



K-stability is a relatively new yet highly fruitful notion in geometry. Initially introduced to detect the existence of Kähler–Einstein (KE) metrics on Fano manifolds, it turned out to be more powerful, most notably it provides the “right” platform for constructing compact moduli spaces for Fano varieties, and others. However, we are yet to have a full understanding of K-stability. For example, it is conjectured that all smooth complete intersection Fano varieties are K-stable, with substantial progress made in the case of hypersurfaces. In this work, we extend the study to Fano complete intersections of codimension two. Throughout, we work over the field of complex numbers $\mathbb{C}$.

What is K-stability?

Calabi Problem

Let (X,J) be a complex manifold. Does there exist a metric ω on X such that (X,ω,J) is a Kähler manifold and the Einstein equation

$$\mathrm{Ric}_\omega = \lambda \omega$$

is satisfied?

If $\lambda < 0$, a KE metric always exists (Aubin, Yau 1978).

If $\lambda = 0$ (Calabi-Yau manifold), a KE metric always exists (Yau 1978).

If $\lambda > 0$ (Fano manifold), the existence is obstructed and we need to investigate further.

For instance, the projective plane $\mathbb{P}^2$ is KE, but the blowup of $\mathbb{P}^2$ at a point is not.

Theorem (Yau–Tian–Donaldson Conjecture)

A Fano manifold X admits a Kähler–Einstein metric if and only if X is K-polystable.

Algebro-Geometric Approach: a valuative criterion

Here we adopt an equivalent criterion as definition of K-stability instead of the original one. The equivalence is proved in [1] and [2].

Let X be an n -dimensional Fano variety. The *stability threshold* of X is defined as

$$\delta(X) = \inf_{E/X} \frac{A_X(E)}{S_X(E)}$$

The infimum is taken over all birational morphisms $\pi : Y \rightarrow X$ and prime divisors $E \subset Y$.

Here $A_X(E)$ denotes the log discrepancy of E , given by $\mathrm{ord}_E(K_Y - \pi^*(K_X)) + 1$; and

$$S_X(E) = \frac{1}{(-K_X)^n} \int_0^\infty \mathrm{vol}_Y(\pi^*(-K_X) - tE) dt.$$

Then: X is K-stable (resp. K-semistable) $\Leftrightarrow \delta(X) > 1$ (resp. ≥ 1).

Let $Z \subset X$ be a subvariety, the local stability threshold at Z is defined as

$$\delta_Z(X) = \inf_{E, C_X(E) \supset Z} \frac{A_X(E)}{S_X(E)},$$

here the infimum is taken over all prime divisor $E \subset Y$ with a birational map $\pi : Y \rightarrow X$ such that $Z \subset \pi(E) =: C_X(E)$.

Clearly, $\delta(X) = \inf_{x \in X} \delta_x(X)$, the infimum is taken over all closed points $x \in X$.

Abban–Zhuang method

The Abban–Zhuang method provides a general approach to proving K-stability by estimating local stability threshold, by reducing the dimension of the varieties. For details, we refer to the original article [3].

Given a subvariety $Z \subset X$ and an admissible flag $X = Y_0 \supset Y_1 \supset \cdots \supset Y_i$, this provides

$$\delta_Z(X) \geq \min \left\{ \min_{0 \leq i \leq j-1} \left\{ \frac{A_{Y_i}(Y_{i+1})}{S(W_{\bullet}^{(i)}; Y_{i+1})} \right\}, \delta_{Z \cap Y_j}(Y_j; W_{\bullet}^{(j)}, \mathcal{F}) \right\}. \quad (1)$$

The notions here are natural generalizations of S -invariant and δ -invariant defined above to multigraded linear series. It allows us to study stability of X via lower-dimensional data.

If we take Z to be a closed point $x \in X$, then (1) provides a lower bound of $\delta_x(X)$. By choosing the flag appropriately one can show $\delta_x(X) > 1$ for all $x \in X$.

