

Several properties on Aluthge transformation

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This report is based on the following preprints:

- [Y1] T.Yamazaki, *Characterizations of $\log A \geq \log B$ and normaloid operators via Heinz inequality*, preprint.
- [Y2] T.Yamazaki, *Parallelisms between Aluthge transformation and powers of operators*, to appear in Acta Sci. Math. (Szeged).
- [Y3] T.Yamazaki, *An expression of spectral radius via Aluthge transformation*, preprint.

ABSTRACT

In 1990, Aluthge defined an operator transformation $\tilde{T}$ of T by $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, where $T = U|T|$ is the polar decomposition of T . This transformation has very interesting properties, and many authors call $\tilde{T}$ *Aluthge transformation* and have studied properties of this transformation.

In this paper, firstly, we shall show properties of Aluthge transformation on operator norm, and a characterization of normaloid operators by giving a definition to n -th Aluthge transformation $\widetilde{\widetilde{T_n}} = \widetilde{(\widetilde{T_{n-1}})}$.

Secondly, we shall point out that there are parallelisms between Aluthge transformation and powers of operators. Moreover we shall show $\lim_{n \rightarrow \infty} \|\widetilde{T_n}\| = r(T)$ which is a parallel result to $\lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}} = r(T)$.

Lastly, we shall discuss relations between the orders $|\tilde{T}|^p \geq |T|^p$ and $|T|^{p-1} \geq |T^*|^{p-1}$ for some positive number p .

1. INTRODUCTION

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H . For each $p > 0$, an operator T is said to be *p-hyponormal* if $|T|^{2p} \geq |T^*|^{2p}$, where $|T| = (T^*T)^{\frac{1}{2}}$. Especially, an operator T is said to be *hyponormal* if T is 1-hyponormal. It is well known that “every *p-hyponormal* operator is *q-hyponormal* for $p \geq q > 0$.” And it is also well known that “for each $q > 0$, there

exists a q -hyponormal and non- p -hyponormal operator for any $p > q > 0$." Especially, there exists a $\frac{1}{2}$ -hyponormal and non-hyponormal operator. Relating to these facts, many authors have studied some operator transformations from $\frac{1}{2}$ -hyponormal operator to hyponormal operator. And the following two operator transformations were obtained:

Let $T = U|T|$ be the polar decomposition of T .

- (i) $S = U|T|^{\frac{1}{2}}$.
- (ii) $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ (Aluthge transformation [1]).

If T is $\frac{1}{2}$ -hyponormal, then both S and $\tilde{T}$ are hyponormal. Moreover, it was shown that $\sigma(T) = \sigma(\tilde{T})$ in [3, 4, 11], where $\sigma(T)$ is the spectrum of T . So we understand that Aluthge transformation is a better transformation than (i).

In this paper, we shall show several properties of Aluthge transformation as follows: Firstly, it is well known that $\|T\| \geq \|\tilde{T}\|$ holds for all operator T . Relating to this fact, we shall show a characterization of the condition $\|T\| = \|\tilde{T}\|$, and generalize this result by giving a definition to " n -th Aluthge transformation". An operator T is said to be *normaloid* if $\|T\| = r(T)$, where $r(T)$ is the spectral radius of T . It is well known that "for each $p > 0$, every p -hyponormal operator is normaloid." Moreover we shall show a characterization of normaloid operators via Aluthge transformation.

Secondly, we shall show a parallel result to powers of p -hyponormal operators for $p \in [0, 1]$ via n -th Aluthge transformation. And we shall show a new expression of spectral radius via Aluthge transformation.

Lastly, we shall discuss relations between the orders $|T|^p \geq |T^*|^p$ and $|\tilde{T}|^{p-1} \geq |T|^{p-1}$ for some positive number p .

2. A CHARACTERIZATION OF NORMALOID OPERATORS

Fujii, Izumino and Nakamoto [6] showed the following characterization of normaloid operators via Aluthge transformation as follows:

Theorem A ([6]). *Let $T \in B(H)$. Then the following assertions are mutually equivalent:*

- (1) T is normaloid.
- (2) $\|T\| = \|\tilde{T}\|$ and $\tilde{T}$ is normaloid (i.e., $\|\tilde{T}\| = r(\tilde{T})$).

In this section, we shall discuss the condition under which $\|T\| = \|\tilde{T}\|$ and normaloidness of T via Aluthge transformation. First, we obtain the following result:

Theorem 1 ([Y1]). Let $T \in B(H)$. Then for each natural number n , the following assertions are equivalent:

- (1) $\|T\| = \|T^{n+1}\|^{\frac{1}{n+1}}$.
- (2) $\|T\| = \|\tilde{T}^n\|^{\frac{1}{n}}$.

