

Relations between two inequalities $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ and $A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$ and their applications

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Abstract

Let A and B be positive invertible operators. Then for each $p \geq 0$ and $r \geq 0$, two inequalities

$$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r \text{ and } A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$$

are equivalent. In this report, we shall show relations between these inequalities in case A and B are not invertible. And we shall show some applications of this result to operator classes.

1 Introduction

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H . An operator T is said to be *positive* (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$.

As a recent development on order preserving operator inequalities, it is known the following Theorem F.

Theorem F (Furuta inequality [9]).

If $A \geq B \geq 0$, then for each $r \geq 0$,

$$(i) \quad (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

$$(ii) \quad (A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

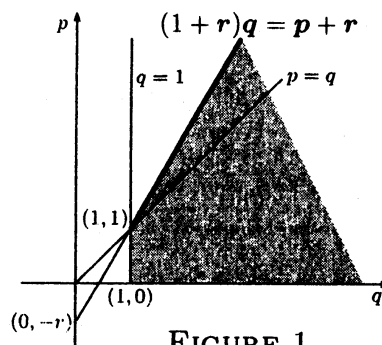


FIGURE 1

Theorem F yields the famous Löwner-Heinz theorem “ $A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$ ” by putting $r = 0$ in (i) or (ii) of Theorem F. Alternative proofs of Theorem F are given in [6] and [18] and also an elementary one page proof in [10]. It was shown by Tanahashi [19] that the domain drawn for p, q and r in the Figure 1 is the best possible one for Theorem F.

As an application of Theorem F, the following result was shown in [7] and [11].

Theorem FC ([7][11]). *Let $A, B > 0$. Then the following assertions are mutually equivalent:*

- (i) $\log A \geq \log B$.
- (ii) $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ for all $p \geq 0$ and $r \geq 0$.
- (iii) $A^r \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$ for all $p \geq 0$ and $r \geq 0$.

We remark that this result is an extension of [4] in case $p = r$, and an excellent proof of this result which used only Theorem F was shown in [22].

On the other hand, the following assertions are well known: Let A and B be positive invertible operators. Then

- (1) $A \geq B \implies \log A \geq \log B$.
- (2) $\log A \geq \log B \implies (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ and $A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$ for all $p \geq 0$ and $r \geq 0$.
- (3) For each $p \geq 0$ and $r \geq 0$, $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r \iff A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$.

(1) holds since $\log t$ is an operator monotone function. (2) is an immediate consequence of Theorem FC. (3) was shown in [11].

Related to these results, it is known in [23] that invertibility of (1) and (2) can be replaced with the condition $N(A) = N(B) = \{0\}$, that is, (1) and (2) hold for some non-invertible operators A and B . But we have not known whether invertibility of A and B in (3) can be replaced with looser condition or not. In this report, we shall show relations between

$$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r \quad \text{and} \quad A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$$

when A and B are not invertible.

Next, An operator T is said to be *hyponormal* if $T^*T \geq TT^*$. An operator T is *invertible log-hyponormal* (defined in [20]) if $\log T^*T \geq \log TT^*$. For each $s > 0$ and $t > 0$, an operator T belongs to *class* $A(s, t)$ (defined in [8]) if $(|T^*|^t |T|^{2s} |T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$, where $|T| = (T^*T)^{\frac{1}{2}}$. Class $A(s, t)$ is introduced as a generalization of *class* A

($|T^2| \geq |T|^2$) defined in [14]. We remark that class A equals class $A(1,1)$ and class A is introduced as a class of operators including invertible log-hyponormal operators and included in the class of *paranormal* operators ($\|T^2x\| \geq \|Tx\|^2$ for all unit vectors $x \in H$). Moreover, for each $s > 0$ and $t > 0$, an operator T belongs to class $wA(s,t)$ (defined in [16]) if $(|T^*|^t |T|^{2s} |T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$ and $|T|^{2s} \geq (|T|^s |T^*|^{2t} |T|^s)^{\frac{s}{s+t}}$. Obviously, for each $s > 0$ and $t > 0$, every class $wA(s,t)$ operator belongs to class $A(s,t)$. As inclusion relations among these classes, the following assertions hold by (1), (2) and (3):

- (1)' Every invertible hyponormal operator is log-hyponormal.
- (2)' Every invertible log-hyponormal operator belongs to class $wA(s,t)$ for all $s > 0$ and $t > 0$.
- (3)' For each $s > 0$ and $t > 0$, invertible class $wA(s,t)$ equals invertible class $A(s,t)$.

