

Nontrivial Solutions of Semilinear Elliptic Equations with Continuous or Discontinuous Nonlinearities

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1 Introduction.

We begin this paper by considering the existence of nontrivial solutions of the boundary value problem of the form

$$-\Delta u = g(u) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \quad (1)$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbf{R}^n$ and g is a real-valued continuous function on $\mathbf{R}$ such that $g(0) = 0$.

Let $0 < \lambda_1 < \lambda_2 \leq \dots \leq \lambda_k \leq \dots$ denote the eigenvalues of the self-adjoint realization in $L^2(\Omega)$ of $-\Delta$ with the Dirichlet boundary condition. Many authors have studied the existence of nontrivial solutions of the problem (1) when $g(t)/t$ crosses finitely many eigenvalues of $-\Delta$ as t varies from $-\infty$ to $+\infty$. Amann and Zehnder [2] proved by generalized Morse theory that (1) has at least one nontrivial solution if $g \in C^2(R, R)$ satisfies

$$\sup_{t \in \mathbf{R}} |g'(t)| < \infty$$

and

$$\lambda_{k-1} \leq g'(0) < \lambda_k \leq \lambda_m < a_* \leq a^* < \lambda_{m+1} \quad \text{for some } m, k \geq 1$$

where

$$a_* = \liminf_{|t| \rightarrow \infty} \frac{g(t)}{t} \quad \text{and} \quad a^* = \limsup_{|t| \rightarrow \infty} \frac{g(t)}{t}.$$

On the other hand, using Leray-Schauder degree, Hirano [6] established the existence of one nontrivial solution of (1) under

$$\lambda_{k-1} < b_* \leq b^* < \lambda_k \leq \lambda_m < a_* \leq a^* < \lambda_{m+1} \quad \text{for some } k, m \geq 1,$$

where a_* and a^* are as above,

$$b_* = \liminf_{|t| \rightarrow 0} \frac{g(t)}{t} \quad \text{and} \quad b^* = \limsup_{|t| \rightarrow 0} \frac{g(t)}{t},$$

without any assumptions of differentiability of g . Hirano's result cannot be applied in the case of resonance at 0. We obtain the existence of one nontrivial solution of (1) under weaker conditions of g near 0 which contain the resonance case at 0 (Theorem 1). Moreover, there are no results for g with $b_* > a^*$ in [6]. We deal with such a function g in Theorem 2.

It is seen in §3, that the assertions of Theorem 1 and Theorem 2 remain valid in the case that g is a piecewise continuous function on any bounded closed interval of R (may be discontinuous at 0), that is,

$$-\Delta u \in [\underline{g}(u), \overline{g}(u)] \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \quad (2)$$

where

$$\underline{g}(t) = \liminf_{s \rightarrow t} g(s) \quad \text{and} \quad \overline{g}(t) = \limsup_{s \rightarrow t} g(s).$$

2 The case that g is continuous.

Our purpose in this section is to prove the following two theorems.

Theorem 1. Let $g : R \rightarrow R$ be a continuous function with $g(0) = 0$. If g satisfies the following condition

$$b^* < \lambda_m < a_* \leq \bar{a} < \lambda_{m+1} \quad (3)$$

for some $m \geq 1$, where

$$a_* = \liminf_{|t| \rightarrow \infty} \frac{g(t)}{t}, \quad \bar{a} = \sup_{t \neq 0} \frac{g(t)}{t} \quad \text{and} \quad b^* = \limsup_{|t| \rightarrow 0} \frac{g(t)}{t},$$

then the equation (1) has at least one nontrivial solution in $H^2(\Omega) \cap H_0^1(\Omega)$.