Remark 1. Put $n = 1$ in Theorem 1, we obtain the following equivalence relation:

$$(2.1) \quad \|T\| = \|T^2\|^{\frac{1}{2}} \iff \|T\| = \|\tilde{T}\|.$$

To prove Theorem 1, we cite the following norm inequality:

Theorem B ([10]). Let A and B be positive operators, and $X \in B(H)$. Then the following inequalities hold:

- (i) $\|A^r X B^r\| \leq \|A X B\|^r \|X\|^{1-r}$ for $r \in [0, 1]$.
- (ii) $\|A^r X B^r\| \geq \|A X B\|^r \|X\|^{1-r}$ for $r > 1$.

Proof of Theorem 1. Let $T = U|T|$ be the polar decomposition of T .

Proof of (1) $\implies$ (2). Assume that $\|T\| = \|T^{n+1}\|^{\frac{1}{n+1}}$. Then we have

$$\begin{aligned} \|T\| &= \|T^{n+1}\|^{\frac{1}{n+1}} \\ &= \| |T|^{\frac{1}{2}} (|T|^{\frac{1}{2}} U |T|^{\frac{1}{2}})^n |T|^{\frac{1}{2}} \|^{\frac{1}{n+1}} \\ &\leq \|T\|^{\frac{1}{n+1}} \|\tilde{T}^n\|^{\frac{1}{n+1}}. \end{aligned}$$

Hence $\|T\| \leq \|\tilde{T}^n\|^{\frac{1}{n}} \leq \|\tilde{T}\| \leq \|T\|$ hold.

Proof of (2) $\implies$ (1). Assume that $\|T\| = \|\tilde{T}^n\|^{\frac{1}{n}}$. Then by (i) of Theorem B for $\frac{1}{2} \in [0, 1]$, we have

$$\begin{aligned} \|T\| &= \|\tilde{T}^n\|^{\frac{1}{n}} \\ &= \|(|T|^{\frac{1}{2}} U |T|^{\frac{1}{2}})^n\|^{\frac{1}{n}} \\ &= \| |T|^{\frac{1}{2}} (U |T|)^{n-1} U |T|^{\frac{1}{2}} \|^{\frac{1}{n}} \\ &\leq \left\{ \| |T| (U |T|)^{n-1} U |T|^{\frac{1}{2}} \| \cdot \| (U |T|)^{n-1} U \|^{\frac{1}{2}} \right\}^{\frac{1}{n}} \\ &\leq \|T^{n+1}\|^{\frac{1}{2n}} \cdot \|T^{n-1}\|^{\frac{1}{2n}} \\ &\leq \|T^{n+1}\|^{\frac{1}{2n}} \cdot \|T\|^{\frac{n-1}{2n}}. \end{aligned}$$

Hence we obtain

$$\|T\| \leq \|T^{n+1}\|^{\frac{1}{n+1}} \leq \|T\|.$$

Therefore the proof of Theorem 1 is complete. $\square$

By considering the following “ n -th Aluthge transformation”, we obtain another generalization of (2.1).

Definition 1 (n -th Aluthge transformation [Y1]). Let $T \in B(H)$ and $T = U|T|$ be the polar decomposition of T . Then for each natural number n , n -th Aluthge transformation $\widetilde{T}_n$ of T is defined by $\widetilde{T}_n = \widetilde{(T_{n-1})}$ and $\widetilde{T}_1 = \widetilde{T}$.

Theorem 2 ([Y1]). Let $T \in B(H)$. Then for each natural number n , the following assertions are equivalent:

- (1) $\|T\| = \|T^{k+1}\|^{\frac{1}{k+1}}$ for all $k = 1, 2, \dots, n$.
- (2) $\|T\| = \|\widetilde{T}_k\|$ for all $k = 1, 2, \dots, n$.

Proof. We shall prove Theorem 2 by induction on n .

In case $n = 1$. Theorem 2 holds by (2.1).

Assume that Theorem 2 holds in case $n = m$.

In case $n = m + 1$.