There are many papers on these classes in case of invertible operators, for example [8], [20] and [24].

On the other hand, even if an operator is non-invertible, log-hyponormality can be defined by $N(T^*) \supset N(T)$ and $\log A \geq \log B$, where A and B are the compressions of T^*T and TT^* to $\overline{R(T)}$, respectively. This definition implicitly appeared in [3] and it was pointed out in [23] that it is the general form of log-hyponormality. Ando [3] showed that every hyponormal operator is log-hyponormal and every log-hyponormal operator is paranormal. Moreover, Uchiyama [23] showed that every log-hyponormal operator is also included in class A (even if an operator is non-invertible). In this report, we shall show that for each $s > 0$ and $t > 0$, class $A(s,t)$ coincides with class $wA(s,t)$, that is, we shall show (3)' without invertibility of operators, and show some properties of class $A(s,t)$ operators. Lastly, we shall show a normality of class $A(s,t)$ operators for $s > 0$ and $t > 0$.

2 Relations between

$$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r \text{ and } A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$$

In this section, we shall show the following result:

Theorem 1. *Let A and B be positive operators. Then for each $p \geq 0$ and $r \geq 0$, the following assertions hold:*

- (i) *If $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$, then $A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$.*
- (ii) *If $A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$ and $N(A) \subset N(B)$, then $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$.*

We remark on Theorem 1 that the assumption of (ii) has a kernel condition $N(A) \subset N(B)$, but the assumption of (i) does not have any kernel conditions. If A and B are invertible, then $N(A) = N(B) = \{0\}$ holds, and the kernel condition of (ii) in Theorem 1 is satisfied. Hence we know that Theorem 1 is a generalization of (3) in the previous section.

To prove Theorem 1, we prepare the following lemma.

Lemma 2. *Let A and B be positive operators. Then the following assertions hold:*

- (i) $\lim_{\varepsilon \rightarrow +0} A^{\frac{1}{2}}(A + \varepsilon I)^{-1}A^{\frac{1}{2}} = \lim_{\varepsilon \rightarrow +0} (A + \varepsilon I)^{-1}A = P_{N(A)^\perp},$
where $P_{\mathcal{M}}$ is the projection onto a closed subspace $\mathcal{M}$.
- (ii) $\lim_{\varepsilon \rightarrow +0} A^{\frac{1}{2}}B^{\frac{1}{2}}\{(B^{\frac{1}{2}}AB^{\frac{1}{2}})^\alpha + \varepsilon I\}^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}} = (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{1-\alpha}$ for $\alpha \in (0, 1)$.

We remark that if A and B are both positive invertible, then

$$A^{\frac{1}{2}}B^{\frac{1}{2}}(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{-\alpha}B^{\frac{1}{2}}A^{\frac{1}{2}} = (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{1-\alpha} \text{ for } \alpha \in (0, 1)$$

by the following Lemma F. Therefore we can regard (ii) of Lemma 2 as a non-invertible version of Lemma F for $\lambda \in (0, 1)$.

Lemma F ([12]). *Let A be a positive invertible operator and B be an invertible operator. Then*

$$(BAB^*)^\lambda = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^*$$

holds for any real number λ .

Proof of Lemma 2.

Proof of (i). We give a proof which is a slightly modification of the proof of [5, Lemma].