Theorem 2. Let $g : R \rightarrow R$ be a continuous function with $g(0) = 0$. If g satisfies that

$$\lambda_{k-1} < \underline{a} \leq a^* < \lambda_k < b_* \quad (4)$$

for some $k \geq 1$, where

$$\underline{a} = \inf_{t \neq 0} \frac{g(t)}{t}, \quad a^* = \limsup_{|t| \rightarrow \infty} \frac{g(t)}{t} \quad \text{and} \quad b_* = \liminf_{|t| \rightarrow 0} \frac{g(t)}{t},$$

then there exists at least one nontrivial solution of (1) in $H^2(\Omega) \cap H_0^1(\Omega)$.

In the following, we write H, H^{-1} and L^2 instead of $H_0^1(\Omega), H^{-1}(\Omega)$ and $L^2(\Omega)$, respectively. We denote by $\|\cdot\|, \|\cdot\|_*$ and $|\cdot|$ the norms of H, H^{-1} and L^2 , respectively. The notation $|\cdot|$ is often used for the absolute value of a real number without notice if there is no possibility of their confusion. The pairing between H and H^{-1} is denoted by $\langle \cdot, \cdot \rangle$. We take $k \in Z^+$ with $b^* < \lambda_k \leq \lambda_m$ if g satisfies the condition (3), and $m \in Z^+$ with $\lambda_k \leq \lambda_m < b_*$ if g satisfies the condition (4). Let H_1, H_2 and H_3 be closed subspaces of H spanned by the eigenfunctions corresponding to the eigenvalues $\{\lambda_{m+1}, \lambda_{m+2}, \dots\}, \{\lambda_k, \dots, \lambda_m\}$ and $\{\lambda_1, \lambda_2, \dots, \lambda_{k-1}\}$, respectively (We consider $\lambda_0 = 0$ and $H_3 = \{0\}$ if $k = 1$.).

For $i = 1, 2, 3$, P_i means the projection from H onto H_i . Define a real valued function f on H by

$$f(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} \int_0^{u(x)} g(t) dt dx \quad \text{for } u \in H. \quad (5)$$

Then we have

$$\langle f'(u), v \rangle = \langle -\Delta u - g(u), v \rangle \quad \text{for any } u, v \in H,$$

and hence weak solutions of (1) coincide with critical points of f .

We need the following two lemmas in order to prove our theorems.

Lemma 1. If g satisfies the conditions (3) or (4), then the Palais-Smale condition holds for the function f defined by (5), that is, for any sequence $\{u_n\}$ in H such that $\{f(u_n)\}$ is bounded and $\|f'(u_n)\|_* \rightarrow 0$, there exists a convergent subsequence of $\{u_n\}$.

Proof. Let $\{u_n\}$ in H satisfy that $\{f(u_n)\}$ is bounded and $\|f'(u_n)\|_* = \|-\Delta u - g(u)\|_* \rightarrow 0$. For each u_n , we put $v_n = P_1 u_n$, $w_n = P_2 u_n$ and $z_n = P_3 u_n$. Then

$$\begin{aligned} & \langle -\Delta u_n - g(u_n), v_n - (w_n + z_n) \rangle \\ &= \|v_n\|^2 - \|w_n + z_n\|^2 - \int_{\Omega} g(u_n)(v_n - (w_n + z_n)) dx. \end{aligned}$$

Suppose that g satisfies the condition (3). Then there exist positive numbers α with $\lambda_m < \alpha < a_*$ and ρ such that $\alpha \leq \frac{g(t)}{t} \leq \bar{a}$ for all $t \in R$ with $|t| \geq \rho$. From the continuity of g , for some constant K , we have $|g(t)| \leq K$ for all t with $|t| < \rho$. If $|u_n(x)| \geq \rho$, then

$$\alpha \leq \frac{g(u_n(x))}{v_n(x) + w_n(x) + z_n(x)} \leq \bar{a}. \quad (6)$$

If $|u_n(x)| < \rho$, then

$$|v_n(x)|^2 - |w_n(x) + z_n(x)|^2 \geq -\rho(|v_n(x)| + |w_n(x) + z_n(x)|).$$

We set

$$\begin{aligned} A &= \{x \in \Omega : |v_n(x)| > |w_n(x) + z_n(x)|\}, \\ A_1 &= \{x \in A : |u_n(x)| \geq \rho\} \quad \text{and} \quad A_2 = \{x \in A : |u_n(x)| < \rho\}. \end{aligned}$$