Proof of (1) $\implies$ (2). Suppose that

$$\|T\| = \|T^2\|^{\frac{1}{2}} = \dots = \|T^{m+2}\|^{\frac{1}{m+2}}.$$

Then we have

$$(2.2) \quad \|T\| = \|\widetilde{T}\| = \dots = \|\widetilde{T}^{m+1}\|^{\frac{1}{m+1}}$$

by Theorem 1. Put $S = \widetilde{T}$ in (2.2). Then (2.2) asserts

$$\|S\| = \|S^2\|^{\frac{1}{2}} = \dots = \|S^{m+1}\|^{\frac{1}{m+1}}.$$

By the induction hypothesis for the case $n = m$, we have

$$(2.3) \quad \|\widetilde{T}\| = \|S\| = \|\widetilde{S}\| = \dots = \|\widetilde{S}_m\| = \|\widetilde{(\widetilde{T}_m)}\| = \|\widetilde{T}_{m+1}\|.$$

Hence we obtain

$$\|T\| = \|\widetilde{T}\| = \dots = \|\widetilde{T}_{m+1}\|$$

by (2.2) and (2.3).

Proof of (2) $\implies$ (1). Suppose that

$$(2.4) \quad \|T\| = \|\widetilde{T}\| = \dots = \|\widetilde{T}_{m+1}\| = \|\widetilde{(\widetilde{T})}_m\|.$$

Put $S = \widetilde{T}$ in (2.4). Then we have

$$\|S\| = \|\widetilde{S}\| = \dots = \|\widetilde{S}_m\|$$

By the induction hypothesis for the case $n = m$, we have

$$\|T\| = \|\widetilde{T}\| = \|S\| = \|S^2\|^{\frac{1}{2}} = \dots = \|S^{m+1}\|^{\frac{1}{m+1}} = \|\widetilde{T}^{m+1}\|^{\frac{1}{m+1}}.$$

Hence we have

$$\|T\| = \|T^k\|^{\frac{1}{k}} \quad \text{for all } k = 1, 2, \dots, m+2$$

by Theorem 1.

Therefore the proof of Theorem 2 is complete. $\square$

By Theorem 2, we obtain the following Corollary 3 which is a characterization of normaloid operators, immediately.

Corollary 3 ([Y1]). *Let $T \in B(H)$. Then the following assertions are equivalent:*

- (1) *T is normaloid.*
- (2) *$\|T\| = \|\widetilde{T_n}\|$ for all natural number n .*

By Corollary 3, we can obtain Theorem A, easily as follows:

T is normaloid

$$\iff \|T\| = \|\widetilde{T}\| = \|\widetilde{T_n}\| = \|\widetilde{(\widetilde{T})_{n-1}}\| \quad \text{for all natural number } n \text{ by Corollary 3}$$

$$\iff \|T\| = \|\widetilde{T}\| \text{ and } \widetilde{T} \text{ is normaloid by Corollary 3.}$$

Proof of Corollary 3. We recall the following well-known result:

$$T \text{ is normaloid} \iff \|T\| = \|T^n\|^{\frac{1}{n}} \text{ for all positive integer } n.$$

Hence we obtain Corollary 3 by Theorem 2. $\square$

3. PARALLEL RESULTS BETWEEN

ALUTHGE TRANSFORMATION AND POWERS OF OPERATORS

It was shown that “there exists a hyponormal operator T such that T^2 is not hyponormal” in [9]. Relating to this fact, Aluthge and Wang [2] showed that “if T is a p -hyponormal operator for $p \in (0, 1]$, then T^n is $\frac{p}{n}$ -hyponormal for all natural number n .” As an extension of this result, the following result was shown in [8]:

Theorem C ([8]). *Let T be a p -hyponormal operator for $p \in (0, 1]$. Then for each natural number n , the following inequalities hold:*

- (i) $|T|^{2(p+1)} \leq |T^2|^{p+1} \leq \dots \leq |T^n|^{\frac{2(p+1)}{n}}$.
- (ii) $|T^*|^{2(p+1)} \geq |T^{*2}|^{p+1} \geq \dots \geq |T^{*n}|^{\frac{2(p+1)}{n}}$.

We remark that as a generalization of the result by Aluthge and Wang, Ito [12] showed that “if T is a p -hyponormal operator for $p > 0$, then T^n is $\min\{1, \frac{p}{n}\}$ -hyponormal for all natural number n .” And he showed an extension of Theorem C. As a parallel result to Theorem C, we obtain the following result:

Theorem 4 ([Y2]). Let T be a $\frac{p}{2}$ -hyponormal operator for $p \in (0, 1]$. Then for each natural number n , the following inequalities hold:

- (i) $|T|^{p+1} \leq |\widetilde{T}|^{p+1} \leq \dots \leq |\widetilde{T}_n|^{p+1}$.
- (ii) $|T^*|^{p+1} \geq |\widetilde{T}^*|^{p+1} \geq \dots \geq |\widetilde{T}_n^*|^{p+1}$.

