37).

Considering (4) and (44), we shall take

$$F^*(\lambda) = \int_A \tilde{K}(t, \lambda) f(t) dt, \quad (46)$$

with

$$f(t) = \int_B K(\lambda, t) F^*(\lambda) d\lambda, \quad (47)$$

and

$$f^*(\lambda_n) = \int_A g_n(t) f(t) dt, \quad (48)$$

with

$$f(t) = \sum_n f^*(\lambda_n) h_n(t). \quad (49)$$

Then we obtain, from (2), (3), (47), (39), (49), and (37), the following expressions :

$$\begin{aligned}\|f\|^2 &= \int_A dt \cdot \int_B d\lambda \cdot K(\lambda, t) F^*(\lambda) \cdot \int_B d\sigma \cdot \hat{K}(t, \sigma) F(\sigma) \\ &= \int_B d\lambda \cdot F^*(\lambda) \cdot \int_B d\sigma \cdot F(\sigma) \cdot \delta(\lambda - \sigma) = \int_B F^*(\lambda) F(\lambda) d\lambda, \quad (50)\end{aligned}$$

and

$$\begin{aligned}\|f\|^2 &= \int_A dt \cdot \sum_m \sum_n f^*(\lambda_m) h_m(t) \cdot f(\lambda_n) g_n(t) \\ &= \sum_m \sum_n f^*(\lambda_m) f(\lambda_n) \cdot \delta_{m,n} = \sum_n f^*(\lambda_n) f(\lambda_n), \quad (51)\end{aligned}$$

with the norm $\|f\|$ of function $f(t)$:

$$\|f\|^2 = \int_A [f(t)]^2 dt. \quad (52)$$

The expressions (37), (45), and the generalized Parseval equalities (50) and (51) are given by Isomichi¹²⁾.

Now we shall differentiate (39) with regard to t , then we have

$$f'(t) = \sum_n f(\lambda_n) g'_n(t). \quad (53)$$

On the other hand, we apply the generalized sampling theorem (39) to $f'(t)$, then we obtain

$$f'(t) = \sum_n f'(\lambda_n) g_n(t), \quad (54)$$

with

$$f'(\lambda_n) = \left[\frac{df(t)}{dt} \right]_{t=\lambda_n}.$$

From (53) and (54), we have

$$f'(t) = \sum_n f'(\lambda_n) g_n(t) = \sum_n f(\lambda_n) g'_n(t). \quad (55)$$

Accordingly, from the m -th derivative of $f(t)$, we obtain :

$$f^{(m)}(t) = \sum_n f^{(m-r)}(\lambda_n) g_n^{(r)}(t) = \sum_n f^{(m-k)}(\lambda_n) g_n^{(k)}(t), \quad (56)$$

for any integers r and k ($0 \leq r \leq m$, $0 \leq k \leq m$), and with

$$f^{(m)}(z) = \frac{d^m}{dz^m} f(z), \quad \text{and} \quad g_n^{(m)}(z) = \frac{d^m}{dz^m} g_n(z). \quad (m = \text{integers})$$

Here, we shall mention an expansion-formula, which contains both Taylor's and sampling formulae. If a function $f(t)$ is expressed in (39), i.e.

$$f(t) = \sum_n f(\lambda_n) g_n(t), \quad (57)$$

and if we can expand $f(t + \xi)$ in a Taylor series, as

$$f(t + \xi) = \sum_{m=0}^{+\infty} \frac{1}{m!} \xi^m \cdot f^{(m)}(t), \quad (58)$$

then we have an expansion-formula :

$$f(t + \xi) = \sum_{m=0}^{+\infty} \sum_n \frac{1}{m!} \xi^m \cdot f^{(m)}(\lambda_n) g_n(t) \quad (59)$$

$$= \sum_{m=0}^{+\infty} \sum_n \frac{1}{m!} \xi^m \cdot f(\lambda_n) g_n^{(m)}(t). \quad (60)$$