By the second inequality in (6), we have

$$\begin{aligned} & \int_A g(u_n)(v_n - (w_n + z_n))dx \\ & \leq \int_{A_1} \bar{\alpha}(|v_n|^2 - |w_n + z_n|^2)dx + \int_{A_2} K(|v_n| + |w_n + z_n|)dx \\ & \leq \int_A (\bar{\alpha}|v_n|^2 - \alpha|w_n + z_n|^2)dx + \int_{A_2} (\bar{\alpha}\rho + K)(|v_n| + |w_n + z_n|)dx. \end{aligned}$$

Putting

$$\begin{aligned} B &= \{x \in \Omega : |v_n(x)| \leq |w_n(x) + z_n(x)|\}, \\ B_1 &= \{x \in B : |u_n(x)| \geq \rho\} \quad \text{and} \quad B_2 = \{x \in B : |u_n(x)| < \rho\}, \end{aligned}$$

it follows that

$$\begin{aligned} & \int_B g(u_n)(v_n - (w_n + z_n))dx \\ & \leq \int_B (\bar{\alpha}|v_n|^2 - \alpha|w_n + z_n|^2)dx + \int_{B_2} (\bar{\alpha}\rho + K)(|v_n| + |w_n + z_n|)dx \end{aligned}$$

from the first inequality in (6). Therefore we have

$$\begin{aligned} & \int_{\Omega} g(u_n)(v_n - (w_n + z_n))dx \\ & \leq \bar{\alpha}\|v_n\|^2 - \alpha\|w_n + z_n\|^2 + 2|\Omega|^{1/2}(\bar{\alpha}\rho + K)\|u_n\|. \end{aligned}$$

Thus it holds that

$$\begin{aligned} & \langle -\Delta u_n - g(u_n), v_n - (w_n + z_n) \rangle \\ & \geq (1 - \frac{\bar{\alpha}}{\lambda_{m+1}})\|v_n\|^2 + (\frac{\alpha}{\lambda_m} - 1)\|w_n + z_n\|^2 - 2|\Omega|^{1/2}(\bar{\alpha}\rho + K)\|u_n\| \\ & \geq \omega_1\|u_n\|^2 - \omega_2\|u_n\| \end{aligned}$$

for some $\omega_1, \omega_2 > 0$. The assumption $\| -\Delta u_n - g(u_n) \|_* \rightarrow 0$ and this inequality imply the boundedness of $\{u_n\}$ in H and hence the existence of a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ which converges weakly to some u in H . Then we have

$$\langle -\Delta u_{n_j} - g(u_{n_j}), u_{n_j} - u \rangle \rightarrow 0.$$

Since H is compactly embedded into L^2 , $\{u_{n_j}\}$ strongly converges to u in L^2 and $\langle g(u_{n_j}), u_{n_j} - u \rangle \rightarrow 0$, so $\langle -\Delta u_{n_j}, u_{n_j} - u \rangle \rightarrow 0$. Since $\{-\Delta u_{n_j}\}$ weakly converges to $-\Delta u$ in H^{-1} , we have

$$\lim_{j \rightarrow \infty} \|u_{n_j}\|^2 = \lim_{j \rightarrow \infty} \langle -\Delta u_{n_j}, u_{n_j} - u \rangle + \lim_{j \rightarrow \infty} \langle -\Delta u_{n_j}, u \rangle = \|u\|^2.$$

Thus we obtain the strong convergence of $\{u_{n_j}\}$ in H . The proof is similar in the case that g satisfies the condition (4).

