

A note on Sturm-type comparison theorems on a half-open interval

広島大・理 内藤 雄基 (Yūki Naito)

1. Introduction and statement of the results

In this note, we investigate comparison theorems of Sturm-type on a half-open interval $[a, \omega)$, $\omega \leq \infty$. We consider two differential equations

$$(1.1) \quad (p(t)x')' + q(t)x = 0, \quad a \leq t < \omega,$$

$$(1.2) \quad (P(t)y')' + Q(t)y = 0, \quad a \leq t < \omega,$$

where $p(t)$, $q(t)$, $P(t)$, and $Q(t)$ are continuous functions on $[a, \omega)$, and

$$p(t) \geq P(t) > 0 \quad \text{and} \quad Q(t) \geq q(t) \quad \text{on } [a, \omega).$$

In this case, (1.2) is called a Sturm majorant for (1.1) on $[a, \omega)$ and (1.1) is called a Sturm minorant for (1.2).

Sturm's comparison theorem can be stated as follows: (See, e.g., [2, Chap.11, Theorem 3.1].)

Theorem A. *Let $x(t) \not\equiv 0$ be a solution of (1.1) and let $x(t)$ has exactly n (≥ 1) zeros $t = t_1 < t_2 < \cdots < t_n$ in $(a, b]$, $b < \omega$. Let $y(t)$ be a solution of (1.2). If either $x(a) = 0$ or $x(a) \neq 0$, $y(a) \neq 0$, and*

$$\frac{p(a)x'(a)}{x(a)} \geq \frac{P(a)y'(a)}{y(a)},$$

then $y(t)$ has one of the following properties:

- (i) $y(t)$ has at least n zeros in (a, t_n) ;
- (ii) $y(t)$ is a constant multiple of $x(t)$ on $[a, t_n]$ and $p(t) \equiv P(t)$, $q(t) \equiv Q(t)$ on $[a, t_n]$.

Let $x(t) > 0$ in (t_n, ω) in Theorem A. In this case, it seems interesting to ask the question whether a solution $y(t)$ of (1.2) has at least one zero in (t_n, ω) or not?

Assume that (1.1) is nonoscillatory at $t = \omega$. It is well known [2, Chap.11, Theorem 6.4] that (1.1) has a principal solution $x_0(t)$ which is essentially unique (up to a constant factor) such that

$$\int^{\omega} \frac{ds}{p(s)[x_0(s)]^2} = \infty$$

and for any solution $x_1(t)$ linearly independent of $x_0(t)$,

$$\lim_{t \rightarrow \omega} \frac{x_0(t)}{x_1(t)} = 0.$$

The solution $x_1(t)$ is called a nonprincipal solution.

Our main results are the following.

Theorem 1. *Assume that (1.1) is nonoscillatory at $t = \omega$. Let $x_0(t)$ be a principal solution of (1.1) satisfying $x_0(t) > 0$ in (a, ω) . Let $y(t)$ be a solution of (1.2). If either $x_0(a) = 0$ or $x_0(a) \neq 0$, $y(a) \neq 0$, and*

$$(1.3) \quad \frac{p(a)x'_0(a)}{x_0(a)} \geq \frac{P(a)y'(a)}{y(a)},$$

then $y(t)$ has one of the following properties:

- (i) $y(t)$ has at least one zero in (a, ω) ;
- (ii) $y(t)$ is a constant multiple of $x_0(t)$ on $[a, \omega)$ and $p(t) \equiv P(t)$, $q(t) \equiv Q(t)$ on $[a, \omega)$.

Theorem 2. *Assume that (1.1) is nonoscillatory at $t = \omega$. Let $x_0(t)$ be a principal solution of (1.1) and let $x(t)$ has exactly n (≥ 1) zeros in (a, ω) . Let $y(t)$ be a solution of (1.2). If either $x_0(a) = 0$ or $x_0(a) \neq 0$, $y(a) \neq 0$, and (1.3) holds, then $y(t)$ has one of the following properties:*

- (i) $y(t)$ has at least $n + 1$ zeros in (a, ω) ;
- (ii) $y(t)$ is a constant multiple of $x_0(t)$ on $[a, \omega)$ and $p(t) \equiv P(t)$, $q(t) \equiv Q(t)$ on $[a, \omega)$.

Remark. For other results concerning comparison theorems of Sturm-type on a half-open interval, we refer to [4] and [5].

When $p(t) \equiv P(t)$ and $q(t) \equiv Q(t)$ on $[a, \omega)$, as a consequence of Theorems 1 and A, we have the following.

Corollary 1. *Assume that (1.1) is nonoscillatory at $t = \omega$. Let $x_0(t)$ be a principal solution of (1.1) and let t_0 ($\geq a$) be the largest zero, i.e., $x_0(t_0) = 0$ and $x_0(t) > 0$ in (t_0, ω) . Then we have the following properties:*

- (i) every nonprincipal solution has exactly one zero in (t_0, ω) ;
- (ii) every solution of (1.1) has exactly one zero on $[t_0, \omega)$.