For the special case of $\xi = 0$, the expressions (59) and (60) reduce to (57). While, for the special case of $t = \lambda_k$ (with k fixed) in (59) and (60), we have

$$\begin{aligned} f(\lambda_k + \xi) &= \sum_{m=0}^{+\infty} \sum_n \frac{1}{m!} \xi^m \cdot f^{(m)}(\lambda_n) g_n(\lambda_k) = \sum_m \sum_n \frac{1}{m!} \xi^m \cdot f^{(m)}(\lambda_n) \cdot \delta_{n,k} \\ &= \sum_{m=0}^{+\infty} \frac{1}{m!} \xi^m \cdot f^{(m)}(\lambda_k), \end{aligned} \quad (61)$$

and

$$f(\lambda_k + \xi) = \sum_{m=0}^{+\infty} \sum_n \frac{1}{m!} \xi^m \cdot f(\lambda_n) g_n^{(m)}(\lambda_k), \quad (62)$$

i.e.

$$f(t) = \sum_{m=0}^{+\infty} \sum_n \frac{1}{m!} (t - \lambda_k)^m f(\lambda_n) g_n^{(m)}(\lambda_k). \quad (62')$$

The expression (61) is nothing but the usual Taylor expansion of $f(\lambda_k + \xi)$, while the expression (62) gives $f(\lambda_k + \xi)$ in terms of the derivatives of sampling functions g_n at the sampling point λ_k .

Example 1

We shall take the Fourier transforms in (1) and (2):

$$K(\lambda, t) = \frac{1}{2\pi} \exp[i\lambda t], \quad A = (-\infty, +\infty),$$

$$\tilde{K}(t, \lambda) = \exp[-it\lambda], \quad B = (-\infty, +\infty),$$

$$\phi_n(\lambda) = \tilde{K}(\lambda_n, \lambda) = \exp[-i\lambda_n \lambda], \quad D = (-\beta, \beta),$$

$$\lambda_n = \frac{n-k}{\beta} \pi, \quad n = \text{integers}, \quad k = \text{real number given arbitrarily},$$

$$\int_{-\beta}^{\beta} \exp[-i\lambda_m \lambda] \cdot \exp[+i\lambda_n \lambda] d\lambda = \gamma_m \cdot \delta_{m,n},$$

$$\gamma_n = 2\beta,$$

then the sampling theorem (12) reads as follows:

$$\begin{aligned} f(t) &= \frac{1}{2\beta} \sum_{n=-\infty}^{+\infty} f\left(-\frac{n-k}{\beta} \pi\right) \cdot \frac{2 \sin(\beta t + (n-k)\pi)}{t + \frac{n-k}{\beta} \pi} \\ &= \sum_{n=-\infty}^{+\infty} f\left(\frac{n+k}{\beta} \pi\right) \cdot \frac{\sin(\beta t - (n+k)\pi)}{\beta t - (n+k)\pi}. \end{aligned} \quad (63)$$

The expression is nothing but Someya's sampling theorem⁹⁾ given in 1949.

The expression (63) gives $f(t)$ in terms of the sampling function

$\sin(\beta t - (n+k)\pi)/(\beta t - (n+k)\pi)$, with values of $f((n\pi+k\pi)/\beta)$

at sampling points $t = (n+k)\pi/\beta$ ($n = \text{integers}$). If we put $k = 0$ in (63), we have Shannon's sampling theorem^{10) 11)}.

Example 2

We shall take the Fourier cosine transforms for (1) and (2):

$$K(\lambda, t) = \frac{2}{\pi} \cos(\lambda t), \quad A = (0, +\infty),$$

$$\tilde{K}(t, \lambda) = \cos(t\lambda), \quad B = (0, +\infty),$$

$$\phi_n(\lambda) = \hat{K}(\lambda_n, \lambda) = \cos(\lambda_n \lambda), \quad D = (0, \beta),$$

$$\gamma_n = \frac{\beta}{2} \left\{ 1 + \frac{\sin(2\lambda_n \beta)}{2\lambda_n \beta} \right\},$$

where λ_n 's are the roots (arranged in ascending order of magnitude) of the equation :

$$\lambda_n \tan(\lambda_n \beta) = M, \quad (64)$$

with a constant M . Then the expression (12) takes form :

$$\begin{aligned} f(t) &= \frac{2}{\beta} \sum_n \frac{1}{1 + \frac{\sin(2\lambda_n \beta)}{2\lambda_n \beta}} \left(\int_0^\beta F(\lambda) \cos(\lambda_n \lambda) d\lambda \right) \cdot \int_0^\beta \cos(t\lambda) \cos(\lambda_n \lambda) d\lambda \\ &= \frac{2}{\beta} \sum_n f(\lambda_n) \cdot \frac{\cos(\lambda_n \beta)}{1 + \frac{\sin(2\lambda_n \beta)}{2\lambda_n \beta}} \cdot \frac{t \tan(\beta t) - M}{t^2 - \lambda_n^2} \cos(\beta t), \quad (65) \end{aligned}$$

with sampling points $t = \lambda_n$ (n - integers), and the sampling function:

$$g_n(t) = \frac{2}{\beta} \cdot \frac{\cos(\lambda_n \beta)}{1 + \frac{\sin(2\lambda_n \beta)}{2\lambda_n \beta}} \cdot \frac{t \tan(\beta t) - M}{t^2 - \lambda_n^2} \cos(\beta t). \quad (66)$$