Lemma 2. Under the assumption (3), there exist positive constants c_i ($i = 1, 2, 3, 4$), ε_j ($j = 1, 2$) and K such that

i) if $\|P_1 u\| \geq c_1$, $\|P_2 u\| \leq c_2$ and $\|P_3 u\| \leq c_3$, then $f(u) \geq \varepsilon_1$;

ii) if $\|P_2 u\| \leq c_4$ and $\|P_3 u\| \leq K\|P_2 u\|$, then $f(u) \geq \varepsilon_2\|P_2 u\|^2$.

Proof. For simplicity, we set $v = P_1 u$, $w = P_2 u$ and $z = P_3 u$. By $\bar{a} < \lambda_{m+1}$, we have

$$\begin{aligned} f(u) &\geq \frac{1}{2}\|v + w + z\|^2 - \frac{1}{2}\bar{a}|v + w + z|^2 \\ &\geq \frac{1}{2}\left\{\left(1 - \frac{\bar{a}}{\lambda_{m+1}}\right)\|v\|^2 - \left(\frac{\bar{a}}{\lambda_k} - 1\right)\|w\|^2 - \left(\frac{\bar{a}}{\lambda_1} - 1\right)\|z\|^2\right\}, \end{aligned}$$

so there exist positive constants c_i ($i = 1, 2, 3$) and ε_1 for which the statement i) holds. From $b^* < \lambda_k$, we obtain positive constants δ and α with $\alpha < \lambda_k$ such that $\frac{g(t)}{t} \leq \alpha$ for

all t with $|t| \leq \delta$. In the case that $|v(x) + w(x) + z(x)| \leq \delta$, we have

$$\begin{aligned} & \frac{1}{2}(\lambda_{m+1}|v|^2 + \lambda_k|w|^2 + \lambda_1|z|^2) - \int_0^{v+w+z} g(t)dt \\ & \geq \frac{1}{2}(\lambda_{m+1} - \alpha)|v|^2 + \frac{1}{2}(\lambda_k - \alpha)|w|^2 + \frac{1}{2}(\lambda_1 - \alpha)|z|^2 - \alpha(vw + wz + zv) \\ & \geq \frac{1}{2}(\lambda_k - \alpha)|w|^2 + \frac{1}{2}(\lambda_1 - \alpha)|z|^2 - \alpha(vw + wz + zv). \end{aligned}$$

Now, we choose $d > 0$ such that

$$(\lambda_{m+1} - \alpha)p^2 + 2(\lambda_{m+1} - \alpha)pq + (\bar{\alpha} - \alpha)q^2 \leq (\lambda_{m+1} - \bar{\alpha})\delta^2$$

for all $p, q \geq 0$ with $p + q \leq d$. Moreover we can take $c > 0$ such that

$$\sup_{x \in \Omega} (|P_2 u(x)| + |P_3 u(x)|) \leq d$$

if $\|P_2 u + P_3 u\| \leq c$. Let $\|w + z\| \leq c$. In the case that $|v(x) + w(x) + z(x)| > \delta$, we have

$$\left| \int_0^{v+w+z} g(t)dt \right| \leq \frac{1}{2}\bar{\alpha}(v + w + z)^2 - \frac{1}{2}(\bar{\alpha} - \alpha)\delta^2$$

and hence

$$\begin{aligned} & \frac{1}{2}(\lambda_{m+1}|v|^2 + \lambda_k|w|^2 + \lambda_1|z|^2) - \int_0^{v+w+z} g(t)dt \\ & \geq \frac{1}{2}(\lambda_{m+1} - \bar{\alpha})\left\{|v| + \frac{\alpha - \bar{\alpha}}{\lambda_{m+1} - \bar{\alpha}}(|w| + |z|)\right\}^2 \\ & \quad - \frac{\bar{\alpha} - \alpha}{2(\lambda_{m+1} - \bar{\alpha})}\{\lambda_{m+1} - \alpha)|w|^2 + 2(\lambda_{m+1} - \alpha)|w||z| + (\bar{\alpha} - \alpha)|z|^2 - (\lambda_{m+1} - \bar{\alpha})\delta^2\} \\ & \quad + \frac{1}{2}(\lambda_k - \alpha)|w|^2 + \frac{1}{2}(\lambda_1 - \bar{\alpha})|z|^2 - \alpha(vw + wz + zv) \\ & \geq \frac{1}{2}(\lambda_k - \alpha)|w|^2 + \frac{1}{2}(\lambda_1 - \bar{\alpha})|z|^2 - \alpha(vw + wz + zv). \end{aligned}$$

