

TOPOLOGICAL ENTROPY OF CIRCLE ENDOMORPHISMS

by

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1° The purpose of this paper is to investigate the topological entropy $h(f)$ of a piecewise monotone and continuous transformation f of the unit circle S^1 and to relate it to the mapping degree $\deg(f)$ of f . For such a transformation f , $L(f)$ denotes the number of the maximal intervals where f is monotone. Then the limit $G(f) = \lim_{n \rightarrow \infty} \frac{1}{n} \log L(f^n)$ exists and is called the growth number of f . Our main results are the following theorems 1 and 2.

THEOREM 1 Let f be a piecewise monotone and continuous transformation of S^1 . Suppose that f is not a local homeomorphism. Then $h(f) = G(f)$.

THEOREM 2 (1) For a piecewise monotone and continuous transformation f , $h(f) \geq \log \deg(f)$,

(2) For a local homeomorphism f of S^1 , $h(f) = \log \deg(f)$.

REMARK Theorem 2 (1) is the simplest case of a general theorem of Manning ([4]).

In the case of transformations of the interval, the concept of growth number is introduced and extensively studied by Milnor and Thurston ([3]).

Now let us recall the definition of the topological entropy of a continuous transformation of a compact space X ([1]).

$$h(f) = \sup \{ H(f, \alpha) \mid \alpha \text{ is a open covering of } X \}$$

$$H(f, \alpha) = \lim_{n \rightarrow \infty} \frac{1}{n} \log N(\alpha \vee f^{-1}\alpha \vee \dots \vee f^{-n+1}\alpha)$$

Here for open coverings α, β of X , $\alpha \vee \beta$ denotes the open covering $\{ A \cap B \mid A \in \alpha, B \in \beta \}$, and $N(\alpha)$ is the smallest cardinality of subcoverings of α .

2° A nonempty finite subset Δ of the unit circle S^1 is called a partition; its cardinality is denoted by $m(\Delta)$; the closure of a connected component of $S^1 - \Delta$ is called a small interval of Δ ; for a piecewise monotone and continuous transformation f , Δ_n denotes the partition $\Delta \cup f^{-1}\Delta \cup f^{-2}\Delta \cup \dots \cup f^{-n+1}\Delta$. A continuous transformation f is called piecewise monotone if there exists a partition Δ such that f is strictly monotone on each small interval of Δ . Such a partition Δ is called a monotone partition for f . Henceforth in this paper, any transformation of S^1 is to be piecewise monotone and continuous.

LEMMA 3 For a monotone partition Δ for f ,

$$h(f) \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \log m(\Delta_n)$$

PROOF Given an arbitrary covering α of S^1 , one can

choose a partition Δ' such that (1) Δ' contains as a subset the prescribed monotone partition Δ for f , (2) each small interval

of Δ' is contained in some member of $\mathcal{O}$. Then $m(\Delta'_n) \geq N(\mathcal{O} \vee f^{-1}\mathcal{O} \vee f^{-2}\mathcal{O} \vee \dots \vee f^{-n+1}\mathcal{O})$. (Each small interval of Δ'_n is contained in at least one member of $\mathcal{O} \vee f^{-1}\mathcal{O} \vee \dots \vee f^{-n+1}\mathcal{O}$.) Let J be a small interval of Δ_n . Then because Δ is a monotone partition for f , f^i maps J homeomorphically into some small interval of Δ for each $i \in \{0, 1, \dots, n-1\}$. Thus $m(\Delta'_n \cap J) \leq nb$, where b is so chosen that $b \geq m(\Delta' \cap I)$ for each small interval I of Δ . Hence $m(\Delta'_n) \leq (nb+1) m(\Delta_n)$. Thus

$$N(\mathcal{O} \vee f^{-1}\mathcal{O} \vee \dots \vee f^{-n+1}\mathcal{O}) \leq (nb+1) m(\Delta_n)$$

Letting $n \rightarrow \infty$, we have

$$H(f, \mathcal{O}) \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \log m(\Delta_n).$$

Now as $\mathcal{O}$ was an arbitrary covering of S^1 , we have done the proof.

3° A partition Δ is called a fine partition for a transformation f of S^1 , in case (1) f embeds each small interval of Δ into S^1 , (2) the length of each small interval of Δ , as well as that of its image by f , is smaller than $\frac{1}{3}$ of the whole length of S^1 . Thus a fine partition for f is automatically a monotone partition for f . The next lemma is a converse of lemma 3.

LEMMA 4 For a fine partition Δ for f ,

$$h(f) \geq \limsup_{n \rightarrow \infty} \frac{1}{n} \log m(\Delta_n).$$

PROOF Our proof is analogous to the argument employed by Bowen in his paper [2] .