Example 3

We shall take the Fourier sine-transforms for (1) and (2) :

$$K(\lambda, t) = \frac{2}{\pi} \sin(\lambda t), \quad A = (0, +\infty),$$

$$\hat{K}(t, \lambda) = \sin(t\lambda), \quad B = (0, +\infty),$$

$$\phi_n(\lambda) = \hat{K}(\lambda_n, \lambda) = \sin(\lambda_n \lambda), \quad D = (0, \beta),$$

$$\gamma_n = \frac{\beta}{2} \left\{ 1 - \frac{\sin(2\lambda_n \beta)}{2\lambda_n \beta} \right\},$$

where λ_n 's are the roots (arranged in ascending order of magnitude) of the equation :

$$\lambda_n \cot(\lambda_n \beta) = N, \quad (67)$$

with a constant N . Then the expression (12) becomes to :

$$f(t) = \frac{2}{\beta} \sum_n \frac{1}{1 - \frac{\sin(2\lambda_n \beta)}{2\lambda_n \beta}} \left(\int_0^\beta F(\lambda) \sin(\lambda_n \lambda) d\lambda \right) \cdot \int_0^\beta \sin(t\lambda) \sin(\lambda_n \lambda) d\lambda$$

$$= -\frac{2}{\beta} \sum_n f(\lambda_n) \cdot \frac{\sin(\lambda_n \beta)}{1 - \frac{\sin(2\lambda_n \beta)}{2\lambda_n \beta}} \cdot \frac{t \cot(\beta t) - N}{t^2 - \lambda_n^2} \sin(\beta t), \quad (68)$$

with sampling points $t = \lambda_n$ ($n = \text{integers}$), and the sampling function :

$$g_n(t) = -\frac{2}{\beta} \cdot \frac{\sin(\lambda_n \beta)}{1 - \frac{\sin(2\lambda_n \beta)}{2\lambda_n \beta}} \cdot \frac{t \cot(\beta t) - N}{t^2 - \lambda_n^2} \sin(\beta t). \quad (69)$$

The formulae (65) and (68) were obtained by Kroll¹⁴⁾ in connection with the solution of an integral equation.

Example 4

Let us take the Hankel transforms of order $\nu \geq -1/2$ for (1) and (2), and take the Fourier-Bessel series¹⁵⁾ for $F(\lambda)$:

$$\sqrt{t} f(t) \in L_1(0, +\infty),$$

$$K(\lambda, t) = t J_\nu(\lambda t), \quad A = (0, +\infty),$$

$$\widehat{K}(t, \lambda) = \lambda J_\nu(t\lambda), \quad B = (0, +\infty),$$

$$\phi_n(\lambda) = J_\nu(j_n \lambda), \quad D = (0, \beta),$$

with the orthogonality relation :

$$\int_0^\beta \lambda J_\nu(j_m \lambda) J_\nu(j_n \lambda) d\lambda = \delta_m \cdot \delta_{m,n},$$

and

$$\delta_n = \frac{\beta^2}{2} J_{\nu+1}^2(j_n \beta),$$

where $j_n \beta$'s ($n = 1, 2, 3, \dots$) are the positive zeros of $J_\nu(x)$, being arranged in ascending order of magnitude, i.e.

$$J_\nu(j_n \beta) = 0. \quad (70)$$

The expression (12) takes form :

$$\begin{aligned}
 f(t) &= \frac{2}{\beta^2} \sum_{n=1}^{+\infty} \frac{1}{J_{\nu+1}(j_n \beta)} \left(\int_0^\beta \lambda F(\lambda) J_\nu(j_n \lambda) d\lambda \right) \cdot \int_0^\beta \lambda J_\nu(t\lambda) J_\nu(j_n \lambda) d\lambda \\
 &= -\frac{2}{\beta} \sum_{n=1}^{+\infty} f(j_n) \cdot \frac{j_n}{J_{\nu+1}(j_n \beta)} \cdot \frac{J_\nu(\beta t)}{t^2 - j_n^2}, \quad (71)
 \end{aligned}$$

with sampling points $t = j_n$ ($n =$ positive integers), and the sampling function :

$$g_n(t) = -\frac{2}{\beta} \cdot \frac{j_n}{J_{\nu+1}(j_n \beta)} \cdot \frac{J_\nu(\beta t)}{t^2 - j_n^2}. \quad (72)$$

The formula (71) was given by the present authors¹⁾.