It follows that

$$\begin{aligned} f(u) & \geq \int_{\Omega} \left\{ \frac{1}{2}(\lambda_{m+1}|v|^2 + \lambda_k|w|^2 + \lambda_1|z|^2) - \int_0^{v+w+z} g(t)dt \right\} dx \\ & \geq \frac{1}{2}(\lambda_k - \alpha)|w|^2 + \frac{1}{2}(\lambda_1 - \bar{\alpha})|z|^2 \\ & \geq \frac{1}{2} \left\{ \frac{\lambda_k - \alpha}{\lambda_m} \|w\|^2 - \frac{\bar{\alpha} - \lambda_1}{\lambda_1} \|z\|^2 \right\} \end{aligned}$$

if $\|w + z\| \leq c$. Taking K, c_4 and ε_2 such that

$$0 < K < \sqrt{\frac{\lambda_1(\lambda_k - \alpha)}{\lambda_m(\bar{a} - \lambda_1)}}, \quad 0 < (1 + K)c_4 \leq c$$

and

$$0 < \varepsilon_2 < \frac{\lambda_k - \alpha}{2\lambda_m} \left(1 - K^2 \frac{\lambda_m(\bar{a} - \lambda_1)}{\lambda_1(\lambda_k - \alpha)}\right),$$

the statement ii) holds.

We are now ready to prove Theorem 1.

Proof of Theorem 1. By $\lambda_m < a_* \leq \bar{a} < \lambda_{m+1}$, there exists $r > 0$ such that $f(w + z) < \inf_{v \in H_1} f(v)$ for all $w \in H_2$ and $z \in H_3$ with $\|w + z\| \geq r$. We define

$$\begin{aligned} \Gamma^* = \{A \subset H : A \text{ is a compact set such that } \sigma(A) \ni 0 \text{ for any} \\ \text{continuous mapping } \sigma : A \rightarrow H_2 \oplus H_3 \text{ satisfying} \\ \sigma(u) = u \text{ for all } u \in A \cap S\} (\neq \emptyset), \end{aligned}$$

where

$$S = \{w + z : w \in H_2, z \in H_3 \text{ and } \|w + z\| = r\}$$

and

$$c^* = \inf_{A \in \Gamma^*} \max_A f \quad (\geq \inf_{v \in H_1} f(v)).$$

It is easily seen that if $A \in \Gamma^*$ and $\eta : A \rightarrow H$ is a continuous mapping such that $\eta(u) = u$ for all $u \in A \cap S$, then $\eta(A) \in \Gamma^*$. Since f satisfies the Palais-Smale condition by Lemma 1, c^* is a critical value of f by a method similar to Rabinowitz's saddle point theorem ([9] and [7]). Assume that 0 is the only critical point of f . Let c_i ($i = 1, 2, 3, 4$), ε_j ($j = 1, 2$) and K be positive numbers in Lemma 2. We set

$$U = \{u \in H; \|P_1 u\| < a, \|P_2 u\| < b \text{ and } \|P_3 u\| < \frac{c}{2}\}$$

