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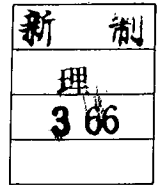
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(主論文)

Spitzer's Markov chains with measurable potentials

By

Munemi MIYAMOTO



京大附図

1. Introduction and summary of results. Spitzer [10] has introduced Markov chains, whose space of "time parametres" is an infinite tree T , and whose state space is a set $\{1, +1\}$. He investigates Gibbs distributions on T that are Markov chains of such construction. Several works [1],[4] and [8] are made on Gibbs distributions on trees.

In a present paper, we generalize Spitzer's results to a case when the state space is a compact set. If the state space consists of two points as in a case of Spitzer, all Markov chains are reversible. So, in that case, the "time parametre" space T need not be equipped with a direction. But, since Markov chains may not be reversible in our case, we must introduce a direction into T . Thus, we consider Markov chains whose space of "time parametres" is an infinite directed tree T , and whose state space is a compact measure space (X, B, μ) .

Let $F(x, y)$ be a measurable function on $X \times X$, boundedness or symmetry $F(x, y) = F(y, x)$ of which we do not assume. A Markov chain on T , whose transition density we denote by $p(x, y)$, is a Gibbs distribution on T with the potential F , if and only if

$$p(x,y) = \lambda(s,n)u(x)^{-1}u(y)^s v(y)^{n-1}e^{-F(x,y)},$$

where u and v are positive solutions of integral equations of the Hammerstein type

$$\begin{cases} u(x) = \lambda(s,n) \int_X e^{-F(x,y)} u(y)^s v(y)^{n-1} \mu(dy), \\ v(x) = \lambda(s,n) \int_X e^{-F(y,x)} u(y)^s v(y)^{n-1} \mu(dy). \end{cases}$$

Numbers s , n and $\lambda(s,n)$ will be defined in the following sections. Let $M(F)$ be the set of Markov chains that are, at the same time, Gibbs distributions with the potential F . Under summability conditions on F , all or no chain in $M(F)$ is reversible. Roughly speaking, all chains in $M(F)$ are reversible if and only if F is nearly symmetric. In a symmetric case, the transition density $p(x,y)$ has the form;

$$p(x,y) = \lambda(s,n)u(x)^{-1}u(y)^{n+s-1}e^{-F(x,y)},$$

where u is a positive solution of the integral equation;

$$u(x) = \lambda(s,n) \int_X e^{-F(x,y)} u(y)^{s+n-1} \mu(dy).$$

Existence of positive solutions of the integral equations is proved by applying theory of cones in a Banach space.

Dobrushin and Shlosman [3] proved that all Gibbs distributions in Z^2 whose state space is the circle S^1 , are invariant under rotation of the circle, if the potential is of finite range, of C^2 -class and rotation-invariant. We present an example of chains in $M(F)$ that are not rotation-invariant although the potential F is rotation-invariant and of C^∞ -class.

Next, we consider a potential βF , where $\beta > 0$ is the reciprocal temperature. We prove uniqueness of $M(\beta F)$ for sufficiently small β . We present an example in which the number of chains in $M(\beta F)$ is exactly calculated for sufficiently large β .

2. Potentials and Gibbs distributions. Let X be a compact metric space. Let B be the topological Borel field of X and let μ be a measure on (X, B) . Let T be the infinite directed tree, in which s branches emanate from every vertex and n branches flow into every vertex. Two vertices $a \neq b$ in T are neighbours if they are connected by a branch, which we denote by $a-b$ or $b-a$. If a branch connecting a and b emanates from a , which is equivalent to that the branch flows into b , we write $a \rightarrow b$ or $b \leftarrow a$. We remark $s, n \geq 1$. For a subset V of T , let ∂V be the set of vertices in V^c that are neighbours of vertices in V . Let $\Omega = X^T$. For $\omega \in \Omega$ and $a \in T$, let $x_a(\omega) = \omega_a$. For $V \subset T$, let $x_V(\omega)$ be the restriction $\omega|_V$ of ω on V , and let B_V be the σ -algebra of Ω generated by x_V . B_Ω is the σ -algebra generated by the cylinder sets.

A potential is a pair $F = (F_1, F_2)$ of real-valued measurable functions F_1 and F_2 , where F_1 and F_2 are defined on X and on $X \times X$, respectively. For a finite subset V of T and for $\underline{x} \in \Omega$, put

$$\begin{aligned} H_V(\underline{x}) = H_V^F(\underline{x}) = & \sum_{a \in V} F_1(x_a) + \sum_{\substack{a, b \in V \\ a \rightarrow b}} F_2(x_a, x_b) \\ & + \sum_{\substack{a \in V, b \in \partial V \\ a \rightarrow b}} F_2(x_a, x_b) + \sum_{\substack{a \in V, b \in \partial V \\ a \leftarrow b}} F_2(x_b, x_a). \end{aligned}$$

The family $\{H_V\}_V$ is called Hamiltonian.

Definition. Two potentials $F = (F_1, F_2)$ and $F' = (F'_1, F'_2)$ are said to be equivalent, which we denote by $F \approx F'$, if $H_V^F(\underline{x}) - H_V^{F'}(\underline{x})$ does not depend on x_V for every finite subset V . We remark that it may depend on $x_{\partial V}$.

Lemma 1. Let $F = (F_1, F_2)$ be a potential and put

$$F'_2(x, y) = F_2(x, y) + \frac{1}{n+s} \{F_1(x) + F_1(y)\},$$

then $F \approx (0, F'_2)$. If F_2 is symmetric, F'_2 is also symmetric.