Example 5

We shall take the Hankel transforms for (1) and (2) just as in Example 4. The function $F(\lambda)$ is assumed to be expanded in the Deni expansion¹⁶⁾ in the domain $\mathcal{D} = (0, \beta)$. We shall take

$$\begin{aligned}
 K(\lambda, t) &= t J_\nu(\lambda t), & A &= (0, +\infty), \\
 \widetilde{K}(t, \lambda) &= \lambda J_\nu(t\lambda), & B &= (0, +\infty), \\
 \phi_m(\lambda) &= J_\nu(\lambda_n \lambda), & \mathcal{D} &= (0, \beta),
 \end{aligned}$$

with the orthogonality relation :

$$\int_0^\beta \lambda J_\nu(\lambda_m \lambda) J_\nu(\lambda_n \lambda) d\lambda = j_n \cdot \delta_{m,n},$$

and

$$j_n = \frac{1}{2\lambda_n^2} \{ (\lambda_n^2 + h^2) \beta^2 - \nu^2 \} J_\nu^2(\lambda_n \beta),$$

where λ'_n 's ($n = 1, 2, 3, \dots$) are the positive roots (arranged in ascending order of magnitude) of the following equation :

$$\lambda_n J_\nu'(\lambda_n \beta) + h J_\nu(\lambda_n \beta) = 0, \quad (73)$$

with a constant h .

Then the expression (12) becomes :

$$f(t) = 2 \sum_{n=1}^{+\infty} \frac{\lambda_n^2}{\{(\lambda_n^2 + h^2)\beta^2 - \nu^2\} J_\nu(\lambda_n \beta)} \left(\int_0^\beta F(\lambda) J_\nu(\lambda_n \lambda) d\lambda \right) \cdot \int_0^\beta J_\nu(\lambda t) J_\nu(\lambda_n \lambda) d\lambda$$

$$= -2\beta \sum_{n=1}^{+\infty} f(\lambda_n) \cdot \frac{\lambda_n^2}{\{(\lambda_n^2 + h^2)\beta^2 - \nu^2\} J_\nu(\lambda_n \beta)} \cdot \frac{t J_\nu'(\beta t) + h J_\nu(\beta t)}{t^2 - \lambda_n^2}, \quad (74)$$

with sampling points $t = \lambda_n$ (n = positive integers), and the sampling function :

$$g_n(t) = -2\beta \frac{\lambda_n^2}{\{(\lambda_n^2 + h^2)\beta^2 - \nu^2\} J_\nu(\lambda_n \beta)} \cdot \frac{t J_\nu'(\beta t) + h J_\nu(\beta t)}{t^2 - \lambda_n^2}. \quad (75)$$

The formula (74) was given by Kroll¹⁴⁾.

Letting h tend to infinity in the expressions (73) and (74), we see that they reduce to the following expressions, respectively :

$$J_\nu(\lambda_n \beta) = 0, \quad (76)$$

and

$$f(t) = -\frac{2}{\beta} \sum_{n=1}^{+\infty} f(\lambda_n) \cdot \frac{\lambda_n}{J_{\nu+1}(\lambda_n \beta)} \cdot \frac{J_\nu(\beta t)}{t^2 - \lambda_n^2}. \quad (77)$$

The expressions (76) and (77) are nothing but the expressions (70) and (71), respectively. Accordingly, the expression (71) is a limiting case of (74).