Let $A = \int_0^{\|A\|} t dF(t)$ be the spectral decomposition of A . Then

$$\lim_{\varepsilon \rightarrow +0} (A + \varepsilon I)^{-1}A = \lim_{\varepsilon \rightarrow +0} \int_0^{\|A\|} \frac{t}{t + \varepsilon} dF(t) = \int_0^{\|A\|} \chi_{(0, \|A\|]}(t) dF(t) = I - F(0),$$

where $\chi_{(0, \|A\|]}(t)$ is the characteristic function on $(0, \|A\|]$. Since $I - F(0) = P_{N(A)^\perp}$, we have

$$\lim_{\varepsilon \rightarrow +0} A^{\frac{1}{2}}(A + \varepsilon I)^{-1}A^{\frac{1}{2}} = \lim_{\varepsilon \rightarrow +0} (A + \varepsilon I)^{-1}A = P_{N(A)^\perp}.$$

Proof of (ii). Let $A^{\frac{1}{2}}B^{\frac{1}{2}} = U|A^{\frac{1}{2}}B^{\frac{1}{2}}|$ be the polar decomposition of $A^{\frac{1}{2}}B^{\frac{1}{2}}$.

Then we have

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow +0} A^{\frac{1}{2}} B^{\frac{1}{2}} \{(B^{\frac{1}{2}} A B^{\frac{1}{2}})^{\alpha} + \varepsilon I\}^{-1} B^{\frac{1}{2}} A^{\frac{1}{2}} \\
&= \lim_{\varepsilon \rightarrow +0} U |A^{\frac{1}{2}} B^{\frac{1}{2}}|^{1-\alpha} |A^{\frac{1}{2}} B^{\frac{1}{2}}|^{\alpha} (|A^{\frac{1}{2}} B^{\frac{1}{2}}|^{2\alpha} + \varepsilon I)^{-1} |A^{\frac{1}{2}} B^{\frac{1}{2}}|^{\alpha} |A^{\frac{1}{2}} B^{\frac{1}{2}}|^{1-\alpha} U^* \\
&= U |A^{\frac{1}{2}} B^{\frac{1}{2}}|^{1-\alpha} P_{N(|A^{\frac{1}{2}} B^{\frac{1}{2}}|)^{\perp}} |A^{\frac{1}{2}} B^{\frac{1}{2}}|^{1-\alpha} U^* \quad \text{by (i)} \\
&= U |A^{\frac{1}{2}} B^{\frac{1}{2}}|^{2(1-\alpha)} U^* = |B^{\frac{1}{2}} A^{\frac{1}{2}}|^{2(1-\alpha)} = (A^{\frac{1}{2}} B A^{\frac{1}{2}})^{1-\alpha}.
\end{aligned}$$

Hence the proof is complete. $\square$

Proof of Theorem 1. Let $\varepsilon > 0$. And also we may assume $p > 0$ and $r > 0$.

Proof of (i). Since $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$, we obtain

$$A^{\frac{p}{2}} B^{\frac{r}{2}} (B^r + \varepsilon I)^{-1} B^{\frac{r}{2}} A^{\frac{p}{2}} \geq A^{\frac{p}{2}} B^{\frac{r}{2}} \{(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} + \varepsilon I\}^{-1} B^{\frac{r}{2}} A^{\frac{p}{2}}. \quad (2.1)$$

In (2.1), by tending $\varepsilon \rightarrow +0$ and Lemma 2, we obtain

$$A^{\frac{p}{2}} P_{N(B)^{\perp}} A^{\frac{p}{2}} \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}.$$

Hence we have

$$A^p \geq A^{\frac{p}{2}} P_{N(B)^{\perp}} A^{\frac{p}{2}} \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}.$$

Proof of (ii). Since $A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$, we obtain

$$B^{\frac{r}{2}} A^{\frac{p}{2}} (A^p + \varepsilon I)^{-1} A^{\frac{p}{2}} B^{\frac{r}{2}} \leq B^{\frac{r}{2}} A^{\frac{p}{2}} \{(A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}} + \varepsilon I\}^{-1} A^{\frac{p}{2}} B^{\frac{r}{2}}. \quad (2.2)$$