Proof. Put $F''_2(x, y) = \frac{1}{n+s} \{F_1(x) + F_1(y)\}$. We have

$$\begin{aligned} & \sum_{\substack{a, b \in V \\ a \rightarrow b}} F''_2(x_a, x_b) + \sum_{\substack{a \in V, b \in \partial V \\ a \rightarrow b}} F''_2(x_a, x_b) + \sum_{\substack{a \in V, b \in \partial V \\ a \leftarrow b}} F''_2(x_b, x_a) \\ &= \sum_{a \in V} F_1(x_a) + \frac{1}{n+s} \sum_{b \in \partial V} \#\{a \in V; a-b\} F_1(x_b). \end{aligned}$$

$$\text{Therefore, } H_V^{(0, F'_2)}(\underline{x}) - H_V^F(\underline{x}) = \frac{1}{n+s} \sum_{b \in \partial V} \#\{a \in V; a-b\} F_1(x_b),$$

which implies $F \approx (0, F'_2)$.

In the following we assume always $F_1 = 0$. We identify a potential $(0, F)$ with the function F .

Definition. 1) A potential F is said to be symmetrizable if there exists a symmetric potential $\hat{F}$ with $F \approx \hat{F}$. We call $\hat{F}$ a symmetrization of F .

2) A potential F is said to be uniformly symmetrizable if there exists a symmetrization $\hat{F}$ of F such that

$$\sup_{x, y} |F(x, y) - \hat{F}(x, y)| < +\infty.$$

We call $\hat{F}$ a uniform symmetrization of F .

Lemma 2. 1) A potential F is symmetrizable if and only if there exists a measurable function f such that

$$F(x,y) - F(y,x) = f(x) - f(y).$$

2) A potential F is uniformly symmetrizable if and only if there exists a bounded measurable function f which satisfies the above equality.

Proof. Assume $F(x,y) - F(y,x) = f(x) - f(y)$. We have

$$\begin{aligned} F(x,y) &= \frac{1}{2}\{F(x,y) + F(y,x)\} + \frac{1}{2}\{F(x,y) - F(y,x)\} \\ &= \frac{1}{2}\{F(x,y) + F(y,x)\} + \frac{1}{2}\{f(x) - f(y)\}. \end{aligned}$$

Put $\hat{F}(x,y) = \frac{1}{2}\{F(x,y) + F(y,x)\} + \frac{s-n}{2(n+s)}\{f(x) + f(y)\}$. Since

$$\begin{aligned} &\sum_{\substack{a,b \in V \\ a \rightarrow b}} \{f(x_a) - f(x_b)\} + \sum_{\substack{a \in V, b \in \partial V \\ a \rightarrow b}} \{f(x_a) - f(x_b)\} \\ &\quad + \sum_{\substack{a \in V, b \in \partial V \\ a \leftarrow b}} \{f(x_b) - f(x_a)\} \\ &= (s - n) \sum_{a \in V} f(x_a) + \sum_{b \in \partial V} [\#\{a \in V; a \rightarrow b\} - \#\{a \in V; a \leftarrow b\}] f(x_b), \end{aligned}$$

and since

$$\begin{aligned} &\sum_{\substack{a,b \in V \\ a \rightarrow b}} \{f(x_a) + f(x_b)\} + \sum_{\substack{a \in V, b \in \partial V \\ a \rightarrow b}} \{f(x_a) + f(x_b)\} \\ &\quad + \sum_{\substack{a \in V, b \in \partial V \\ a \leftarrow b}} \{f(x_b) + f(x_a)\} \\ &= (s+n) \sum_{a \in V} f(x_a) + \sum_{b \in \partial V} \#\{a \in V; a \rightarrow b\} f(x_b), \end{aligned}$$

we have $H_V^F(\underline{x}) - H_V^{\hat{F}}(\underline{x}) =$

$$= \frac{1}{2} \sum_{b \in \partial V} [\#\{a \in V; a \leftarrow b\} - \#\{a \in V; a \rightarrow b\} - \frac{s-n}{s+n} \#\{a \in V; a=b\}] f(x_b),$$

which implies $F \approx \hat{F}$. If f is bounded, from an equality

$$F(x, y) - \hat{F}(x, y) = \frac{1}{n+s} \{nf(x) - sf(y)\},$$

it follows $\sup_{x, y} |F(x, y) - \hat{F}(x, y)| < +\infty$.

Conversely, assume $F \approx \hat{F}$, where $\hat{F}$ is symmetric. Let $a_i \rightarrow a$ ($1 \leq i \leq n$) and $a'_j \leftarrow a$ ($1 \leq j \leq s$). By the equivalence of potentials, the difference $H_{\{a\}}^F(\underline{x}) - H_{\{a\}}^{\hat{F}}(\underline{x})$ does not depend on x_a , which we denote by $\Delta(x_{a_1}, x_{a_2}, \dots, x_{a_n}, x_{a'_1}, x_{a'_2}, \dots, x_{a'_s})$. Fixing any $x_0 \in X$, we take arbitrary x and y from X . Put $x_a = y$, $x_{a_1} = x$, $x_{a_i} = x_0$ ($2 \leq i \leq n$) and $x_{a'_j} = x_0$ ($1 \leq j \leq s$). Put $\Delta(x) = \Delta(x, x_0, \dots, x_0)$. We have

$$\begin{aligned} \Delta(x) &= \Delta(x, x_0, \dots, x_0) \\ &= H_{\{a\}}^F(\underline{x}) - H_{\{a\}}^{\hat{F}}(\underline{x}) \\ &= \sum_{i=1}^n \{F(x_{a_i}, x_a) - \hat{F}(x_{a_i}, x_a)\} + \sum_{j=1}^s \{F(x_a, x_{a'_j}) - \hat{F}(x_a, x_{a'_j})\} \\ &= \{F(x, y) - \hat{F}(x, y)\} + (n-1)\{F(x_0, y) - \hat{F}(x_0, y)\} \\ &\quad + s\{F(y, x_0) - \hat{F}(y, x_0)\}. \end{aligned}$$

Consequently,

$$F(x, y) = \hat{F}(x, y) - (n-1)\{F(x_0, y) - \hat{F}(x_0, y)\} - s\{F(y, x_0) - \hat{F}(y, x_0)\} + \Delta(x).$$

Exchanging x and y , we have

$$F(y,x) = \hat{F}(x,y) - (n-1)\{F(x_0,x) - \hat{F}(x_0,x)\} - s\{F(x,x_0) - \hat{F}(x,x_0)\} + \Delta(y),$$

from which follows an equality

$$F(x,y) - F(y,x) = f(x) - f(y),$$

where $f(x) = \Delta(x) + (n-1)\{F(x_0,x) - \hat{F}(x_0,x)\} + s\{F(x,x_0) - \hat{F}(x,x_0)\}$.