To prove Theorem 4, we prepare the following results and definition:

Theorem D ([1, 7, 11, 16]). Let T be a $\frac{p}{2}$ -hyponormal operator for $p > 0$ (i.e., $|T|^p \geq |T^*|^p$). Then the following inequalities hold:

- (i) In case $p \in (0, 1]$. $|\widetilde{T}|^{p+1} \geq |T|^{p+1} \geq |(\widetilde{T})^*|^{p+1}$
(i.e., $\widetilde{T}$ is $\frac{p+1}{2}$ -hyponormal).
- (ii) In case $p \in [1, 2]$. $|\widetilde{T}|^2 \geq |T|^2 \geq |(\widetilde{T})^*|^2$ (i.e., $\widetilde{T}$ is hyponormal).

Theorem D was shown in [1] when U is unitary, where $T = U|T|$ is the polar decomposition of T . And Theorem D was shown in [11, 16]. Moreover, a generalization of Theorem D was shown in [7, 11, 16].

Definition 2 (*-Aluthge transformation [Y2]). Let $T = U|T|$ be the polar decomposition of an operator T . Then *-Aluthge transformation of T is defined as follows:

- (i) $\widetilde{T}^{(*)} \stackrel{\text{def}}{=} (\widetilde{T}^*)^* = |T^*|^{\frac{1}{2}} U |T^*|^{\frac{1}{2}}$ (*-Aluthge transformation).
- (ii) For each natural number n ,

$$\widetilde{T}_n^{(*)} \stackrel{\text{def}}{=} \widetilde{(\widetilde{T}_{n-1}^{(*)})}^{(*)} = (\widetilde{T}_n^*)^* \text{ and } \widetilde{T}_1^{(*)} \stackrel{\text{def}}{=} \widetilde{T}^{(*)}$$
 (n -th *-Aluthge transformation).

As relations between $\widetilde{T}$ and $\widetilde{T}^{(*)}$, we obtain the following results, immediately.

Theorem 5 ([Y2]). Let $T \in B(H)$. Then the following assertions hold:

- (i) $\sigma(\widetilde{T}) = \sigma(\widetilde{T}^{(*)}) = \sigma(T)$.
- (ii) $w(\widetilde{T}) = w(\widetilde{T}^{(*)})$, where $w(T)$ is the numerical radius of T .
- (iii) $\|\widetilde{T}\| = \|\widetilde{T}^{(*)}\|$.

We remark that (i) of Theorem 5 asserts more generalization form of $\sigma(\widetilde{T}) - \{0\} = \sigma(\widetilde{T}^{(*)}) - \{0\} = \sigma(T) - \{0\}$. And $\sigma(\widetilde{T}) = \sigma(T)$ has been already shown in [3, 4, 11].

Proposition 6 ([Y2]). Let $T \in B(H)$. Then for each $p > 0$,

$$\widetilde{T} \text{ is } p\text{-hyponormal} \iff \widetilde{T}^{(*)} \text{ is } p\text{-hyponormal}.$$

Remark 2. If T is $\frac{p}{2}$ -hyponormal for $p \in (0, 2]$, then we obtain the following assertions, easily. $\widetilde{T}_n$ is $\frac{p}{2}$ -hyponormal for all natural number n by using Theorem D

several times, and also $\widetilde{T}_n^{(*)}$ is $\frac{p}{2}$ -hyponormal for all natural number n by Theorem D and Proposition 6 several times.

Proof of Theorem 4. We shall prove Theorem 4 by induction on n .

Proof of (i). (a) By (i) of Theorem D, we have $|\widetilde{T}|^{p+1} \geq |T|^{p+1}$.

(b) Assume that $|\widetilde{T}_{n-1}|^{p+1} \geq \dots \geq |\widetilde{T}|^{p+1} \geq |T|^{p+1}$.

(c) Proof of $|\widetilde{T}_n|^{p+1} \geq |\widetilde{T}_{n-1}|^{p+1}$.

Put $S = \widetilde{T}_{n-1}$. Then S is also $\frac{p}{2}$ -hyponormal by Remark 2. Hence we have

$$|\widetilde{T}_n|^{p+1} = |\widetilde{S}|^{p+1} \geq |S|^{p+1} = |\widetilde{T}_{n-1}|^{p+1} \quad \text{by (i) of Theorem D.}$$

Proof of (ii). Let $T = U|T|$ be the polar decomposition of T .