In (2.2), by tending $\varepsilon \rightarrow +0$ and Lemma 2, we obtain

$$B^{\frac{r}{2}} P_{N(A)^{\perp}} B^{\frac{r}{2}} \leq (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}}. \quad (2.3)$$

On the other hand,

$$N(A) \subset N(B) \iff P_{N(A)^{\perp}} \geq P_{N(B)^{\perp}}. \quad (2.4)$$

By (2.3) and (2.4), we have

$$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^{\frac{r}{2}} P_{N(A)^{\perp}} B^{\frac{r}{2}} \geq B^{\frac{r}{2}} P_{N(B)^{\perp}} B^{\frac{r}{2}} = B^r.$$

Therefore the proof is complete. $\square$

Remark. We recall the assumptions of (i) and (ii) in Theorem 1. Here we assume $p = r = 1$ in Theorem 1.

$$(i-a) \quad (B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \geq B.$$

$$(ii-a) \quad A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}} \text{ and } N(A) \subset N(B).$$

We proved that (i-a) ensures $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$ and (ii-a) ensures (i-a) in Theorem 1, so we might expect that (i-a) and (ii-a) are equivalent. But we have the following counterexample.

Example 1. $(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \geq B$ and $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$, but $N(A) \not\subset N(B)$.

Let $A = 2 \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $A^{\frac{1}{2}} = \sqrt{\frac{2}{5}} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, $B^{\frac{1}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = B$. Hence

$$\sqrt{2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \geq B$$

and

$$A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}} = \frac{\sqrt{2}}{5} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

But $\begin{pmatrix} -2 \\ 1 \end{pmatrix} \in N(A)$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix} \notin N(B)$, so that $N(A) \not\subset N(B)$.

Moreover, we have the following example in [16].

Example 2 ([16]). $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$, but $(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \not\geq B$ and $N(A) \not\subset N(B)$.

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then we can check $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$, $(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \not\geq B$ and $N(A) \not\subset N(B)$, easily.

Therefore we recognize that $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$ requires some condition to be equivalent to $(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \geq B$. So we consider the following condition.

$$(ii-a') \quad A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}} \text{ and } N(A^{\frac{1}{2}}B^{\frac{1}{2}}) \subset N(B).$$

We can easily check that $N(A) \subset N(B)$ ensures $N(A^{\frac{1}{2}}B^{\frac{1}{2}}) \subset N(B)$. And also $(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \geq B$ ensures $N(A^{\frac{1}{2}}B^{\frac{1}{2}}) \subset N(B)$ since $N(A^{\frac{1}{2}}B^{\frac{1}{2}}) = N(B^{\frac{1}{2}}AB^{\frac{1}{2}}) = N((B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}}) \subset N(B)$, so that (i-a) ensures (ii-a') by (i) in Theorem 1.

But, unfortunately, we understand that (ii-a') does not ensure (i-a) by the following example.

Example 3. $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$ and $N(A^{\frac{1}{2}}B^{\frac{1}{2}}) \subset N(B)$, but $(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \not\geq B$.

Let $A = \frac{1}{2} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $A^{\frac{1}{2}} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, $B^{\frac{1}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = B$. Hence

$$A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}} = \frac{1}{\sqrt{50}} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

and $N(A^{\frac{1}{2}}B^{\frac{1}{2}}) = N(B) = \left\{ t \begin{pmatrix} 0 \\ 1 \end{pmatrix} : t \in \mathbb{C} \right\}$ since $A^{\frac{1}{2}}B^{\frac{1}{2}} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$. But

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \not\geq B.$$

At the end of this remark, we note that Chō-Huruya-Kim [5] gave an example such that $N(T) \not\subset N(T^*)$, $N(T) \not\supset N(T^*)$ and $|\tilde{T}| \geq |T| \geq |(\tilde{T})^*|$ (i.e., T is w -hyponormal) by using A and B in Example 1 stated above, where $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ and $T = U|T|$ is the polar decomposition of T .