If we moreover take the flag of maximum length, i.e. $X = Y_0 \supset Y_1 \supset \cdots \supset Y_n = \{x\}$. The finite numbers obtained from right hand side of (1) can be thought of as $(\frac{1}{b_1}, \dots, \frac{1}{b_n})$, where $(b_1, \dots, b_n)$ are coordinates of the barycenter of the Okounkov body associated to the flag w.r.t. $-K_X$. For example, we have $b_1 = \frac{A_X(Y_1)}{S_X(Y_1)}$.

Known results

- All smooth quadric hypersurfaces are K-polystable.
- A general degree d hypersurface in $\mathbb{P}^n$ is K-stable when $2 < d \leq n$.
- ([4]) All Fano hypersurfaces of Fano index 1 are K-stable;
- ([3]) All smooth Fano hypersurfaces of Fano index 2 are K-stable;
- ([5]) All smooth Fano hypersurfaces of Fano index $r \geq 3$ and dimension $n \geq r^3$ are K-stable.

Here the Fano index of X is defined as the largest integer r such that the anticanonical divisor $-K_X \sim_{\mathbb{Q}} rL$ for some Cartier divisor L .

Main Results

1. A general Fano codimension 2 complete intersection of index 1 is K-stable.
2. Every smooth hypersurface of index 1 in a quadric (that is, an n -dimensional smooth complete intersection of hypersurfaces of degree 2 and n in $\mathbb{P}^{n+2}$) is K-stable.

Proof of main result 1

Let $Y = \{g_m(x_1, \dots, x_{n+2}) = 0, x_0^k = f_k(x_1, \dots, x_{n+2})\} \subset \mathbb{P}^{n+2}$ be an n -dimensional Fano complete intersection of a degree m and a degree k hypersurface, where g_m, f_k are homogeneous polynomials of degree m and k , respectively. We have $m + k = n + 2$ when the Fano index of Y is 1. Then Y is a cyclic Galois covering over an n -fold $X = \{g_m = 0\} \subset \mathbb{P}^{n+1}$, branched along divisor $D = \{f_k = 0\}|_X$. Choose g_m and f_k such that both X and D are smooth subvarieties of $\mathbb{P}^{n+1}$. By adjunction, $K_X = \mathcal{O}_{\mathbb{P}^{n+1}}(-n - 2 + m)|_X = \mathcal{O}_{\mathbb{P}^{n+1}}(-k)|_X$, and clearly $D \in |-K_X|$.

It is known that Fermat hypersurfaces are K-polystable. Therefore we pick $g_m = x_1^m + \cdots + x_{n+2}^m$. By the following lemma we can conclude that Y is K-stable. By openness of K-stability, a general Y is K-stable. $\square$

Lemma 1. ([6]) Let $Y \rightarrow X$ be a cyclic Galois covering of smooth Fano varieties with smooth branch divisor $D \in |- \lambda K_X|$ for $\lambda \geq 1$. If X is K-semistable, then Y is K-stable.

Sketch of proof of main result 2

The cases $n = 2, 3$ are proved by Tian and by Zhuang [7], respectively. We only consider the case $n \geq 4$.

Step 1:

Let $x \in X$ be a closed point, let E_x be the exceptional divisor of the ordinary blowup of x , then we have the following inequality ([7])

$$S(-K_X; E_x) = S(H; E_x) \leq \frac{1}{n+1} \tau(H; E_x) + \frac{n-1}{n+1} \eta(H; E_x) \leq \frac{2n}{n+1} + \frac{n-1}{n+1} \cdot \sqrt{2n} < n,$$

when $n \geq 3$. Therefore $A_X(E_x) - S_X(E_x) > 0$ for any $x \in X$.

