

Infinite Symmetry of the One-Dimensional System with Inverse Square Interactions*

Kazuhiro HIKAMI[†] and Miki WADATI

*Department of Physics, Faculty of Science,
University of Tokyo,
Hongo 7-3-1, Bunkyo-ku, Tokyo 113, Japan*

Abstract

We consider the one-dimensional quantum particle system with inverse square interactions, and reveal the underlying infinite symmetry. We construct explicitly the $\mathcal{W}$ -operators to show that the system represents the $w_{1+\infty}$ -algebra. Both the Calogero-type ($1/x^2$ -interactions) and Sutherland-type ($1/\sin^2 x$ -interactions) models are included in these operators : the former is the spin-1 operators and the latter the zero-mode operators. The symmetry of the Calogero-Moser-type (Calogero-type model confined in the harmonic potential) is also discussed.

*A talk given at 基研研究会 “低次元系物性と場の理論”

[†]HIKAMI@tkyvax.phys.s.u-tokyo.ac.jp

1 Introduction

In this paper we report the symmetry of the quantum N -particle system with inverse square interactions. This system is first introduced by Calogero [1], and further studied by Sutherland [2], Moser [3], and others (see, for review, [4]). The symmetry of this system, however, has not been clarified yet. Based on the quantum inverse scattering method we shall find the symmetry of the long-range interaction system.

The Hamiltonian, which we consider, is

$$\begin{aligned}\mathcal{H}_C &= \sum_{j=1}^N p_j^2 + \sum_{1 \leq j < k \leq N} V(x_j - x_k) \\ &= \sum_j p_j^2 + \sum_{j < k} \frac{2(a^2 - a)}{(x_j - x_k)^2},\end{aligned}\quad (1.1)$$

where p_j is the momentum operator satisfying the canonical commutation relation,

$$[x_j, p_k] \equiv x_j p_k - p_k x_j = i\delta_{jk}. \quad (1.2)$$

We call (1.1) as the Calogero-type model. Well known is the fact that the ground state, Ψ_G , of this model is written as the Jastrow-type wave function,

$$\Psi_G = \prod_{j < k} (x_j - x_k)^a. \quad (1.3)$$

The Calogero-type model on a line is integrable and remains so under the periodic boundary condition. When the periodic boundary condition is imposed, the potential $V(x)$ is changed into

$$V(x) = \sum_{n=-\infty}^{\infty} (x + n\pi)^{-2} = \sin^{-2} x, \quad (1.4)$$

where we set the size of the system to be π for simplicity. Then the Hamiltonian becomes

$$\mathcal{H}_S = \sum_{j=1}^N p_j^2 + \sum_{j < k} \frac{2(a^2 - a)}{\sin^2(x_j - x_k)}, \quad (1.5)$$

which we call the Sutherland-type model.

One of the extensions is the addition of the harmonic potential,

$$\mathcal{H}_{CM} = \sum_j (p_j^2 + \omega^2 x_j^2) + \sum_{j \neq k} \frac{a^2 - a}{(x_j - x_k)^2}, \quad (1.6)$$

which is referred to as the Calogero-Moser type model. The energy spectrum is gapful to form the Landau level while the Sutherland-type is gapless.

This paper is organized as follows. In section 2 we construct the conserved operators and give extended Lax equations. In section 3 we show that the $w_{1+\infty}$ algebra is realized. The Calogero-type and the Sutherland-type models appear as one of these operators. In section 4

the Calogero-Moser type model is also discussed. Section 5 is devoted to the conclusions and the discussions.

2 Lax Equation

We introduce the Lax operators L and M_2 , which are $N \times N$ operator-valued matrices :

$$L_{jk} = \delta_{jk} \cdot p_j + (1 - \delta_{jk}) \cdot \frac{ia}{x_j - x_k}, \quad (2.1)$$

$$(M_2)_{jk} = \delta_{jk} \cdot \sum_{l \neq j} \frac{2a}{(x_j - x_l)^2} - (1 - \delta_{jk}) \cdot \frac{2a}{(x_j - x_k)^2}. \quad (2.2)$$

Remark that the operator M_2 satisfies "the sum-to-zero condition",

$$\sum_{l=1}^N (M_2)_{jl} = \sum_{l=1}^N (M_2)_{lk} = 0. \quad (2.3)$$

This condition plays a crucial role throughout this paper. In addition to the Lax operators, we define "the coordinate operator" as

$$X = \text{diag}(ix_1, ix_2, \dots, ix_N). \quad (2.4)$$

The imaginary unit i is included for our convenience. These three operator-valued matrices satisfy the following Heisenberg equations of motion :

$$[\mathcal{H}_C, L] = [L, M_2], \quad (2.5)$$

$$[\mathcal{H}_C, X] = [X, M_2] + 2L. \quad (2.6)$$

The former is the Lax equation, and the latter "the additional equation".