and

$$V = \{u \in H : \|P_1u\| < a, \|P_2u\| < b \text{ and } \|P_3u\| < c\},$$

where

$$a = c_1, b = \min\{c_2, c_4\} \text{ and } c = \min\{c_3, Kb\}.$$

We may suppose that $r > \sqrt{b^2 + c^2}$ with no loss of generality. Putting $\gamma = \min\{\varepsilon_1, \varepsilon_2 b^2\}$, it follows that $f \geq \gamma$ on $\{u \in H : \|P_1u\| \geq a, \|P_2u\| \leq b \text{ and } \|P_3u\| \leq c\} \cup \{u \in H : \|P_1u\| \leq a, \|P_2u\| = b \text{ and } \|P_3u\| \leq c\}$. From $c^* = 0$, for $0 < \varepsilon < \gamma$, there exists $A \in \Gamma^*$ with $\max_A f < \varepsilon$. Now, we define $T : H \rightarrow H$ by

$$T(u) = \begin{cases} u & \text{if } u \notin V \\ \varphi(\|P_3u\|)(P_1 + P_2)u + P_3u & \text{if } u \in V, \end{cases}$$

where $\varphi : [0, +\infty) \rightarrow [0, 1]$ is defined by

$$\varphi(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{\varepsilon}{2} \\ \frac{2}{\varepsilon}t - 1 & \text{if } \frac{\varepsilon}{2} < t \leq c \\ 1 & \text{if } c < t. \end{cases}$$

Then, T is continuous on $\{u \in H : \|P_1u\| = a, \|P_2u\| \leq b \text{ and } \|P_3u\| \leq c\}^c \cap \{u \in H : \|P_1u\| \leq a, \|P_2u\| = b \text{ and } \|P_3u\| \leq c\}^c$. By $\dim H_2 \neq 0$, we can choose $w_0 \in H_2$ with $0 < \|w_0\| < \frac{b}{2}$. Define $\tilde{T} : T(A) \rightarrow H$ by

$$\tilde{T}(u) = \begin{cases} u & \text{if } \|P_3u\| \geq \frac{\varepsilon}{2} \\ P_1u + Q((P_2 + P_3)u) & \text{if } \|P_3u\| < \frac{\varepsilon}{2}, \end{cases}$$

where $Q((P_2 + P_3)u)$ means the intersection of the half-line $\{t(P_2 + P_3)u + (1 - t)w_0 : t \geq 0\}$ and the relative boundary of $\{w + z : w \in H_2, z \in H_3, \|w\| < b \text{ and } \|z\| < \frac{c}{2}\}$ in $H_2 \oplus H_3$. Putting $\sigma = (P_2 + P_3) \circ \tilde{T} \circ T$, σ is a continuous mapping from A into $H_2 \oplus H_3$ such that $\sigma(u) = u$ for all $u \in A \cap S$. Since $f \geq \gamma > \varepsilon$ on $\{u \in H : \|P_1u\| \geq a, \|P_2u\| \leq b \text{ and } \|P_3u\| \leq c\}$, we have $\sigma(A) \not\subset 0$. This is contrary to $A \in \Gamma^*$. This

completes the proof.

Next we prove Theorem 2.

Proof of Theorem 2. From $\lambda_{k-1} < \underline{a} \leq a^* < \lambda_k$, we take $r > 0$ largely enough such that $f(z) < \inf_{v \in H_1, w \in H_2} f(v+w)$ for all $z \in H_3$ with $\|z\| \geq r$. We set $B = \{z \in H_3 : \|z\| \leq r\}$ and $S = \{z \in H_3 : \|z\| = r\}$. Define

$$\Gamma = \{g : g \text{ is a continuous mapping from } B \text{ into } H \text{ such that } g(z) = z \text{ for all } z \in S\} \quad (\neq \emptyset)$$

and

$$c = \inf_{g \in \Gamma} \max_{z \in B} f(g(z)) \quad (\geq \inf_{v \in H_1, w \in H_2} f(v+w)).$$

Similarly to the proof of Theorem 1, c is a critical value of f . Now, suppose that f does not have any nonzero critical points in H . From $\underline{a} > \lambda_{k-1}$, it follows that

$$f(z) \leq \frac{1}{2}\|z\|^2 - \frac{1}{2}\lambda_{k-1}|z|^2 \leq 0 \quad \text{for all } z \in H_3.$$

By $b_* > \lambda_m$, there exists $\delta > 0$ such that $\frac{g(t)}{t} \geq \lambda_m$ for all t with $|t| \leq \delta$. Then, we obtain $c_1 > 0$ such that $\sup_{x \in \Omega} |w(x) + z(x)| \leq \delta$ if $w \in H_2, z \in H_3$ and $\|w + z\| \leq c_1$.