(a) Proof of $|T^*|^{p+1} \geq |\widetilde{T}^*|^{p+1}$.

$$\begin{aligned} |T^*|^{p+1} &= U|T|^{p+1}U^* \\ &\geq U(\widetilde{T})^*|^{p+1}U^* \quad \text{by (i) of Theorem D} \\ &= U(|T|^{\frac{1}{2}}U|T|U^*|T|^{\frac{1}{2}})^{\frac{p+1}{2}}U^* \\ &= (U|T|^{\frac{1}{2}}U|T|U^*|T|^{\frac{1}{2}}U^*)^{\frac{p+1}{2}} \\ &= (|T^*|^{\frac{1}{2}}U|T^*|U^*|T^*|^{\frac{1}{2}})^{\frac{p+1}{2}} \\ &= |\widetilde{T}^*|^{p+1}. \end{aligned}$$

(b) Assume that $|T^*|^{p+1} \geq |\widetilde{T}^*|^{p+1} \geq \dots \geq |\widetilde{T}_{n-1}^*|^{p+1}$.

(c) Proof of $|\widetilde{T}_{n-1}^*|^{p+1} \geq |\widetilde{T}_n^*|^{p+1}$.

Put $S = (\widetilde{T}_{n-1}^*)^* = \widetilde{T}_{n-1}^{(*)}$. Then S is also $\frac{p}{2}$ -hyponormal by Remark 2. By (a), we obtain

$$|\widetilde{T}_{n-1}^*|^{p+1} = |S^*|^{p+1} \geq |\widetilde{S}^*|^{p+1} = |\widetilde{T}_n^*|^{p+1}.$$

Therefore the proof of Theorem 4 is complete. $\square$

By considering Theorem 4, we can understand that n -th Aluthge transformation and powers of operators have similar properties. On the other hand, $\lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}} = r(T)$ is a very famous and useful result. So we shall show the parallel result to this one as follows:

Theorem 7 ([Y3]). *Let $T \in B(H)$. Then $\lim_{n \rightarrow \infty} \|\widetilde{T}_n\| = r(T)$.*

To prove Theorem 7, we prepare the following lemmas:

Lemma 8 ([Y3]). For a natural number n and $k = 0, 1, \dots, n+1$, let

$$(3.1) \quad {}_n D_k = \frac{n!(n-2k+1)}{k!(n-k+1)!}.$$

Then the following assertions hold:

- (i) ${}_n D_0 = 1$ for all natural number n .
- (ii) ${}_n D_k + {}_n D_{k+1} = {}_{n+1} D_{k+1}$ for all natural number n and $k = 0, 1, \dots, n$.
- (iii) ${}_{2n+1} D_n = {}_{2n+2} D_{n+1}$ for all natural number n .
- (iv) $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (n-2k+1) {}_n D_k = 2^n$,
where $\lfloor \frac{n}{2} \rfloor$ is the largest integer satisfying $\lfloor \frac{n}{2} \rfloor \leq \frac{n}{2}$.
- (v) $\lim_{n \rightarrow \infty} \frac{(n-2k+1) {}_n D_k}{2^n} = 0$ for all positive integer k .

Lemma 9 ([Y3]). Let $T \in B(H)$. Then

$$\|\widetilde{T}^n\| \leq \|T^{n+1}\|^{\frac{1}{2}} \|T^{n-1}\|^{\frac{1}{2}}$$

holds for all natural number n .

Proof. Let $T = U|T|$ be the polar decomposition of T . Then we have

$$\begin{aligned} \|\widetilde{T}^n\| &= \|(|T|^{\frac{1}{2}} U |T|^{\frac{1}{2}})^n\| = \| |T|^{\frac{1}{2}} (U |T|)^{n-1} U |T|^{\frac{1}{2}} \| \\ &\leq \| |T| (U |T|)^{n-1} U |T| \|^{\frac{1}{2}} \| (U |T|)^{n-1} U \|^{\frac{1}{2}} \quad \text{by (i) of Theorem B} \\ &\leq \|T^{n+1}\|^{\frac{1}{2}} \|T^{n-1}\|^{\frac{1}{2}}. \end{aligned}$$

□

Lemma 10 ([Y3]). Let $T \in B(H)$ and $m = \lfloor \frac{n}{2} \rfloor$. Then

$$\|\widetilde{T}_n\| \leq \|T^{n+1}\|^{\frac{n D_0}{2^n}} \|T^{n-1}\|^{\frac{n D_1}{2^n}} \dots \|T^{n-2k+1}\|^{\frac{n D_k}{2^n}} \dots \|T^{n-2m+1}\|^{\frac{n D_m}{2^n}}.$$

Proof. We shall prove Lemma 10 by induction on n .