If $\sup_{x,y} |F(x,y) - \hat{F}(x,y)| < +\infty$, then $\Delta(x)$ is bounded, therefore

f is also bounded.

For a finite subset V of T , put $\mu_V(dx_V) = \prod_{a \in V} \mu(dx_a)$.

Definition. A potential F is said to be admissible if for any finite subset V of T

$$\Xi(V, x_{\partial V}) \equiv \int_V e^{-H_V^F(\underline{x})} \mu_V(dx_V) < +\infty \quad \text{a.e. } (\mu_{\partial V}).$$

Lemma 3. A potential F is admissible, if

$$(A,1) \quad \int \int e^{(n+s)F(x,y)} \mu(dx) \mu(dy) < +\infty,$$

or if

$$(A,2) \quad \sup_x \{ \int e^{-F(x,y)} \mu(dy), \int e^{-F(y,x)} \mu(dy) \} < +\infty.$$

Proof. Admissibility under (A,1) is a direct consequence of 1) in the following Lemma 3'. Under (A,2) we have

$\int e^{-H_V^F(\underline{x})} \mu_{V \cup \partial V}(dx_{V \cup \partial V}) < +\infty$ by 2) in Lemma 3', if we put $F_{a,b} = F$

for $a, b \in V \cup \partial V$ with $\{a,b\} \not\subset \partial V$, and if we put $F_{a,b} = 0$ for $a, b \in \partial V$

Lemma 3'. Let be given a family $\{F_{a,b}; a \rightarrow b \in T\}$ of functions $F_{a,b}$
 $= F_{a,b}(x,y)$. For a finite subset V of T , put

$$\tilde{H}_V(\underline{x}) = \sum_{\substack{a,b \in V \\ a \rightarrow b}} F_{a,b}(x_a, x_b) + \sum_{\substack{a \in V, b \in \partial V \\ a \rightarrow b}} F_{a,b}(x_a, x_b) + \sum_{\substack{a \in V, b \in \partial V \\ a \leftarrow b}} F_{b,a}(x_b, x_a),$$

$$\tilde{\tilde{H}}_V(\underline{x}) = \sum_{\substack{a,b \in V \\ a \rightarrow b}} F_{a,b}(x_a, x_b).$$

1) If for each $a \rightarrow b \in T$,

$$(A,1)' \quad \int \int e^{(n+s)F_{a,b}(x,y)} \mu(dx) \mu(dy) < +\infty,$$

then it holds $\int e^{-\tilde{H}_V(\underline{x})} \mu_V(dx_V) < +\infty$ a.e. $(\mu_{\partial V})$.

2) If for each $a \rightarrow b \in T$,

$$(A,2)' \quad \sup_x \{ \int e^{-F_{a,b}(x,y)} \mu(dy), \int e^{-F_{a,b}(y,x)} \mu(dy) \} < +\infty,$$

then it holds $\int e^{-\tilde{\tilde{H}}_V(\underline{x})} \mu_V(dx_V) < +\infty$.

Proof is carried out by induction in $\#V$.

1) Let V be a set consisting of a single vertex a . Let $a_i \rightarrow a$
 $(1 \leq i \leq n)$ and $a_j' \leftarrow a$ $(1 \leq j \leq s)$. We have

$$\tilde{H}_{\{a\}}(\underline{x}) = \sum_{i=1}^n F_{a_i, a}(x_{a_i}, x_a) + \sum_{j=1}^s F_{a, a_j'}(x_a, x_{a_j'}),$$

$$\int e^{-\tilde{H}_{\{a\}}(\underline{x})} \mu(dx_a) = \int \prod_{i=1}^n e^{-F_{a_i, a}(x_{a_i}, x_a)} \prod_{j=1}^s e^{-F_{a, a_j'}(x_a, x_{a_j'})} \mu(dx_a)$$

$$\leq \left\{ \prod_{i=1}^n \int e^{-(n+s)F_{a_i, a}(x_{a_i}, x_a)} \mu(dx_a) \prod_{j=1}^s \int e^{-(n+s)F_{a, a'_j}(x_a, x_{a'_j})} \mu(dx_a) \right\}^{\frac{1}{n+s}}$$

$$< +\infty \quad \text{a.e.}(\mu_{\partial\{a\}}).$$

We assume that the statement is true if $\#V \leq k$. Let $\#V = k+1$.

Fix any $a_0 \in V$ and let $V_0 = V \setminus \{a_0\}$. Put

$$F'_{a, a_0}(x) = -\frac{1}{n+s} \log \int e^{-(n+s)F_{a, a_0}(x, z)} \mu(dz), \text{ if } a \rightarrow a_0,$$

$$F'_{a_0, a}(x) = -\frac{1}{n+s} \log \int e^{-(n+s)F_{a_0, a}(z, x)} \mu(dz), \text{ if } a \leftarrow a_0,$$

$$F'_{a, b}(x, y) = F_{a, b}(x, y), \text{ if otherwise.}$$

It is clear that $\int \int e^{-(n+s)F'_{a, b}(x, y)} \mu(dx) \mu(dy) < +\infty$. We have

$$\begin{aligned} \tilde{H}_V(\underline{x}) &= \sum_{\substack{a \in V_0 \cup \partial V \\ a \rightarrow a_0}} F_{a, a_0}(x_a, x_{a_0}) + \sum_{\substack{a \in V_0 \cup \partial V \\ a \leftarrow a_0}} F_{a_0, a}(x_{a_0}, x_a) \\ &+ \sum_{\substack{a, b \in V_0 \\ a \rightarrow b}} F'_{a, b}(x_a, x_b) + \sum_{\substack{a \in V_0, b \in \partial V_0 \setminus \{a_0\} \\ a \rightarrow b}} F'_{a, b}(x_a, x_b) \\ &+ \sum_{\substack{a \in V_0, b \in \partial V_0 \setminus \{a_0\} \\ a \leftarrow b}} F'_{b, a}(x_b, x_a). \end{aligned}$$