Step 2:

Let $C \subset X$ be a curve. Fix two closed points $P, Q \in C$, and consider the flag

$$Y_{\bullet} : X = Y_0 \supset Y_1 \supset \cdots \supset Y_{n-1} \ni Y_n$$

such that Y_1 is a general smooth hyperplane section intersecting C at finitely many points including P and Q , and Y_i is a smooth general hyperplane section in Y_{i-1} passing through P, Q for $1 \leq i \leq n-2$. When the line $\overline{PQ}$ joining P and Q is not contained in X , we pick Y_{n-1} in the same way, and pick Y_n to be a closed point in Y_{n-1} not in C . When the line $\overline{PQ}$ is contained in X but distinct from C , we pick Y_{n-1} to be this line and Y_n to be a closed point in Y_{n-1} not in C . When C is a line, we pick Y_{n-1} as C itself and finish the flag at Y_{n-1} .

The Abban-Zhuang method gives the inequality

$$\delta_C(X; -K_X) \geq \min \left\{ \min_{0 \leq i \leq n-2} \left\{ \frac{A_{Y_i}(Y_{i+1})}{S(W_{\bullet}^{(i)}; Y_{i+1})} \right\}, \delta_{C \cap Y_{n-1}}(Y_{n-1}; W_{\bullet}^{(j)}, \mathcal{F}) \right\} \quad (2)$$

(or $\delta_C(X; -K_X) \geq \min \left\{ \min_{0 \leq i \leq n-3} \left\{ \frac{A_{Y_i}(Y_{i+1})}{S(W_{\bullet}^{(i)}; Y_{i+1})} \right\}, \delta_C(Y_{n-2}; W_{\bullet}^{(j)}, \mathcal{F}) \right\}$ when C is a line.)

We compute that $S(W_{\bullet}^{(i)}; Y_{i+1}) = \frac{1}{n+1}$ for $0 \leq i \leq n-3$, which is also true for $i = n-2$ if $\overline{PQ}$ is not contained in X ; otherwise $S(W_{\bullet}^{(i)}; Y_{n-2}) \leq \frac{3}{n+1}(\frac{4}{3} - \frac{5}{6n}) < \frac{n}{n+1}$. From this we can deduce that the last term in inequality (2) is $\geq \frac{n+1}{n}$. Therefore $\delta_C(X; -K_X) \geq \frac{n+1}{n}$.

Using the following lemma, we obtain the desired conclusion. $\square$

Lemma 2. ([3]) Let X be a Fano manifold of dimension n . Assume that

1. $\delta_Z(X) \geq \frac{n+1}{n}$ for any subvariety $Z \subset X$ of dimension ≥ 1 ;
2. $A_X(E_x) - S_X(E_x) > 0$ for any $x \in X$, where E_x denotes the exceptional divisor of the ordinary blowup of x .

Then X is K-stable.

References

- [1] K. Fujita and Y. Odaka, “On the K-stability of Fano varieties and anticanonical divisors,” *Tohoku Math. J.* (2), vol. 70, no. 4, pp. 511–521, 2018.
- [2] H. Blum and M. Jonsson, “Thresholds, valuations, and K-stability,” *Advances in mathematics* (New York, 1965), vol. 365, p. 107062, 2020.
- [3] H. Abban and Z. Zhuang, “K-stability of Fano varieties via admissible flags,” *Forum of Mathematics, Pi*, vol. 10, 2022.
- [4] K. Fujita, “K-stability of Fano manifolds with not small alpha invariants,” *Journal of the Institute of Mathematics of Jussieu*, vol. 18, no. 3, p. 519–530, 2019.
- [5] H. Abban and Z. Zhuang, “Seshadri constants and K-stability of Fano manifolds,” *Duke Mathematical Journal*, vol. 172, no. 6, pp. 1109–1144, 2023.
- [6] R. Dervan, “On K-stability of finite covers,” *The Bulletin of the London Mathematical Society*, vol. 48, no. 4, pp. 717–728, 2016.
- [7] Z. Zhuang, “Birational superrigidity and K-stability of Fano complete intersections of index 1,” *Duke mathematical journal*, vol. 169, no. 12, pp. 2205–2229, 2020.