With the Lax equations (2.5) and the sum-to-zero condition (2.3), we can prove that a set of the conserved operators $\mathcal{I}_n$ is constructed from the Lax operator L as

$$\mathcal{I}_n = \sum_{j,k} (L^n)_{jk}. \quad (2.7)$$

For instance, some of them are

$$\mathcal{I}_1 = \sum_j p_j \equiv \text{total momentum},$$

$$\mathcal{I}_2 = \mathcal{H}_C,$$

$$\mathcal{I}_3 = \sum_j p_j^3 + \sum_{j \neq k} \frac{3(a^2 - a)}{(x_j - x_k)^2} p_j.$$

For further discussions, it is important to notice that the extended Lax equations and the additional equations are satisfied for $\mathcal{I}_n$;

$$[\mathcal{I}_n, L] = [L, M_n], \quad (2.8)$$

$$[\mathcal{I}_n, X] = [X, M_n] + nL^{n-1}, \quad (2.9)$$

where M_n are $N \times N$ matrices satisfying the sum-to-zero condition,

$$\sum_l (M_n)_{jl} = \sum_l (M_n)_{lk} = 0. \quad (2.10)$$

These equations are proved for the classical [6, 7] and quantum [8] cases. Equation (2.8) leads to the involutiveness of the conserved operators,

$$[\mathcal{I}_n, \mathcal{I}_m] = 0, \quad (2.11)$$

which supports the integrability of the Calogero-type model in the Liouville sense.

3 $\mathcal{W}$ Algebra

The conserved quantities can be extended to the “higher spin” using the coordinate matrix X . We define the $\mathcal{W}$ -operators recursively by

$$\begin{aligned} \mathcal{W}_n^{(s)} &= \frac{1}{2(n+s)} \left[\sum_j x_j^2, \mathcal{W}_{n+2}^{(s-1)} \right], \\ \mathcal{W}_n^{(1)} &\equiv \mathcal{I}_n. \end{aligned} \quad (3.1)$$

The superscript s indicates spin. We note that the $\mathcal{W}$ -operators have the form,

$$\mathcal{W}_n^{(s)} = \sum_j (ix_j)^{s-1} p_j^{n+s-1} + \dots, \quad (3.2)$$

and are only defined for $n \geq -s + 1$.

These $\mathcal{W}$ -operators constitute the $w_{1+\infty}$ algebra [11] (strictly speaking, $w_{1+\infty}^+$ algebra, a subalgebra of $w_{1+\infty}$ algebra),

$$[\mathcal{W}_n^{(s)}, \mathcal{W}_m^{(s')}] = ((s' - 1)n - (s - 1)m) \mathcal{W}_{n+m}^{(s+s'-2)} + (\text{lower terms}), \quad (3.3)$$

We write some of the $\mathcal{W}$ -operators explicitly :

$$\mathcal{W}_{-1}^{(2)} = \sum_j ix_j, \quad \mathcal{W}_0^{(2)} = \sum_j ix_j p_j + \frac{N}{2}, \quad (3.4)$$

$$\mathcal{W}_{-2}^{(3)} = \sum_j (ix_j)^2, \quad \mathcal{W}_{-1}^{(3)} = \sum_j (-x_j^2 p_j + ix_j),$$

$$\mathcal{W}_0^{(3)} = \sum_j (-x_j^2 p_j^2 + 2ix_j p_j) + \sum_{j \neq k} (a^2 - a) \frac{x_j x_k}{(x_j - x_k)^2} + \frac{N}{6} (3 + 2a(1 - a)(N - 1)) \quad (3.5)$$

One notices (3.3) that not only spin-1 operators $\mathcal{W}_n^{(1)}$ but also the zero-mode operators $\mathcal{W}_0^{(s)}$ constitute the involutive sets. The significance of the zero-mode operators is found by changing the variables. We introduce as z_j [10],

$$x_j \equiv \exp(2iz_j). \quad (3.6)$$

Note that z_j satisfies the periodic boundary condition (recall that the system size is π). By substituting (3.6) into (3.4) and (3.5), we have

$$\mathcal{W}_0^{(2)} = \frac{1}{2} \sum_j (-i\partial_j) + \frac{N}{2}, \quad (3.7)$$

$$\begin{aligned} \mathcal{W}_0^{(3)} = & \frac{1}{4} \left(- \sum_j \partial_j^2 + \sum_{j \neq k} \frac{a^2 - a}{\sin^2(z_j - z_k)} \right) + \frac{1}{2} \sum_j (-i\partial_j) \\ & + \frac{N}{6} (3 + 2a(1-a)(N-1)), \end{aligned} \quad (3.8)$$

where ∂_j denotes $\partial/\partial z_j$. This result shows that the zero-mode operators $\mathcal{W}_0^{(s)}$ are nothing but the conserved operators of the Sutherland-type model $\mathcal{H}_S$.