Denote the sum of the first two terms and the sum of the last three terms by $\tilde{H}_1(\underline{x})$ and by $\tilde{H}_2(\underline{x})$, respectively. Remark that $\#\{a \in V_0 \cup \partial V ; a \rightarrow a_0\} = n + s$. We have by Hölder's inequality

$$\begin{aligned}
\int e^{-\tilde{H}_1(\underline{x})} \mu(dx_{a_0}) &= \int \prod_{\substack{a \in V_0 \cup \partial V \\ a \rightarrow a_0}} e^{-F_{a,a_0}(x_a, x_{a_0})} \prod_{\substack{a \in V_0 \cup \partial V \\ a \leftarrow a_0}} e^{-F_{a_0,a}(x_{a_0}, x_a)} \mu(dx_{a_0}) \\
&\leq \left\{ \prod_{\substack{a \in V_0 \cup \partial V \\ a \rightarrow a_0}} \int e^{-(n+s)F_{a,a_0}(x_a, x_{a_0})} \mu(dx_{a_0}) \prod_{\substack{a \in V_0 \cup \partial V \\ a \leftarrow a_0}} \int e^{-(n+s)F_{a_0,a}(x_{a_0}, x_a)} \mu(dx_{a_0}) \right\}^{\frac{1}{n+s}}
\end{aligned}$$

$$= \exp \left\{ \sum_{\substack{a \in V_0 \cup \partial V \\ a \rightarrow a_0}} F'_{a,a_0}(x_a) - \sum_{\substack{a \in V_0 \cup \partial V \\ a \leftarrow a_0}} F'_{a_0,a}(x_a) \right\}.$$

On the other hand,

$$\begin{aligned}
\tilde{H}_2(\underline{x}) &+ \sum_{\substack{a \in V_0 \cup \partial V \\ a \rightarrow a_0}} F'_{a,a_0}(x_a) + \sum_{\substack{a \in V_0 \cup \partial V \\ a \leftarrow a_0}} F'_{a_0,a}(x_a) \\
&= \tilde{H}'_{V_0}(\underline{x}) + \sum_{\substack{a \in \partial V \\ a \rightarrow a_0}} F'_{a,a_0}(x_a) + \sum_{\substack{a \in \partial V \\ a \leftarrow a_0}} F'_{a_0,a}(x_a),
\end{aligned}$$

where $\tilde{H}'_{V_0}(\underline{x})$ is the Hamiltonian determined by $\{F'_{a,b}\}$, i.e.,

$$\begin{aligned}
\tilde{H}'_{V_0}(\underline{x}) &= \sum_{\substack{a,b \in V_0 \\ a \rightarrow b}} F'_{a,b}(x_a, x_b) + \sum_{\substack{a \in V_0, b \in \partial V_0 \\ a \rightarrow b}} F'_{a,b}(x_a, x_b) \\
&+ \sum_{\substack{a \in V_0, b \in \partial V_0 \\ a \leftarrow b}} F'_{b,a}(x_b, x_a).
\end{aligned}$$

Therefore, we have

$$\int e^{-\tilde{H}_V(\underline{x})} \mu_V(dx_V) = \int e^{-\tilde{H}_2(\underline{x})} \mu_{V_0}(dx_{V_0}) \int e^{-\tilde{H}_1(\underline{x})} \mu(dx_{a_0})$$

$$\leq \exp\left\{-\sum_{\substack{a \in \partial V \\ a \rightarrow a_0}} F'_{a, a_0}(x_a) - \sum_{\substack{a \in \partial V \\ a \leftarrow a_0}} F'_{a_0, a}(x_a)\right\} \int e^{-\tilde{H}'_{V_0}(\underline{x})} \mu_{V_0}(dx_{V_0}).$$

The last integral is finite a.e. $(\mu_{\partial V_0})$ by the assumption of induction.

2) If $\#V = 1$, $\tilde{H}_V(\underline{x}) = 0$. Consequently, $\int e^{-\tilde{H}_V(\underline{x})} \mu_V(dx_V) < +\infty$ is trivial. We assume that the statement is true if $\#V \leq k$. Let $\#V = k+1$. It is easy to see that there exists $a_0 \in V$ such that $\#(V \cap \partial a_0) = 0$ or 1 . Put $V_0 = V \setminus \{a_0\}$. If $\#(V \cap \partial a_0) = 0$, $\tilde{H}_V(\underline{x}) = \tilde{H}_{V_0}(\underline{x})$. Therefore, by the assumption of induction,

$$\begin{aligned} \int e^{-\tilde{H}_V(\underline{x})} \mu_V(dx_V) &= \int \int e^{-\tilde{H}_{V_0}(\underline{x})} \mu_{V_0}(dx_{V_0}) \mu(dx_{a_0}) \\ &= \mu(X) \int e^{-\tilde{H}_{V_0}(\underline{x})} \mu_{V_0}(dx_{V_0}) < +\infty. \end{aligned}$$

If $V \cap \partial a_0 = \{b\}$ and if, for example, $a_0 \rightarrow b$, then

$$\tilde{H}_V(\underline{x}) = \tilde{H}_{V_0}(\underline{x}) + F_{a_0, b}(x_{a_0}, x_b).$$

Therefore,

$$\begin{aligned} \int e^{-\tilde{H}_V(\underline{x})} \mu_V(dx_V) &= \int \int e^{-\tilde{H}_{V_0}(\underline{x}) - F_{a_0, b}(x_{a_0}, x_b)} \mu(dx_{a_0}) \mu_{V_0}(dx_{V_0}) \\ &\leq \sup_x \int e^{-F_{a_0, b}(x_{a_0}, x)} \mu(dx_{a_0}) \int e^{-\tilde{H}_{V_0}(\underline{x})} \mu_{V_0}(dx_{V_0}) < +\infty. \end{aligned}$$

In the following we consider only admissible potentials without mentioning.