(a) $\|\widetilde{T}\| \leq \|T^2\|^{\frac{1}{2}}$ holds by Lemma 9.

(b) Assume that

$$(3.2) \quad \begin{aligned} \|\widetilde{T}_{n-1}\| &\leq \|T^n\|^{\frac{n-1 D_0}{2^{n-1}}} \|T^{n-2}\|^{\frac{n-1 D_1}{2^{n-1}}} \\ &\quad \times \dots \times \|T^{n-2k}\|^{\frac{n-1 D_k}{2^{n-1}}} \dots \|T^{n-2m}\|^{\frac{n-1 D_m}{2^{n-1}}}, \end{aligned}$$

where $m = \lfloor \frac{n-1}{2} \rfloor$.

(c-1) In case $n = 2m + 1$ for $m = 1, 2, \dots$. Then $\lfloor \frac{n}{2} \rfloor = \lfloor \frac{n-1}{2} \rfloor = m$. Hence by (3.2), we have

$$\begin{aligned}
\|\widetilde{T}_n\| &= \|\widetilde{(T)}_{n-1}\| \\
&\leq \|\widetilde{T}^n\|^{\frac{n-1D_0}{2^{n-1}}} \|\widetilde{T}^{n-2}\|^{\frac{n-1D_1}{2^{n-1}}} \\
&\quad \times \dots \times \|\widetilde{T}^{n-2k+2}\|^{\frac{n-1D_{k-1}}{2^{n-1}}} \|\widetilde{T}^{n-2k}\|^{\frac{n-1D_k}{2^{n-1}}} \dots \|\widetilde{T}^3\|^{\frac{n-1D_{m-1}}{2^{n-1}}} \|\widetilde{T}\|^{\frac{n-1D_m}{2^{n-1}}} \\
&\leq \left(\|T^{n+1}\|^{\frac{1}{2}} \|T^{n-1}\|^{\frac{1}{2}} \right)^{\frac{n-1D_0}{2^{n-1}}} \left(\|T^{n-1}\|^{\frac{1}{2}} \|T^{n-3}\|^{\frac{1}{2}} \right)^{\frac{n-1D_1}{2^{n-1}}} \\
&\quad \times \dots \times \left(\|T^{n-2k+3}\|^{\frac{1}{2}} \|T^{n-2k+1}\|^{\frac{1}{2}} \right)^{\frac{n-1D_{k-1}}{2^{n-1}}} \left(\|T^{n-2k+1}\|^{\frac{1}{2}} \|T^{n-2k-1}\|^{\frac{1}{2}} \right)^{\frac{n-1D_k}{2^{n-1}}} \\
&\quad \times \dots \times \left(\|T^4\|^{\frac{1}{2}} \|T^2\|^{\frac{1}{2}} \right)^{\frac{n-1D_{m-1}}{2^{n-1}}} \|T^2\|^{\frac{n-1D_m}{2^{n-1}}} \\
&= \|T^{n+1}\|^{\frac{n-1D_0}{2^n}} \|T^{n-1}\|^{\frac{n-1D_0+n-1D_1}{2^n}} \\
&\quad \times \dots \times \|T^{n-2k+1}\|^{\frac{n-1D_{k-1}+n-1D_k}{2^n}} \dots \|T^2\|^{\frac{n-1D_{m-1}+n-1D_m}{2^n}} \\
&= \|T^{n+1}\|^{\frac{nD_0}{2^n}} \|T^{n-1}\|^{\frac{nD_1}{2^n}} \dots \|T^{n-2k+1}\|^{\frac{nD_k}{2^n}} \dots \|T^2\|^{\frac{nD_m}{2^n}}
\end{aligned}$$

by (i) and (ii) of Lemma 8, and the last inequality holds by Lemma 9.