3 Applications

In this section, we shall show some applications of Theorem 1 to operator classes. In section 1, we introduced definitions of some operator classes, here we recall definitions of these classes as follows:

Definition 1. Let $s > 0$, $t > 0$ and $T = U|T|$ be the polar decomposition of T .

(i) T belongs to class $A(s, t) \iff (|T^*|^t |T|^{2s} |T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$.

(ii) T belongs to class $wA(s, t)$

$$\iff (|T^*|^t |T|^{2s} |T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t} \text{ and } |T|^{2s} \geq (|T|^s |T^*|^{2t} |T|^s)^{\frac{s}{s+t}}$$

$$\iff |\tilde{T}_{s,t}|^{\frac{2t}{s+t}} \geq |T|^{2t} \text{ and } |T|^{2s} \geq |(\tilde{T}_{s,t})^*|^{\frac{2s}{s+t}},$$

where $\tilde{T}_{s,t} = |T|^s U |T|^t$ (generalized Aluthge transformation).

(iii) T belongs to class $A \iff |T^2| \geq |T|^2$, that is, T belongs to class $A(1, 1)$.

(iv) T is w -hyponormal $\iff |\tilde{T}| \geq |T| \geq |(\tilde{T})^*|$, that is, T belongs to class $wA(\frac{1}{2}, \frac{1}{2})$, where $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ (Aluthge transformation).

(i), (ii), (iii) and (iv) of Definition 1 were defined in [8], [16], [14] and [2], respectively. We remark that Aluthge transformation has many interesting properties, and many authors study this transformation, for instance, [1], [13], [15] and [17]. These classes

include invertible log-hyponormal operators, and are included in normaloid (i.e., $\|T\| = r(T)$, where $r(T)$ is the spectral radius of T). It has been known that for each $s > 0$ and $t > 0$, class $A(s, t)$ includes class $wA(s, t)$ by the definitions (i) and (ii). And also for each $s > 0$ and $t > 0$, every invertible class $A(s, t)$ operator is an invertible class $wA(s, t)$ operator, which was shown in [8] and [16]. More precise inclusion relations among class $wA(s, t)$, and powers of class $wA(s, t)$ operators were already shown as follows:

Theorem A ([16], [26]).

- (i) For each $s > 0$ and $t > 0$, every class $wA(s, t)$ operator is a class $wA(\alpha, \beta)$ operator for any $\alpha \geq s$ and $\beta \geq t$.
- (ii) Let T be a class $wA(s, t)$ operator for $s \in (0, 1]$ and $t \in (0, 1]$. Then for each natural number n , T^n belongs to class $wA(\frac{s}{n}, \frac{t}{n})$.

We remark that Theorem A holds for classes of invertible class $A(s, t)$ operators instead of class $wA(s, t)$, which were shown in [8] and [25]. We can summarize inclusion relations among these classes as the following Figure 2. Dotted lines in the diagram mean that we need invertibility of operators to prove the relations.