Put

$$q_{V, x_{\partial V}}^F(x_V) = \mathbb{E}(V, x_{\partial V})^{-1} e^{-H_V^F(\underline{x})},$$

which is a probability density on (X^V, μ_V) . We call $q_{V, x_{\partial V}}^F$ conditional Gibbs density. We remark that $q_{V, x_{\partial V}}^F = q_{V, x_{\partial V}}^{F'}$ for all finite subset V and for a.a. $(\mu_{\partial V})$ $x_{\partial V}$, if and only if $F \approx F'$.

Definition ([2], [8]). A probability measure P on (Ω, B_Ω) is called Gibbs distribution with a potential F , if for each finite subset V of T , conditional probability distribution $P(\cdot | B_V^c)$ relative to B_V^c is absolutely continuous with respect to μ_V and

$$\frac{dP(\cdot | B_V^c)}{d\mu_V} = q_{V, x_{\partial V}}^F \quad \text{a.e.}(P).$$

Let $G(F)$ be the set of Gibbs distributions with the potential F .

3. Markov chains on the directed tree T . Let $p(x, y)$ be a positive transition density on (X, B, μ) and let $h(x)$ be the invariant probability density of $p(x, y)$. Put

$$\hat{p}(x, y) = h(y)p(y, x)h(x)^{-1},$$

which is called reversed transition density of p . We say that p is reversible if $p = \hat{p}$.

Let V be a connected finite subset of T . Let us introduce the second direction $\rightarrow$ in V . Fix any $a_0 \in V$. If $a - b$ and there exists a chain $a_0 - a_1 - \dots - a_k - a - b$, we write $a \rightarrow b$ or $b \leftarrow a$. In particular, $a_0 \rightarrow a$ if $a_0 - a$. We remark that if $a - b \in V$, either $a \rightarrow b$ or $a \leftarrow b$. Put

$$p_V(x_V) = h(x_{a_0}) \prod_{\substack{a,b \in V \\ a \rightarrow b \\ a \leftarrow b}} p(x_a, x_b) \prod_{\substack{a,b \in V \\ a \rightarrow b \\ a \leftarrow b}} \hat{p}(x_a, x_b),$$

$$P_V\{\omega \in \Omega; x_V(\omega) \in E\} = \int_E P_V(x_V) \mu_V(dx_V) \text{ for } E \in B_V.$$

It is easy to see that p_V does not depend on the choice of the centre a_0 and that $\{P_V\}$ is a consistent cylinder measure. By Kolmogorov's extension theorem, $\{P_V\}$ extends to a measure p on (Ω, B_Ω) . We identify the measure p with its transition density $p(x, y)$.

Definition. A measure p constructed above is called Spitzer's Markov chain with a potential F if $p \in G(F)$. Denote by $M(F)$ the set of Spitzer's Markov chains with the potential F .

Theorem 1. A transition density $p = p(x, y)$ belongs to $M(F)$, if and only if $p(x, y)$ has the expression;

$$p(x, y) = \lambda(s, n) u(x)^{-1} u(y)^s v(y)^{n-1} e^{-F(x, y)},$$

where $\lambda(s, n)$ is the Perron-Frobenius eigenvalue of the kernel $e^{-F(x, y)}$ if $s = n = 1$, and $\lambda(s, n) = 1$ if otherwise, and u and v are positive measurable functions satisfying

$$(*) \quad \begin{cases} u(x) = \lambda(s, n) \int_X e^{-F(x, y)} u(y)^s v(y)^{n-1} \mu(dy), \\ v(x) = \lambda(s, n) \int_X e^{-F(y, x)} u(y)^s v(y)^{n-1} \mu(dy), \\ \int_X u(x)^s v(x)^n \mu(dx) < +\infty. \end{cases}$$

The invariant probability density $h(x)$ has the form;

$$h(x) = c u(x)^S v(x)^n,$$

where c is a normalizing constant.

Proof. 1°. Assume $p(x, y) \in M(F)$. Let $a_i \rightarrow a$ ($1 \leq i \leq n$) and $a'_j \leftarrow a$ ($1 \leq j \leq s$) as before. Choose a as the centre of $\{a, a_1, a_2, \dots, a_n, a'_1, a'_2, \dots, a'_s\}$ in the definition of the direction $\rightarrow$. We have

$$\begin{aligned} q_{a, x_{\partial a}}(x) &= \Xi(a, x_{\partial a})^{-1} \exp\left\{ \sum_{i=1}^n F(x_{a_i}, x) - \sum_{j=1}^s F(x, x_{a'_j}) \right\} \\ &= Z(x_{\partial a})^{-1} h(x) \prod_{i=1}^n \hat{p}(x, x_{a_i}) \prod_{j=1}^s p(x, x_{a'_j}), \end{aligned}$$

where $Z(x_{\partial a}) = \int h(x) \prod_{i=1}^n \hat{p}(x, x_{a_i}) \prod_{j=1}^s p(x, x_{a'_j}) \mu(dx)$. Put $U(x, y) =$

$p(x, y) e^{F(x, y)}$. Then,

$$\begin{aligned} & Z(x_{\partial a})^{-1} h(x) \prod_{i=1}^n \hat{p}(x, x_{a_i}) \prod_{j=1}^s p(x, x_{a'_j}) \\ &= Z(x_{\partial a})^{-1} \prod_{i=1}^n h(x_{a_i}) h(x)^{1-n} \prod_{i=1}^n U(x_{a_i}, x) \prod_{j=1}^s U(x, x_{a'_j}) \times \\ & \quad \times \exp\left\{ \sum_{i=1}^n F(x_{a_i}, x) - \sum_{j=1}^s F(x, x_{a'_j}) \right\}. \end{aligned}$$

Consequently, $W \equiv h(x)^{1-n} \prod_{i=1}^n U(x_{a_i}, x) \prod_{j=1}^s U(x, x_{a'_j})$ does not depend on x .