(c-2) In case $n = 2m + 2$ for $m = 0, 1, 2, \dots$. Then $\lfloor \frac{n}{2} \rfloor = m + 1$ and $\lfloor \frac{n-1}{2} \rfloor = m$. Hence by (3.2), we have

$$\begin{aligned}
\|\widetilde{T}_n\| &= \|\widetilde{(T)}_{n-1}\| \\
&\leq \|\widetilde{T}^n\|^{\frac{n-1D_0}{2^{n-1}}} \|\widetilde{T}^{n-2}\|^{\frac{n-1D_1}{2^{n-1}}} \\
&\quad \times \dots \times \|\widetilde{T}^{n-2k+2}\|^{\frac{n-1D_{k-1}}{2^{n-1}}} \|\widetilde{T}^{n-2k}\|^{\frac{n-1D_k}{2^{n-1}}} \dots \|\widetilde{T}^4\|^{\frac{n-1D_{m-1}}{2^{n-1}}} \|\widetilde{T}^2\|^{\frac{n-1D_m}{2^{n-1}}} \\
&\leq \left(\|T^{n+1}\|^{\frac{1}{2}} \|T^{n-1}\|^{\frac{1}{2}} \right)^{\frac{n-1D_0}{2^{n-1}}} \left(\|T^{n-1}\|^{\frac{1}{2}} \|T^{n-3}\|^{\frac{1}{2}} \right)^{\frac{n-1D_1}{2^{n-1}}} \\
&\quad \times \dots \times \left(\|T^{n-2k+3}\|^{\frac{1}{2}} \|T^{n-2k+1}\|^{\frac{1}{2}} \right)^{\frac{n-1D_{k-1}}{2^{n-1}}} \left(\|T^{n-2k+1}\|^{\frac{1}{2}} \|T^{n-2k-1}\|^{\frac{1}{2}} \right)^{\frac{n-1D_k}{2^{n-1}}} \\
&\quad \times \dots \times \left(\|T^5\|^{\frac{1}{2}} \|T^3\|^{\frac{1}{2}} \right)^{\frac{n-1D_{m-1}}{2^{n-1}}} \left(\|T^3\|^{\frac{1}{2}} \|T\|^{\frac{1}{2}} \right)^{\frac{n-1D_m}{2^{n-1}}} \\
&= \|T^{n+1}\|^{\frac{n-1D_0}{2^n}} \|T^{n-1}\|^{\frac{n-1D_0+n-1D_1}{2^n}} \\
&\quad \times \dots \times \|T^{n-2k+1}\|^{\frac{n-1D_{k-1}+n-1D_k}{2^n}} \dots \|T^3\|^{\frac{n-1D_{m-1}+n-1D_m}{2^n}} \|T\|^{\frac{n-1D_m}{2^n}} \\
&= \|T^{n+1}\|^{\frac{nD_0}{2^n}} \|T^{n-1}\|^{\frac{nD_1}{2^n}} \dots \|T^{n-2k+1}\|^{\frac{nD_k}{2^n}} \dots \|T^3\|^{\frac{nD_m}{2^n}} \|T\|^{\frac{nD_{m+1}}{2^n}}
\end{aligned}$$

by (i), (ii) and (iii) of Lemma 8, and the last inequality holds by Lemma 9. $\square$

Lemma 11 ([Y3]). Let $\{a_n\}_{n=1}^{\infty}$ be a sequence satisfying $\lim_{n \rightarrow \infty} a_n = a$, and for each natural number n , $\{c_{n,k}\}_{k=1}^n$ be a positive sequence satisfying

$$(3.3) \quad c_{n,1} + \cdots + c_{n,k} + \cdots + c_{n,n} = 1 \quad \text{for all natural number } n$$

and $\lim_{n \rightarrow \infty} c_{n,k} = 0$ for fixed $k = 1, 2, \dots$. Then

$$\lim_{n \rightarrow \infty} (c_{n,1}a_1 + \cdots + c_{n,k}a_k + \cdots + c_{n,n}a_n) = a.$$

Proof of Theorem 7. Let $m = [\frac{n}{2}]$. Then by Lemma 10, (iv) of Lemma 8 and Arithmetic mean-Geometric mean inequality, we have

$$\begin{aligned} r(T) = r(\widetilde{T}_n) &\leq \|\widetilde{T}_n\| \leq \|T^{n+1}\|^{\frac{nD_0}{2^n}} \|T^{n-1}\|^{\frac{nD_1}{2^n}} \\ &\quad \dots \|T^{n-2k+1}\|^{\frac{nD_k}{2^n}} \dots \|T^{n-2m+1}\|^{\frac{nD_m}{2^n}} \\ &\leq \frac{(n+1)_n D_0}{2^n} \|T^{n+1}\|^{\frac{1}{n+1}} + \frac{(n-1)_n D_1}{2^n} \|T^{n-1}\|^{\frac{1}{n-1}} \\ &\quad + \dots + \frac{(n-2k+1)_n D_k}{2^n} \|T^{n-2k+1}\|^{\frac{1}{n-2k+1}} \\ &\quad + \dots + \frac{(n-2m+1)_n D_m}{2^n} \|T^{n-2m+1}\|^{\frac{1}{n-2m+1}} \\ &\longrightarrow r(T) \quad \text{as } n \rightarrow \infty \end{aligned}$$

by $\lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}} = r(T)$, (iv) and (v) of Lemma 8 and Lemma 11. $\square$