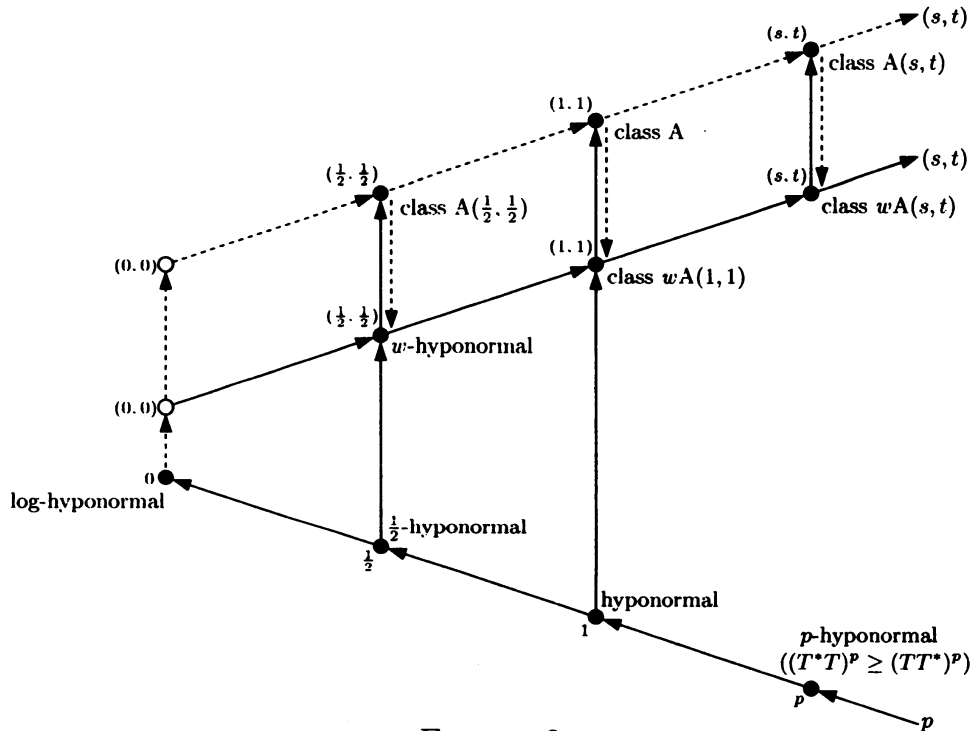


FIGURE 2

Here, in general, we can obtain that class $A(s, t)$ coincides with class $wA(s, t)$ by (i) of Theorem 1 as follows:

Theorem 3. For each $s > 0$ and $t > 0$, the following assertions hold:

- (i) Class $A(s, t)$ coincides with class $wA(s, t)$.
- (ii) Class A coincides with class $wA(1, 1)$.
- (iii) Class $A(\frac{1}{2}, \frac{1}{2})$ coincides with the class of w -hyponormal operators, i.e., class $wA(\frac{1}{2}, \frac{1}{2})$.

We can prove Theorem 3 by only applying (i) of Theorem 1 to definitions of these classes, so we omit to prove. By (iii) of Theorem 3, we have

$$\begin{aligned} |\tilde{T}| \geq |T| &\iff (|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}})^{\frac{1}{2}} \geq |T^*| \iff T : \text{class } A(\frac{1}{2}, \frac{1}{2}) \\ &\iff T : w\text{-hyponormal} \iff |\tilde{T}| \geq |T| \geq |(\tilde{T})^*|. \end{aligned}$$

Hence

$$|\tilde{T}| \geq |T| \implies |T| \geq |(\tilde{T})^*|,$$

that is, we may as well define w -hyponormality by only $|\tilde{T}| \geq |T|$.

Next, we shall show some properties of class $A(s, t)$ operators without the assumption of invertibility, which are known as properties of invertible class $A(s, t)$ operators and class $wA(s, t)$ operators.

Theorem 4.

- (i) For each $s > 0$ and $t > 0$, every class $A(s, t)$ operator is a class $A(\alpha, \beta)$ operator for any $\alpha \geq s$ and $\beta \geq t$.
- (ii) Let T be a class $A(s, t)$ operator for $s \in (0, 1]$ and $t \in (0, 1]$. Then for each natural number n , T^n belongs to class $A(\frac{s}{n}, \frac{t}{n})$.

Proof is very easy by (i) of Theorem 3 and Theorem A, so we omit the proof, too. By putting $s = t = 1$ in (ii) of Theorem 4 and noting that class $A(\frac{1}{2}, \frac{1}{2})$ equals w -hyponormality by (iii) of Theorem 3, we obtain the following result on powers of class A operators without the assumption of invertibility.

Corollary 5. Let T be a class A operator. Then for each natural number n , T^n belongs to class $A(\frac{1}{n}, \frac{1}{n})$. Especially T^2 is w -hyponormal.