Fix x_0 in X and take arbitrary y from X . Let $x_{a_i} = x_0$

($1 \leq i \leq n$) and let $x_{a'_j} = x_0$ or y ($1 \leq j \leq s$). Put $v = \#\{j: x_{a'_j} = y\}$.

We have

$$\begin{aligned} W &= h(x)^{1-n} U(x_0, x)^n U(x, y)^v U(x, x_0)^{s-v} \\ &= h(x)^{1-n} U(x_0, x)^n U(x, x_0)^s \left\{ \frac{U(x, y)}{U(x, x_0)} \right\}^v. \end{aligned}$$

Letting $v = 0$, we see that $h(x)^{1-n} U(x_0, x)^n U(x, x_0)^s$ does not depend on x . Next, letting $v = 1$, we see that $\frac{U(x, y)}{U(x, x_0)}$ does not depend on x , which we denote by $V(y)$. Putting $U(x) = U(x, x_0)$, we have $U(x, y) = U(x)V(y)$. Therefore, $p(x, y) = U(x)V(y)e^{-F(x, y)}$ and $c_1 \equiv h(x)^{1-n} U(x)^s V(x)^n$ does not depend on x .

Case, $n = 1$. Put

$$u(x) = \begin{cases} U(x)^{-1}, & \text{if } s = 1, \\ c_1^{\frac{1}{s-1}} U(x)^{-1}, & \text{if } s \geq 2. \end{cases}$$

From $c_1 = U(x)^s V(x)$, it follows that

$$V(x) = c_1 U(x)^{-s} = \begin{cases} c_1 u(x), & \text{if } s = 1, \\ c_1^{\frac{1}{s-1}} u(x)^s, & \text{if } s \geq 2. \end{cases}$$

We have

$$\begin{aligned} p(x, y) &= U(x)V(y)e^{-F(x, y)} \\ &= \begin{cases} c_1 u(x)^{-1} u(y)e^{-F(x, y)}, & \text{if } s = 1, \\ u(x)^{-1} u(y)^s e^{-F(x, y)}, & \text{if } s \geq 2. \end{cases} \end{aligned}$$

The equality $\int p(x, y) \mu(dy) = 1$ implies that

$$u(x) = \begin{cases} c_1 \int e^{-F(x,y)} u(y) \mu(dy), & \text{if } s = 1, \\ \int e^{-F(x,y)} u(y)^s \mu(dy), & \text{if } s \geq 2. \end{cases}$$

Since $u(x) > 0$, c_1 is the Perron-Frobenius eigenvalue $\lambda(1,1)$ of $e^{-F(x,y)}$. Thus we have

$$p(x,y) = \lambda(s,1) u(x)^{-1} u(y)^s e^{-F(x,y)},$$

$$u(x) = \lambda(s,1) \int e^{-F(x,y)} u(y)^s \mu(dy).$$

Put $v(x) = u(x)^{-s} h(x)$. The equality $h(x) = \int h(y) p(y,x) \mu(dy)$

implies $v(x) = \lambda(s,1) \int e^{-F(y,x)} u(y)^{s-1} v(y) \mu(dy)$.

From $\int h d\mu = 1$, it follows $\int u^s v d\mu = 1$. Thus, the proof is completed in case $n = 1$.

Case, $n \geq 2$. Put $u(x) = U(x)^{1/n}$ and $v(y) = \{U(y)^s V(y)\}^{\frac{1}{n-1}}$, i.e.,

$$U(x) = u(x)^n, \quad V(y) = u(y)^s v(y)^{n-1}.$$

Consequently, $p(x,y) = u(x)^{-1} u(y)^s v(y)^{n-1} e^{-F(x,y)}$. The equality $\int p(x,y) \mu(dy) = 1$ means

$$u(x) = \int e^{-F(x,y)} u(y)^s v(y)^{n-1} \mu(dy).$$

On the other hand,

$$\begin{aligned} c_1 &= h(x)^{1-n} U(x)^s V(x)^n \\ &= \{h(x)^{1/n} u(x)^s v(x)^{n-1}\}^{n-1}, \end{aligned}$$

which means $h(x) = c_2 u(x)^s v(x)^n$ with a constant c_2 . The equality $\int h d\mu = 1$ implies $\int u^s v^n d\mu < +\infty$. From $h(x) = \int h(y) p(y,x) \mu(dy)$,

it follows that

$$v(x) = \int e^{F(y,x)} u(y)^{s-1} v(y)^n \mu(dy).$$

The proof is completed in case $n \geq 2$.

2°. Assume conversely that positive functions u and v satisfy (*).

Put

$$p(x,y) = \lambda(s,n) u(x)^{-1} u(y)^s u(y)^{n-1} e^{-F(x,y)},$$

$$h(x) = c u(x)^s v(x)^n \quad \text{with } c = (\int u^s v^n d\mu)^{-1}.$$

The reversed transition density $\hat{p}(x,y) = h(y)p(y,x)h(x)^{-1}$ is equal to

$$\hat{p}(x,y) = \lambda(s,n) v(x)^{-1} v(y)^n u(y)^{s-1} e^{-F(y,x)}.$$

Let V be a connected finite subset of T and fix $a_0 \in V$ as the centre of $V \cup \partial V$ in the definition of the direction $\rightarrow$. We have

$$\begin{aligned} p_{V \cup \partial V}(x_{V \cup \partial V}) &= h(x_{a_0}) \prod_{\substack{a,b \in V \cup \partial V \\ a \rightarrow b \\ a \leftarrow b}} p(x_a, x_b) \prod_{\substack{a,b \in V \cup \partial V \\ a \rightarrow b \\ a \leftarrow b}} \hat{p}(x_a, x_b) \\ &= c \lambda(s,n)^{\#\{a-b \in V \cup \partial V\}} \hat{\Xi}(V, x_{V \cup \partial V})^{-1} \exp \left\{ - \sum_{\substack{a,b \in V \cup \partial V \\ a \rightarrow b}} F(x_a, x_b) \right\}, \end{aligned}$$

where we put

$$\begin{aligned} \hat{\Xi}(V, x_{V \cup \partial V})^{-1} &= u(x_{a_0})^s v(x_{a_0})^n \prod_{\substack{a,b \in V \cup \partial V \\ a \rightarrow b \\ a \leftarrow b}} \{u(x_a)^{-1} u(x_b)^s v(x_b)^{n-1}\} \times \\ &\times \prod_{\substack{a,b \in V \cup \partial V \\ a \rightarrow b \\ a \leftarrow b}} \{v(x_a)^{-1} v(x_b)^n u(x_b)^{s-1}\}. \end{aligned}$$