If we put $\nu = 1/2$ in the expression (74), then we obtain

$$f(t) = -2\beta \sum_{n=1}^{+\infty} f(\lambda_n) \cdot \frac{\lambda_n^2 \sqrt{\lambda_n}}{\{(\lambda_n^2 + h^2)\beta^2 - \frac{1}{4}\} \sin(\lambda_n \beta)} \cdot \frac{t \cos(\beta t) + (h - \frac{1}{2\beta}) \sin(\beta t)}{\sqrt{t} (t^2 - \lambda_n^2)}, \quad (78)$$

where λ_n 's are positive roots of the following equations, being arranged in ascending order of magnitude :

$$\lambda_n \cos(\lambda_n \beta) + (h - \frac{1}{2\beta}) \sin(\lambda_n \beta) = 0. \quad (79)$$

In the expression (78), if we replace $f(t)$ and $f(\lambda_n)$ by $f(t)/\sqrt{t}$ and $f(\lambda_n)/\sqrt{\lambda_n}$ respectively, then we obtain :

$$f(t) = -2\beta \sum_{n=1}^{+\infty} f(\lambda_n) \cdot \frac{\lambda_n^2}{\{(\lambda_n^2 + h^2)\beta^2 - \frac{1}{4}\} \sin(\lambda_n \beta)} \cdot \frac{t \cot(\beta t) + (h - \frac{1}{2\beta})}{t^2 - \lambda_n^2} \sin(\beta t). \quad (80)$$

Further, if we put

$$h - \frac{1}{2\beta} = -N, \quad (81)$$

in the expressions (79) and (80), then the expressions (79) and (80) reduce to (67) and (68), respectively.

If we put

$$h = \frac{1}{2\beta}, \quad N = 0, \quad (82)$$

in the expressions (80) and (79), then we obtain :

$$f(t) = -\frac{2}{\beta} \sum_{n=1}^{+\infty} f(\lambda_n) \cdot \frac{1}{\sin(\lambda_n \beta)} \cdot \frac{t \cos(\beta t)}{t^2 - \lambda_n^2}, \quad (83)$$

with

$$\cos(\lambda_n \beta) = 0, \quad (84)$$

i.e.

$$\lambda_n = \frac{2n-1}{2\beta} \pi, \quad (n = 1, 2, 3, \dots) \quad (85)$$

and

$$\sin(\lambda_n \beta) = (-1)^{n+1}, \quad (n = 1, 2, 3, \dots) \quad (86)$$

Then the expression (83) reduces to :

$$f(t) = 2 \sum_{n=1}^{+\infty} f\left(\frac{2n-1}{2\beta} \pi\right) \cdot (-1)^n \cdot \frac{\beta t \cos(\beta t)}{\beta^2 t^2 - \frac{(2n-1)^2 \pi^2}{4}}. \quad (87)$$

The expression (87) corresponds to Someya's sampling theorem (63) with $h = \pm 1/2$ for an odd function $f(t)$.

If we put $\gamma = -1/2$ in (74), we obtain :

$$f(t) = 2\beta \sum_{n=1}^{+\infty} f(\lambda_n) \cdot \frac{\lambda_n^2 \sqrt{\lambda_n}}{\{(\lambda_n^2 + h^2)\beta^2 - \frac{1}{4}\} \cos(\lambda_n \beta)} \cdot \frac{t \sin(\beta t) - (h - \frac{1}{2\beta}) \cos(\beta t)}{\sqrt{t} (t^2 - \lambda_n^2)}, \quad (88)$$

with

$$\lambda_n \sin(\lambda_n \beta) - (h - \frac{1}{2\beta}) \cos(\lambda_n \beta) = 0. \quad (89)$$

In the expression (88), if we replace $f(t)$ and $f(\lambda_n)$ by $f(t)/\sqrt{t}$ and $f(\lambda_n)/\sqrt{\lambda_n}$ respectively, then we obtain

$$f(t) = 2\beta \sum_{n=1}^{+\infty} f(\lambda_n) \cdot \frac{\lambda_n^2}{\{(\lambda_n^2 + h^2)\beta^2 - \frac{1}{4}\} \cos(\lambda_n \beta)} \cdot \frac{t \tan(\beta t) - (h - \frac{1}{2\beta}) \cos(\beta t)}{t^2 - \lambda_n^2}. \quad (90)$$

Further, if we put

$$h - \frac{1}{2\beta} = M, \quad (91)$$

in the expressions (89) and (90), then the expressions (89) and (90) reduce to (64) and (65), respectively.