4. RELATIONS BETWEEN THE ORDERS $|\widetilde{T}|^{p-1} \geq |T|^{p-1}$ AND $|T|^p \geq |T^*|^p$

In this section, we shall discuss properties of the order $|\widetilde{T}|^p \geq |T|^p$ for some $p > 0$. Relating to this order, Theorem D is very famous. As a converse of Theorem D, we obtain the following result:

Theorem 12 ([Y2]). Let T be an invertible operator. Then the following assertions hold:

- (i) For each $p \in [2, 4]$, $|\widetilde{T}|^p \geq |T|^p$ ensures $|T|^{p-1} \geq |T^*|^{p-1}$.
- (ii) For each $p \geq 4$, $|\widetilde{T}|^p \geq |T|^p$ ensures $|T|^3 \geq |T^*|^3$.

To prove Theorem 12, we need the following result:

Theorem E ([5, 13, 14, 15]). Let A and B be positive invertible operators. Then the following assertions hold:

- (i) $A \geq B > 0$ ensures $(B^{-\frac{t}{2}} A^p B^{-\frac{t}{2}})^{\frac{1-t}{p-t}} \geq B^{1-t}$ for $1 \geq p \geq \frac{1}{2}$ with $p > t \geq 0$.
- (ii) $A \geq B > 0$ ensures $(B^{-\frac{t}{2}} A^p B^{-\frac{t}{2}})^{\frac{2p-t}{p-t}} \geq B^{2p-t}$ for $\frac{1}{2} \geq p > t \geq 0$.

Proof of Theorem 12. Let $T = U|T|$ be the polar decomposition of T . Then U is unitary since T is invertible.

Proof of (i). By applying (i) of Theorem E to $|\tilde{T}|^p \geq |T|^p$, we have

$$(4.1) \quad (|T|^{-\frac{t_1 p}{2}} |\tilde{T}|^{p p_1} |T|^{-\frac{t_1 p}{2}})^{\frac{1-t_1}{p_1-t_1}} \geq |T|^{p(1-t_1)}$$

for $1 \geq p_1 \geq \frac{1}{2}$ with $p_1 > t_1 \geq 0$.

Put $p_1 = \frac{2}{p}$ and $t_1 = \frac{1}{p}$ in (4.1). Then we have

$$(|T|^{-\frac{1}{2}} |\tilde{T}|^2 |T|^{-\frac{1}{2}})^{p-1} \geq |T|^{p-1}.$$

It is equivalent to

$$(|T|^{-\frac{1}{2}} |T|^{\frac{1}{2}} U^* |T| U |T|^{\frac{1}{2}} |T|^{-\frac{1}{2}})^{p-1} \geq |T|^{p-1},$$

that is, $U^* |T|^{p-1} U \geq |T|^{p-1}$ since U is unitary. Hence we have $|T|^{p-1} \geq U |T|^{p-1} U^* = |T^*|^{p-1}$.

Proof of (ii). By applying (ii) of Theorem E to $|\tilde{T}|^p \geq |T|^p$, we have

$$(4.2) \quad (|T|^{-\frac{t_1 p}{2}} |\tilde{T}|^{p p_1} |T|^{-\frac{t_1 p}{2}})^{\frac{2p_1-t_1}{p_1-t_1}} \geq |T|^{p(2p_1-t_1)}$$

for $\frac{1}{2} \geq p_1 > t_1 \geq 0$.

Put $p_1 = \frac{2}{p}$ and $t_1 = \frac{1}{p}$ in (4.2). Then we have

$$(|T|^{-\frac{1}{2}} |\tilde{T}|^2 |T|^{-\frac{1}{2}})^3 \geq |T|^3.$$

It is equivalent to

$$(|T|^{-\frac{1}{2}} |T|^{\frac{1}{2}} U^* |T| U |T|^{\frac{1}{2}} |T|^{-\frac{1}{2}})^3 \geq |T|^3,$$

that is, $U^* |T|^3 U \geq |T|^3$ since U is unitary. Hence we have $|T|^3 \geq U |T|^3 U^* = |T^*|^3$. $\square$

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