As usual, let $a_i \rightarrow a_0$ ($1 \leq i \leq n$) and $a'_j \leftarrow a_0$ ($1 \leq j \leq s$). Remark that

$\partial a_0 = \{a_1, \dots, a_n, a'_1, \dots, a'_s\} \in V \cup \partial V$. We have

$$\begin{aligned}
\hat{\Xi}(V, x_{V \cup \partial V})^{-1} &= u(x_{a_0})^s v(x_{a_0})^n \prod_{j=1}^s \{u(x_{a_0})^{-1} u(x_{a'_j})^s v(x_{a'_j})^{n-1}\} \times \\
&\times \prod_{i=1}^n \{v(x_{a_0})^{-1} v(x_{a_i})^n u(x_{a_i})^{s-1}\} \prod_{\substack{a, b \in V \cup \partial V, a \neq a_0 \\ a \rightarrow b \\ a \leftarrow b}} \{u(x_a)^{-1} u(x_b)^s v(x_b)^{n-1}\} \times \\
&\times \prod_{\substack{a, b \in V \cup \partial V, a \neq a_0 \\ a \rightarrow b \\ a \leftarrow b}} \{v(x_a)^{-1} v(x_b)^n u(x_b)^{s-1}\} \\
&= \prod_{j=1}^s \{u(x_{a'_j})^s v(x_{a'_j})^{n-1}\} \prod_{i=1}^n \{v(x_{a_i})^n u(x_{a_i})^{s-1}\} \times \\
&\times \prod_{\substack{a, b \in V \cup \partial U, a \neq a_0 \\ a \rightarrow b \\ a \leftarrow b}} \{u(x_a)^{-1} u(x_b)^s v(x_b)^{n-1}\} \prod_{\substack{a, b \in V \cup \partial V, a \neq a_0 \\ a \rightarrow b \\ a \leftarrow b}} \{v(x_a)^{-1} v(x_b)^n u(x_b)^{s-1}\}.
\end{aligned}$$

Therefore, $\hat{\Xi}(V, x_{V \cup \partial V})^{-1}$ does not depend on x_{a_0} . Since $\hat{\Xi}(V, x_{V \cup \partial V})^{-1}$ does not depend on the choice of the centre $a_0 \in V$ of the direction $\rightarrow$, it does not depend on x_V . Thus, we have $p_{V \cup \partial V}(x_{V \cup \partial V}) =$
 $= \hat{\Xi}(V, x_{\partial V})^{-1} \exp \left\{ - \sum_{\substack{a, b \in V \cup \partial V \\ a \rightarrow b}} F(x_a, x_b) \right\}$, where $\hat{\Xi}(V, x_{\partial V})$ depends only

on $x_{\partial V}$. It is easy to see that the extension of the cylinder measure $\{p_{V \cup \partial V}\}$ belongs to $G(F)$. The proof of Theorem 1 is completed.

We remark that the expression of $p(x, y)$ in Theorem 1 is not unique. If u and v satisfy (*), then also $\hat{u} = c^{n-1}u$ and $\hat{v} = c^{-(s-1)}v$ satisfy (*) and determine the same $p(x, y)$ as u and v . In order to make the expression unique, we need summability of $u^s v^{n-1}$ and $u^{s-1} v^n$, which does not follow from $\int u^s v^n d\mu < +\infty$.

Lemma 4. Put $X(x, M) = \{y \in X; F(x, y) \leq M\}$ and $X^*(x, M) = \{y \in X; F(y, x) \leq M\}$.

We assume that there exist M and an integer k such that

$$(A, 3) \begin{cases} \mu^k\{(x_1, x_2, \dots, x_k); \mu(X \setminus \bigcup_{i=1}^k X(x_i, M)) = 0\} > 0, \\ \mu^k\{(x_1, x_2, \dots, x_k); \mu(X \setminus \bigcup_{i=1}^k X^*(x_i, M)) = 0\} > 0. \end{cases}$$

If u and v satisfy $(*)$ in Theorem 1, it holds that

$$\int u^s v^{n-1} d\mu < +\infty \quad \text{and} \quad \int u^{s-1} v^n d\mu < +\infty.$$

Proof. Since $u(x) = \int e^{-F(x, y)} u(y)^s v(y)^{n-1} \mu(dy) \geq e^{-M} \int_{X(x, M)} u(y)^s v(y)^{n-1} \mu(dy)$,

$$\int u^s v^{n-1} d\mu \leq \sum_{i=1}^k \int_{X(x_i, M)} u^s v^{n-1} d\mu \leq e^M \sum_{i=1}^k u(x_i) < +\infty.$$

Theorem 1'. We assume that there exist M and an integer k such that (A, 3) holds. A transition density $p = p(x, y)$ belongs to $M(F)$, if and only if $p(x, y)$ has the expression:

$$p(x, y) = \lambda(s, n) u(x)^{-1} u(y)^s v(y)^{n-1} e^{-F(x, y)},$$

where u and v are positive measurable functions satisfying

$$(*)' \begin{cases} u(x) = \lambda(s, n) \int e^{-F(x, y)} u(y)^s v(y)^{n-1} \mu(dy), \\ v(x) = \lambda(s, n) \int e^{-F(y, x)} u(y)^{s-1} v(y)^n \mu(dy), \\ \int u(x)^s v(x)^{n-1} \mu(dx) = \int u(x)^{s-1} v(x)^n \mu(dx), \\ \int u(x) \mu(dx) = \int v(x) \mu(dx) = 1, \text{ if } s = n = 1, \\ \int u(x)^s v(x)^n \mu(dx) < +\infty. \end{cases}$$

The expression is unique.