Given a positive integer N , we shall construct an open covering, say $\mathcal{O}_N$, out of the partition $\Delta = \{x_1, x_2, \dots, x_r\}$. For each point x_j of Δ , consider the open interval U_j whose endpoints are those points of Δ_N which are next to x_j . These U_j 's, together with interiors of small intervals I_j 's of Δ ($1 \leq j \leq r$), constitute an open covering $\mathcal{O}_N$ of S^1 . For each member $B = A_0 \cap f^{-1}A_1 \cap \dots \cap f^{-n+1}A_{n-1}$ of $\mathcal{O}_N \vee f^{-1}\mathcal{O}_N \vee \dots \vee f^{-n+1}\mathcal{O}_N$ (A_i is either some U_j or some I_j), we shall count the number of the small intervals of Δ_n which intersects B . Notice that a small interval of Δ_n is of the form $I_{i_0} \cap f^{-1}I_{i_1} \cap f^{-2}I_{i_2} \cap \dots \cap f^{-n+1}I_{i_{n-1}}$. (Here we use the assumption that Δ is fine.) Now look at the sequence $A_0, A_1, \dots, A_{n-1}$ which defines B . We shall construct a subsequence in the following way. Let i_1 be the smallest number, if any, such that A_{i_1} is equal to some U_j , then delete the $(N-1)$ terms $A_{i_1+1}, \dots, A_{i_1+N-1}$ from the sequence. Consider the new sequence, and let i_2 be the next number, if any, such that $A_{i_2} = U_j$ for some U_j . Then delete the $N-1$ terms $A_{i_2+1}, \dots, A_{i_2+N-1}$ from the sequence. Proceeding in this fashion, one obtains finally a subsequence, say, $A_{j_1}, A_{j_2}, \dots, A_{j_s}$. Notice that if $A_{j_k} = U_j$, then A_{j_k} is contained in a union of two small intervals of Δ_N . Thus $f^{-j_1}(A_{j_1}) \cap \dots \cap f^{-j_s}(A_{j_s})$, hence a fortiori B , is contained in a union of at most $2^{\frac{n}{N}+1}$ small intervals of Δ_n . Thus
$$N(\mathcal{O}_N \vee \dots \vee f^{-n+1}\mathcal{O}_N) \cdot 2^{\frac{n}{N}+1} \geq m(\Delta_n).$$

Letting $n \rightarrow \infty$, one gets

$$H(f, \mathcal{O}_N) + \frac{1}{N} \log 2 \geq \limsup_{n \rightarrow \infty} \frac{1}{n} \log m(\Delta_n).$$

As N is arbitrary, one obtains

$h(f) \geq \limsup_{n \rightarrow \infty} \frac{1}{n} \log m(\Delta_n)$, as is desired.

4° In this paragraph we shall complete the proof of theorems 1 and 2. To begin with one has the following easy consequence of lemmas 3 and 4.

PROPOSITION 5 Let Δ be an arbitrary monotone partition for a transformation f of S^1 . Then the following limit exists and is equal to $h(f)$.

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log m(\Delta_n)$$

PROOF Given a monotone partition Δ for f , there exists a fine partition Δ' for f such that $\Delta' \supset \Delta$. Then

$$\begin{aligned} h(f) &\geq \limsup_{n \rightarrow \infty} \frac{1}{n} \log m(\Delta'_n) \geq \liminf_{n \rightarrow \infty} \frac{1}{n} \log m(\Delta'_n) \\ &\geq \liminf_{n \rightarrow \infty} \frac{1}{n} \log m(\Delta_n) \geq h(f) . \end{aligned}$$

Hence $h(f) = \lim_{n \rightarrow \infty} \frac{1}{n} \log m(\Delta_n)$, as is desired.

PROOF OF THEOREM 1 Let f be a transformation of S^1 , which is not a local homeomorphism. Let Δ_f be a monotone partition for f with the smallest cardinality. Of course such a partition is unique and each small interval of Δ_f is a maximal interval where f is monotone. Then clearly $L(f) = m((\Delta_f)_n)$. Thus Theorem 1 is implied by Proposition 5.

PROOF OF THEOREM 2 (1) Notice that $m(f^{-1}\{x\}) \geq \deg(f)$ for each point x of S^1 . Thus for a monotone partition Δ for f , one has $m(\Delta_n) \geq m(f^{-n+1}\Delta) \geq m(f^{-n+1}\{x_0\}) \geq (\deg(f))^{n-1}$, where $x_0 \in \Delta$. Hence $h(f) = \lim_{n \rightarrow \infty} \frac{1}{n} \log m(\Delta_n) \geq \log \deg(f)$.

PROOF OF THEOREM 2 (2) Let x be an arbitrary point of S^1 . Then $\{x\}$ is a monotone partition for a local homeomorphism f . Thus $h(f) = \lim_{n \rightarrow \infty} \frac{1}{n} \log m(\{x\}_n)$. Now $m(f^{-i}\{x\}) = (\deg(f))^i$. Thus $(\deg(f))^{n-1} \leq m(\{x\}_n) \leq 1 + \deg(f) + \deg(f)^2 + \dots + \deg(f)^{n-1}$. Hence we get $h(f) = \log \deg(f)$.

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