Therefore we have

$$f(w+z) \leq \frac{1}{2}\|w+z\|^2 - \frac{1}{2}\lambda_m|w+z|^2 \leq 0$$

for all $w \in H_2$ and $z \in H_3$ with $\|w+z\| \leq c_1$. We may assume $c_1 < r$ without loss of generality. Choosing $c_2 > 0$ arbitrarily, we put

$$U = \{u \in H : \|P_1 u\| < c_2 \text{ and } \|(P_2 + P_3)u\| < \frac{c_1}{2}\}.$$

Since $\dim H_2 \neq 0$, by an argument similar to the proof of Theorem 1, we can construct a continuous mapping $g : B \rightarrow H$ such that $g(z) = z$ for all $z \in S$, $g(B) \cap U = \emptyset$ and

$f(g(z)) \leq 0$ for all $z \in B$. From the well-known deformation lemma, for sufficiently small $\varepsilon_0 > 0$, there exist a continuous mapping $\eta : H \rightarrow H$ and a positive number $\varepsilon < \varepsilon_0$ satisfying the following conditions

- i) $\eta(u) = u$ if $u \notin f^{-1}([-\varepsilon_0, \varepsilon_0])$;
- ii) $\eta(f^{-1}((-\infty, \varepsilon]) \setminus U) \subset f^{-1}((-\infty - \varepsilon])$.

Putting $\tilde{g} = \eta \circ g$, it is clear that $\tilde{g} \in \Gamma$. On the other hand, $\max_{z \in B} f(\tilde{g}(z)) \leq -\varepsilon$ since $g(B) \cap U = \emptyset$. This is contrary to $c = 0$. This completes the proof.

3 The case that g is discontinuous.

In this section, we consider the existence of one nontrivial solution of the equation (2). Let $g : R \rightarrow R$ be a piecewise continuous function on any bounded closed interval (may be discontinuous at 0) with $0 \in [g(0), \bar{g}(0)]$. Then, it is easily seen that the functional f defined by (5) is locally Lipschitz continuous if g satisfies the conditions (3) or (4). Then, we cannot apply the usual critical point theory for differentiable functionals since f may be nondifferentiable. In order to solve the problem (2), Chang [4] made use of the generalized gradients for locally Lipschitz continuous functionals introduced by Clarke [5]. In fact, it was shown that

$$\partial f(u) \subset -\Delta u - [g(u), \bar{g}(u)] \quad \text{for each } u \in H,$$

where $\partial f(u)$ means the generalized gradient of f at u .

Further, he proved in [4] that the deformation lemma holds in this case. On the other hand, Mizoguchi [8] obtained the existence of one nontrivial solution of (2) under the same conditions as Theorem 1 in [6].

We remark that g is automatically continuous at 0 in [8]. According to the proofs of Theorem 1 and Theorem 2, we see that the equation (2) has at least one nontrivial solution if the conditions (3) or (4) are assumed.

Theorem 3. Let $g : R \rightarrow R$ be a piecewise continuous function on any bounded closed interval with $0 \in [\underline{g}(0), \overline{g}(0)]$. If g satisfies the condition (3), then the equation (2) has at least one nontrivial solution in $H^2(\Omega) \cap H_0^1(\Omega)$.

Theorem 4. Let $g : R \rightarrow R$ be a piecewise continuous function on any bounded closed interval with $0 \in [\underline{g}(0), \overline{g}(0)]$. If g satisfies the condition (4), then there exists at least one nontrivial solution of (2) in $H^2(\Omega) \cap H_0^1(\Omega)$.

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