At the end of this section, we shall summarize relations among these classes which are obtained in this section as follows: Please compare Figure 3 with Figure 2 stated

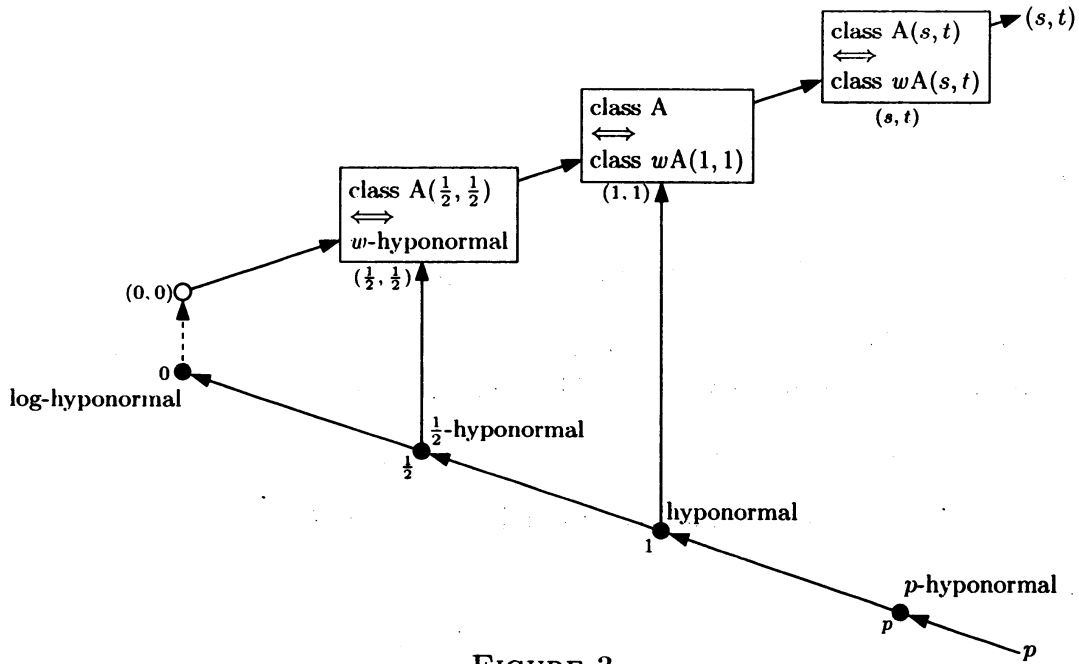


FIGURE 3

4 Normality

In this section, we shall show a normality of some non-normal operators. It is well known that if T and T^* are hyponormal, then T is normal. But in the case T and T^* belong to weaker class than hyponormal, this assertion is not obvious. Many authors obtained many results on this problem, and the following results were known until now.

Theorem B ([21]). *If T is a class A operator and T^* is a w -hyponormal operator, then T is normal.*

Theorem C ([3]). *If T and T^* are paranormal operators satisfying $N(T) = N(T^*)$, then T is normal.*

Here, we shall generalize Theorem B as follows:

Theorem 6. *Let $s_1 > 0$, $s_2 > 0$, $t_1 > 0$ and $t_2 > 0$. If T belongs to class $A(s_1, t_1)$ and T^* belongs to class $A(s_2, t_2)$, then T is normal.*

Put $s_1 = t_1 = 1$ and $s_2 = t_2 = \frac{1}{2}$ in Theorem 6, we have Theorem B by Theorem 3, put $s_1 = t_1 = s_2 = t_2 = 1$ in Theorem 6, we obtain the following result on class A :

Corollary 7. *If T and T^* belong to class A , then T is normal.*

To prove Theorem 6, we need the following results:

Lemma 8. Let A and B be self-adjoint operators, and $X \in B(H)$ satisfying

$$X^*AX \geq X^*BX.$$

If $N(A) \supset N(X^*)$ and $N(B) \supset N(X^*)$, then $A \geq B$.