Equation (1.1) is said to be disconjugate on an interval J if every solution of (1.1) has at most one zero on J . (See [1] and [2].) By Corollary 1, we obtain a criterion for (1.1) to be disconjugate.

Corollary 2. Assume that (1.1) is nonoscillatory at $t = \omega$. Let $x_0(t)$ be a principal solution of (1.1) and let $t_0 (\geq a)$ be the largest zero. Then (1.1) is disconjugate on $[t_1, \omega)$ if and only if $t_0 \leq t_1$.

Finally, we give a comparison theorem for disconjugacy.

Corollary 3. Assume that (1.2) is nonoscillatory at $t = \omega$. (Then (1.1) is nonoscillatory at $t = \omega$.) Let $x_0(t)$ and $y_0(t)$ be principal solutions of (1.1) and (1.2), respectively. Let t_0 and t_1 ($t_0, t_1 \geq a$) be the largest zeros of $x_0(t)$ and $y_0(t)$, respectively. Then, we have either (i) $t_0 < t_1$ or (ii) $t_0 = t_1$ and $p(t) \equiv P(t)$, $q(t) \equiv Q(t)$ on $[t_0, \omega)$. In particular, if (1.2) is disconjugate on an interval J , then (1.1) is disconjugate on J .

Remark. The comparison theorems for disconjugacy have been shown in [1] by different methods.

2. Proofs of Theorems

We prepare the following lemmas.

Lemma 1. Assume that $q(t) \leq 0$ on $[a, \omega)$ in (1.1). Then (1.1) is nonoscillatory at $t = \omega$ and a principal solution $x_0(t)$ of (1.1) satisfies $x_0(t) > 0$ and $x'_0(t) \leq 0$ on $[a, \omega)$.

Lemma 2. Assume that (1.1) is nonoscillatory at $t = \omega$. Let $x_0(t)$ be a principal solution of (1.1) and let $y(t)$ be a solution of (1.2) satisfying $y(t) > 0$ on $[T, \omega)$, $T \geq a$. Then $x_0(t) > 0$ on $[T, \omega)$ and

$$\frac{p(t)x'_0(t)}{x_0(t)} \leq \frac{P(t)y'(t)}{y(t)} \quad \text{on } [T, \omega).$$

Lemmas 1 and 2 are shown in [2, Chap.11, Corollary 6.4] and [2, Chap.11, Corollary 6.5], respectively. However, for the sake of the completeness, we give (slight simple) proofs of them.

Proof of Lemma 1. Let $x_i(t)$, $i = 1, 2$, be solutions of (1.1) determined by $x_i(a) = 1$ and $x'_i(a) = i$. It is easy to see that $(p(t)x'_i(t))' \geq 0$ and $x_i(t) > 0$ on $[a, \omega)$, $i = 1, 2$. Since $x_1(t)$ and $x_2(t)$ are linearly independent, either $x_1(t)$ or $x_2(t)$ is a nonprincipal solution. Without loss of generality, we may assume that $x_1(t)$ is a nonprincipal solution. By [2, Chap.11, Corollary 6.3],

$$x_0(t) = x_1(t) \int_t^\omega \frac{ds}{p(s)[x_1(s)]^2}, \quad a \leq t < \omega,$$

is well defined and a principal solution of (1.1). We see that $x_0(t) > 0$ on $[a, \omega)$. We obtain

$$x'_0(t) = x'_1(t) \int_t^\omega \frac{ds}{p(s)[x_1(s)]^2} - \frac{1}{p(t)x_1(t)}, \quad a \leq t < \omega.$$

Since $p(t)x'_1(t)$ is nondecreasing and $x_1(t)$ is positive,

$$p(t)x'_0(t) \leq \int_t^\omega \frac{x'_1(s)}{[x_1(s)]^2} ds - \frac{1}{x_1(t)} = -\lim_{s \rightarrow \omega} \frac{1}{x_1(s)} \leq 0, \quad a \leq t < \omega.$$

Thus, we have $x'_0(t) \leq 0$ on $[a, \omega)$. $\square$

Proof of Lemma 2. Let

$$u(t) = \exp \left(\int_T^t \frac{P(s)y'(s)}{p(s)y(s)} ds \right), \quad T \leq t < \omega.$$

Then $u(t) > 0$ on $[T, \omega)$ and satisfies

$$(2.1) \quad \frac{p(t)u'(t)}{u(t)} = \frac{P(t)y'(t)}{y(t)} \quad \text{and} \quad (p(t)u')' + Q_0(t)u = 0 \quad \text{for } T \leq t < \omega,$$

where

$$Q_0(t) = Q(t) + \left(\frac{1}{P(t)} - \frac{1}{p(t)} \right) \left(\frac{P(t)y'(t)}{y(t)} \right)^2, \quad T \leq t < \omega.$$