If we put

$$h = \frac{1}{2\beta}, \quad M = 0, \quad (92)$$

in the expressions (90) and (89), then we obtain the following expressions :

$$f(t) = \frac{2}{\beta} \sum_{n=1}^{+\infty} f(\lambda_n) \cdot \frac{1}{\cos(\lambda_n \beta)} \cdot \frac{t \sin(\beta t)}{t^2 - \lambda_n^2}, \quad (93)$$

with

$$\sin(\lambda_n \beta) = 0, \quad (94)$$

i.e.

$$\lambda_n = \frac{n\pi}{\beta}, \quad (n = 1, 2, 3, \dots) \quad (95)$$

and

$$\cos(\lambda_n \beta) = (-1)^n. \quad (n = 1, 2, 3, \dots) \quad (96)$$

Then the expression (93) reduces to :

$$f(t) = 2 \sum_{n=1}^{+\infty} f\left(\frac{n\pi}{\beta}\right) \cdot (-1)^n \cdot \frac{\beta t \sin(\beta t)}{\beta^2 t^2 - n^2 \pi^2}. \quad (97)$$

The expression (97) corresponds to Shannon's sampling theorem for an even function $f(t)$.

Example 6

We shall take other Hankel transforms due to Weber¹⁷⁾ for (1) and (2), namely

$$t f(t) \in L_1(p, +\infty), \quad \alpha \geq p > 0,$$

$$F(\lambda) = \int_A K(\lambda, t) f(t) dt, \quad \text{for } \nu \geq -\frac{1}{2} \quad (98)$$

and

$$G_\nu(\alpha t) f(t) = \int_B \widehat{K}(t, \lambda) F(\lambda) d\lambda, \quad \text{for } t \in A \quad (99)$$

with

$$K(\lambda, t) = t T_\nu(t\alpha, \lambda t), \quad A = (p, +\infty),$$

$$\widehat{K}(t, \lambda) = \lambda T_\nu(t\alpha, t\lambda), \quad B = (\alpha, +\infty),$$

$$T_\mu(x, z) = Y_\nu(x) J_\mu(z) - J_\nu(x) Y_\mu(z), \quad (100)$$

and

$$G_\nu(x) = J_\nu^2(x) + Y_\nu^2(x), \quad (101)$$

under the condition that the integral :

$$\int_p^{+\infty} t f(t) dt < +\infty,$$

is absolutely convergent.

We shall take another Fourier-Bessel expansion¹⁷⁾ for $F(\lambda)$ of the form :

$$F(\lambda) = \sum_{n=1}^{+\infty} a_n \phi_n(\lambda), \quad \text{for } \lambda \in D \subseteq B$$

with

$$\phi_n(\lambda) = T_\nu(\lambda_n \alpha, \lambda_n \lambda), \quad D = (\alpha, \beta),$$

$$\int_\alpha^\beta \lambda T_\nu(\lambda_m \alpha, \lambda_m \lambda) T_\nu(\lambda_n \alpha, \lambda_n \lambda) d\lambda = \gamma_m \cdot \delta_{m,n}, \quad (102)$$

and

$$\gamma_n = \frac{\beta^2}{2} T_{\nu+1}^2(\lambda_n \alpha, \lambda_n \beta) - \frac{\alpha^2}{2} T_{\nu+1}^2(\lambda_n \alpha, \lambda_n \alpha) = \frac{2}{\pi^2 \lambda_n^2} \left[\frac{J_\nu^2(\lambda_n \alpha)}{J_\nu^2(\lambda_n \beta)} - 1 \right],$$

where λ_n 's are the positive roots of the equation :

$$T_\nu(\lambda_n \alpha, \lambda_n \beta) = 0, \quad (103)$$

i.e.

$$Y_\nu(\lambda_n \alpha) J_\nu(\lambda_n \beta) = J_\nu(\lambda_n \alpha) Y_\nu(\lambda_n \beta).$$

From (99), the expression (12) becomes to :

$$\begin{aligned} f(t) &= \frac{\pi^2}{2} \sum_{n=1}^{+\infty} \frac{\lambda_n^2 J_\nu^2(\lambda_n \beta)}{J_\nu^2(\lambda_n \alpha) - J_\nu^2(\lambda_n \beta)} \cdot G_\nu(\lambda_n \alpha) \cdot f(\lambda_n) \cdot \beta \frac{d}{d\beta} T_\nu(\lambda_n \alpha, \lambda_n \beta) \cdot \frac{T_\nu(\alpha t, \beta t)}{(t^2 - \lambda_n^2) G_\nu(\alpha t)} \\ &= -\pi \sum_{n=1}^{+\infty} f(\lambda_n) \cdot \frac{\lambda_n^2 G_\nu(\lambda_n \alpha) J_\nu(\lambda_n \alpha) J_\nu(\lambda_n \beta)}{J_\nu^2(\lambda_n \alpha) - J_\nu^2(\lambda_n \beta)} \cdot \frac{T_\nu(\alpha t, \beta t)}{(t^2 - \lambda_n^2) G_\nu(\alpha t)}, \end{aligned} \quad (105)$$