Proof. Let $H = \overline{R(X)} \oplus N(X^*)$ and $x = Xy + z$, where $y \in H$ and $z \in N(X^*)$. Put $T = A - B$. Then $T = T^*$ and $N(T) \supset N(X^*)$. Hence we have

$$(Tx, x) = (TXy, Xy) + (TXy, z) + (Tz, Xy) + (Tz, z) = (X^*TXy, y) \geq 0,$$

that is, $A \geq B$. □

Proposition 9. Let $A \geq 0$ and $B \geq 0$. If

$$B^{\frac{1}{2}}AB^{\frac{1}{2}} \geq B^2 \tag{4.1}$$

and

$$A^{\frac{1}{2}}BA^{\frac{1}{2}} \geq A^2, \tag{4.2}$$

then $A = B$.

Proof. Put $E = P_{N(A)^\perp}$ and $F = P_{N(B)^\perp}$. (4.1) is equivalent to $B^{\frac{1}{2}}FAFB^{\frac{1}{2}} \geq B^2 = B^{\frac{1}{2}}BB^{\frac{1}{2}}$. By Lemma 8, we have $FAF \geq B$ since $N(FAF) \supset N(B^{\frac{1}{2}})$ and $N(B) = N(B^{\frac{1}{2}})$. Then we have

$$(A^{\frac{1}{2}}FA^{\frac{1}{2}})^2 = A^{\frac{1}{2}}FAFA^{\frac{1}{2}} \geq A^{\frac{1}{2}}BA^{\frac{1}{2}} \geq A^2 \quad \text{by (4.2),}$$

and we obtain the following (4.3) by Löwner-Heinz theorem.

$$A^{\frac{1}{2}}FA^{\frac{1}{2}} \geq A. \tag{4.3}$$

(4.3) is equivalent to $A^{\frac{1}{2}}EFEA^{\frac{1}{2}} \geq A^{\frac{1}{2}}EA^{\frac{1}{2}}$. By Lemma 8, we have $EFE \geq E$ since $N(EFE) \supset N(A^{\frac{1}{2}})$ and $N(E) = N(A^{\frac{1}{2}})$. Therefore we obtain $EFE = E$, and then $F \geq E$, so that $N(A) \supset N(B)$. Hence

$$A \geq B$$

by applying Lemma 8 to (4.1).

By the same way, we also get $B \geq A$, so that $A = B$. □

Proof of Theorem 6. Let $p = \max\{s_1, s_2, t_1, t_2\}$.

Firstly, if T belongs to class $A(s_1, t_1)$, then T belongs to class $A(p, p)$ by (i) of Theorem 4. This class coincides with class $wA(p, p)$ by (i) of Theorem 3. Hence we have

$$(|T^*|^p |T|^{2p} |T^*|^p)^{\frac{1}{2}} \geq |T^*|^{2p} \quad \text{and} \quad |T|^{2p} \geq (|T|^p |T^*|^{2p} |T|^p)^{\frac{1}{2}}. \tag{4.4}$$

Secondly, if T^* belongs to class $A(s_2, t_2)$, then T^* belongs to class $A(p, p)$ by (i) of Theorem 4. This class coincides with class $wA(p, p)$ by (i) of Theorem 3. Hence we have

$$(|T|^p |T^*|^{2p} |T|^p)^{\frac{1}{2}} \geq |T|^{2p} \quad \text{and} \quad |T^*|^{2p} \geq (|T^*|^p |T|^{2p} |T^*|^p)^{\frac{1}{2}}. \quad (4.5)$$

Therefore

$$|T|^p |T^*|^{2p} |T|^p = |T|^{4p} \quad \text{and} \quad |T^*|^{4p} = |T^*|^p |T|^{2p} |T^*|^p$$

hold by (4.4) and (4.5), and then $|T| = |T^*|$ by Proposition 9. $\square$

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