Let $z(t) = x_0(t)/u(t)$ on $[T, \omega)$. Then $z(t)$ is a solution of

$$(2.2) \quad (p(t)[u(t)]^2 z')' + [u(t)]^2 (q(t) - Q_0(t)) z = 0, \quad T \leq t < \omega.$$

Since $x_0(t)$ is a principal solution, we have

$$\int^\omega \frac{ds}{p(s)[x_0(s)]^2} = \int^\omega \frac{ds}{p(s)[u(s)]^2[z(s)]^2} = \infty.$$

Thus $z(t)$ is a principal solution of (2.2). We note that $Q_0(t) \geq Q(t) \geq q(t)$ on $[T, \omega)$. Then, by Lemma 1, we have $z(t) > 0$ and $z'(t) \leq 0$ on $[T, \omega)$, which implies $x_0(t) > 0$ on $[T, \omega)$. From the left side of (2.1) and

$$\frac{x'(t)}{x(t)} = \frac{u'(t)}{u(t)} + \frac{z'(t)}{z(t)}, \quad T \leq t < \omega,$$

we conclude that

$$\frac{p(t)x'(t)}{x(t)} \leq \frac{p(t)u'(t)}{u(t)} = \frac{P(t)y'(t)}{y(t)}, \quad T \leq t < \omega.$$

□

Proof of Theorem 1. Assume that $y(t) > 0$ in (a, ω) . By Picone's identity [3], we have

$$(2.3) \quad \frac{d}{dt} \left[\frac{x_0}{y} (px'_0y - Px_0y') \right] = (Q - q)x_0^2 + (p - P)x_0'^2 + \frac{P(x'_0y - x_0y')^2}{y^2}.$$

We observe that if $x_0(a) = 0$ then

$$\lim_{t \rightarrow a} \frac{x_0(t)}{y(t)} (p(t)x'_0(t)y(t) - P(t)x_0(t)y'(t)) = -P(a)x_0(a)y'(a) \lim_{t \rightarrow a} \frac{x_0(t)}{y(t)} = 0,$$

and that if $x_0(a) \neq 0$, $y(a) \neq 0$, and (1.3) holds, then

$$\lim_{t \rightarrow a} \frac{x_0(t)}{y(t)} (p(t)x'_0(t)y(t) - P(t)x_0(t)y'(t)) = [x_0(a)]^2 \left(\frac{p(a)x'_0(a)}{x_0(a)} - \frac{P(a)y'(a)}{y(a)} \right) \geq 0.$$

Therefore, integrating (2.3) over $[\tau, t]$ and letting $\tau \rightarrow a$, it follows that

$$[x_0(t)]^2 \left(\frac{p(t)x'_0(t)}{x_0(t)} - \frac{P(t)y'(t)}{y(t)} \right) \geq \int_a^t \left[(Q - q)x_0^2 + (p - P)x_0'^2 + \frac{P(x'_0y - x_0y')^2}{y^2} \right] ds$$

for $a < t < \omega$. From Lemma 2, we have

$$\int_a^t \left[(Q - q)x_0^2 + (p - P)x_0'^2 + \frac{P(x'_0y - x_0y')^2}{y^2} \right] ds \leq 0, \quad a < t < \omega,$$

which implies that $q(t) \equiv Q(t)$, $p(t) \equiv P(t)$, and $x_0(t)y'(t) \equiv x'_0(t)y(t)$ on $[a, \omega)$. Hence, $y(t)$ is a constant multiple of $x_0(t)$ on $[a, \omega)$. This completes the proof of Theorem 1.

□

Proof of Theorem 2. Let $t = t_1 < t_2 < \dots < t_n$ be zeros of $x_0(t)$ in (a, ω) . We note that $y(t)$ satisfies either (i) or (ii) in Theorem A on $[a, t_n]$.

By applying Theorem 1 on $[t_n, \omega)$, we have either $y(t)$ has at least one zero in (t_n, ω) or $y(t)$ is a multiple constant of $x_0(t)$ on $[t_n, \omega)$ and $p(t) \equiv P(t)$ and $q(t) \equiv Q(t)$ on $[t_n, \omega)$. In the former case, $y(t)$ has at least $n + 1$ zeros in (a, ω) . In the latter case, since $y(t_n) = 0$, we have either $y(t)$ has at least $n + 1$ zeros in (a, ω) or $y(t)$ is a multiple constant of $x_0(t)$ on $[a, \omega)$ and $p(t) \equiv P(t)$ and $q(t) \equiv Q(t)$ on $[a, \omega)$. This completes the proof of Theorem 2.

□

References

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