with sampling points $t = \lambda_n$ ($n = 1, 2, 3, \dots$), and the sampling function :

$$g_n(t) = -\pi \frac{\lambda_n^2 G_v(\lambda_n \alpha) J_v(\lambda_n \alpha) J_v(\lambda_n \beta)}{J_v^2(\lambda_n \alpha) - J_v^2(\lambda_n \beta)} \cdot \frac{T_v(\alpha t, \beta t)}{(t^2 - \lambda_n^2) G_v(\alpha t)}. \quad (106)$$

In the expression (105), if we replace $f(t)$ and $f(\lambda_n)$ by $f(t)/G_v(\alpha t)$ and $f(\lambda_n)/G_v(\alpha \lambda_n)$ respectively, then we obtain

$$f(t) = -\pi \sum_{n=1}^{+\infty} f(\lambda_n) \cdot \frac{\lambda_n^2 J_v(\lambda_n \alpha) J_v(\lambda_n \beta)}{J_v^2(\lambda_n \alpha) - J_v^2(\lambda_n \beta)} \cdot \frac{T_v(\alpha t, \beta t)}{t^2 - \lambda_n^2}, \quad (107)$$

with sampling points $t = \lambda_n$ and the sampling function :

$$g_n(t) = -\pi \frac{\lambda_n^2 J_v(\lambda_n \alpha) J_v(\lambda_n \beta)}{J_v^2(\lambda_n \alpha) - J_v^2(\lambda_n \beta)} \cdot \frac{T_v(\alpha t, \beta t)}{t^2 - \lambda_n^2}. \quad (108)$$

The expressions (63), (65), (68), (71), (74), (78), (80), (87), (88), (90), (97), (105), and (107), are examples of the sampling formulae derived from our generalized sampling theorem (12).

References

- 1) É.I. Takizawa, K. Kobayasi, and J.-L. Hwang : Chinese Journ. Phys. 5(1967), 21.
- 2) A. Kohlenberg : J. Appl. Phys. 24(1953), 1432.
- 3) L.J. Fogel : IRE Trans. on Information Theory IT-1(1955), 47.
- 4) D.L. Jagerman and L.J. Fogel : IRE Trans. on Information Theory IT-2(1956), 139.
- 5) F.E. Bond and C.R. Cahn : IRE Trans. on Information Theory IT-4(1958), 110.
- 6) D.A. Linden and N.M. Abramson : Information and Control 3(1960), 26.
- 7) A.V. Balakrishnan : IRE Trans. on Information Theory IT-3(1957), 143.
- 8) J.L. Yen : IRE Trans. on Circuit Theory CT-3(1956), 251.
- 9) I. Someya : Transmission of Wave-Forms (1949), Syûkyôsha, Tokyo.
(in Japanese)
- 10) C.E. Shannon : Bell System Techn. Journ. 27(1948), 379 ; 623.
- 11) C.E. Shannon and W. Weaver : Mathematical Theory of Communication (1949), Univ. of Illinois Press.
- 12) Y. Isomichi : Research Group of Information Theory, Report IT 67-36
Soc. of Electrocommunication (1967), 1. (in Japanese)
- 13) É.I. Takizawa : Information Theory and its Exercises (1966),
Hirokawa Book Co. Tokyo, p.275. (in Japanese)

- 14) W. Kroll : Chinese Journ. Phys. 5 (1967), 86.
- 15) G.N. Watson : Theory of Bessel Functions (1944), Cambridge Univ. Press. p.576.
- 16) G.N. Watson : Theory of Bessel Functions (1944), Cambridge Univ. Press. p.580.
- 17) I.N. Sneddon : Fourier Transforms (1951), McGraw-Hill. p.56.
G.N. Watson : Theory of Bessel Functions (1944), Cambridge Univ. Press. p.470.
C.J. Tranter : Integral Transforms in Mathematical Physics (1956), Methuen, London. p.89.