4 Calogero Model in the Harmonic Potential

An interesting extension is the Calogero-type model confined in the harmonic potential,

$$\mathcal{H}_{CM} = \sum_j p_j^2 + \sum_{j \neq k} \frac{a^2 - a}{(x_j - x_k)^2} + \sum_j \omega^2 x_j^2, \quad (4.1)$$

where the subscript CM means the Calogero-Moser model. In this case the Lax equation is modified into

$$[\mathcal{H}_{CM}, L \pm \omega X] = [L \pm \omega X, M_2] \pm 2\omega(L \pm \omega X). \quad (4.2)$$

The W operators are defined by

$$\widehat{W}_n^{(s)} = \sum_{j,k} (L + \omega X)^{s-n-1} \cdot (L - \omega X)^{s+n-1}, \quad (4.3)$$

which constitute the w_∞ algebra [5],

$$[\widehat{W}_n^{(s)}, \widehat{W}_m^{(s')}] = 4\omega((s'-1)n - (s-1)m) \widehat{W}_{n+m}^{(s+s'-2)} + (\text{lower terms}). \quad (4.4)$$

In these operators, $\widehat{W}_n^{(s)}$ is defined for $-s+1 \leq n \leq s-1$ for n and s being integer or half-odd-integer. It is important to notice that the Hamiltonian $\mathcal{H}_{CM}$ is $\widehat{W}_0^{(2)}$, and that operators satisfy

$$[\mathcal{H}_{CM}, \widehat{W}_n^{(s)}] = -4n\omega \widehat{W}_n^{(s)}. \quad (4.5)$$

These relations suggest that the $\widehat{W}_n^{(s)}$ can be regarded as the creation and annihilation operators of the Calogero-Moser type Hamiltonian.

5 Conclusion

We have revealed the symmetry of the quantum one-dimensional models with inverse square interactions. From the viewpoint of the $w_{1+\infty}$ algebra, both of the Calogero and Sutherland type models can be understood as follows ;

	Calogero type	Sutherland type
Conserved operators	Spin-1 $\mathcal{W}_n^{(1)}$	Zero-mode $\mathcal{W}_0^{(s)}$

At the moment, it is still unclear whether the system with the φ -function interactions has such symmetry or not.

The Calogero model can be generalized to $su(\nu)$ case. Each particle has an $su(\nu)$ spin as an internal degree of freedom. The Hamiltonian is

$$\mathcal{H}_{su(\nu)} = \sum_j^N p_j^2 + \sum_{j \neq k} \frac{a^2 - aP_{jk}}{(x_j - x_k)^2}, \quad (5.1)$$

where P is the permutation operator in the $su(\nu)$ spin space. For $su(2)$ case, it is expressed in terms of the Pauli spin matrices,

$$P_{jk} = \frac{1}{2}(1 + \sigma_j \cdot \sigma_k). \quad (5.2)$$

As one expect, the $w_{1+\infty}$ symmetry is extended to the coloured symmetry, the $su(\nu)$ - $w_{1+\infty}$ algebra [11].

References

- [1] F. Calogero, J. Math. Phys. **10** (1969) 2197 ; J. Math. Phys. **12** (1971) 418.
- [2] B. Sutherland, J. Math. Phys. **12** (1971) 246,251 ; Phys. Rev. **A4** (1971) 2019 ; Phys. Rev. **A5** (1972) 1372.
- [3] J. Moser, Adv. Math. **16** (1975) 197.
- [4] M. A. Olshanetsky and A. M. Perelomov, Phys. Rep. **94** (1983) 313.
- [5] J. Avan and A. Jevicki, Phys. Lett. **272B** (1991) 17.
- [6] G. Barucchi and T. Regge, J. Math. Phys. **18** (1977) 1149.
- [7] K. Hikami and M. Wadati, J. Phys. Soc. Jpn. **62** (1993) No. 11.
- [8] M. Wadati, K. Hikami and H. Ujino, to appear in Chaos, Solitons & Fractals.
H. Ujino, M. Wadati and K. Hikami, J. Phys. Soc. Jpn. **62** (1993) 3035.
- [9] A. Bilal, Phys. Lett. **227B** (1989) 406 .
I. Bakas, Phys. Lett. **228B** (1989) 57.
- [10] D. Bernard, M. Gaudin, F. D. M. Haldane and V. Pasquier, preprint SPhT-93-006.
- [11] K. Hikami and M. Wadati, J. Phys. Soc. Jpn. **62** (1993) No. 12. ; preprint, to appear in Chaos, Solitons & Fractals.