Proof. By Theorem 1, a transition density $p(x, y) \in M(F)$ has the following expression with $\hat{u}$ and $\hat{v}$ satisfying $(*)$

$$p(x,y) = \lambda(s,n) \hat{u}(x) \hat{u}(y)^{s-1} \hat{v}(y)^{n-1} e^{-F(x,y)}.$$

In case $n = s = 1$, functions $u = (\int \hat{u} d\mu)^{-1} \hat{u}$ and $v = (\int \hat{v} d\mu)^{-1} \hat{v}$ satisfy $(*)'$, and in case $s+n > 2$, functions $u = c^{n-1} \hat{u}$ and $v = c^{s-1} \hat{v}$ with $c = \{(\int \hat{u}^{s-1} \hat{v}^n d\mu)(\int \hat{u}^s \hat{v}^{n-1} d\mu)^{-1}\}^{\frac{1}{s+n-2}}$ satisfy $(*)'$. In both cases, u and v determine the same $p(x,y)$ as $\hat{u}$ and $\hat{v}$.

Next, assume that

$$\begin{aligned} p(x,y) &= \lambda(s,n) u(x) u(y)^{s-1} v(y)^{n-1} e^{-F(x,y)} \\ &= \lambda(s,n) \tilde{u}(x) \tilde{u}(y)^{s-1} \tilde{v}(y)^{n-1} e^{-F(x,y)}, \end{aligned}$$

where u, v and $\tilde{u}, \tilde{v}$ satisfy $(*)'$. We have $\tilde{u}(x)u(x)^{-1} = \tilde{u}(y)u(y)^{-1} \tilde{v}(y)^{n-1}v(y)^{-(n-1)}$, which implies $u(x) = c \tilde{u}(x)$ in case $n = 1$, and implies $u(x) = c \tilde{u}(x)$ and $v(x) = c^{-\frac{s-1}{n-1}} \tilde{v}(x)$ in case $n \geq 2$. From $\int u d\mu = \int \tilde{u} d\mu = 1$ in case $s = n = 1$, or from $\int u^s v^{n-1} d\mu = \int \tilde{u}^s \tilde{v}^{n-1} d\mu$ and $\int \tilde{u}^s \tilde{v}^{n-1} d\mu = \int \tilde{u}^s \tilde{v}^{n-1} d\mu$ in case $s+n > 2$, it follows that $c = 1$. Therefore the expression is unique.

In the following, we indentify a transition density $p(x,y) \in M(F)$ with a pair (u,v) of positive solutions of $(*)'$. The set of pairs of positive solutions of $(*)'$ is denoted also by $M(F)$.

Theorem 2. The set $M(F)$ is not empty, either if

$$(A,4) \quad \int e^{-F(x,y)} \mu(dy) \text{ and } \int e^{-F(y,x)} \mu(dy) \text{ do not depend on } x,$$

or if

$$(A,5) \quad \sup_x \{ \int e^{-(n+s)F(x,y)} \mu(dy), \int e^{-(n+s)F(y,x)} \mu(dy) \} < +\infty$$

and

$$(A,6) \quad \sup_x \{ \int e^{(n+s)(n+s-2)F(x,y)} \mu(dy), \int e^{(n+s)(n+s-2)F(y,x)} \mu(dy) \} < +\infty$$

Proof. We assume (A,4). Put $c_1 = \int e^{-F(x,y)} \mu(dy)$ and $c_2 = \int e^{-F(y,x)} \mu(dy)$. From $\int \int e^{-F(x,y)} \mu(dx) \mu(dy) = c_1 \mu(X) = c_2 \mu(X)$, it follows $c_1 = c_2$. In case $s = n = 1$, $u(x) = v(x) = \mu(X)^{-1}$ is a positive solution of $(*)'$. In case $s+n > 2$, $u(x) = v(x) = c_1^{-\frac{1}{n+s-2}}$ is a positive solution of $(*)'$.

In order to look for positive solutions of $(*)'$ under the assumptions (A,5) and (A,6), we apply theory of cones in a Banach space. In case $s = n = 1$, $(*)'$ is a system of linear equations with positive kernels. Such equations have positive eigenfunctions, if the kernels are square-integrable ([7]), which follows from (A,5). Therefore, it is enough to investigate only a case $s+n > 2$. We first prove existence of positive solutions of $(*)'$ under the assumptions (A,5) and $\sup_{x,y} F(x,y) < +\infty$ instead of (A,6).

Let L be the set of pairs (u,v) of functions u and v such that

$$\|u\| \equiv \left\{ \int |u(x)|^{n+s} \mu(dx) \right\}^{\frac{1}{n+s}} < +\infty \text{ and } \|v\| \equiv \left\{ \int |v(x)|^{n+s} \mu(dx) \right\}^{\frac{1}{n+s}} < +\infty.$$

If we put $\|(u,v)\| = \|u\| + \|v\|$ for $(u,v) \in L$, $(L, \|\cdot\|)$ becomes a Banach space. Put for $(u,v) \in L$

$$A_1(u,v)(x) = \int e^{-F(x,y)} u(y)^s v(y)^{n-1} \mu(dy),$$

$$A_2(u,v)(x) = \int e^{-F(y,x)} u(y)^{s-1} v(y)^n \mu(dy),$$

$$A(u,v) = (A_1(u,v), A_2(u,v)).$$

Lemma 5. (Theorem 3.2 in Ch.1 of Krasnosel'skii [6]). Under the assumption (A,1), A is a completely continuous mapping from L into L .

Put

$$K_1 = \{u(x) = \int e^{-F(x,y)} a(y) \mu(dy); a(y) \geq 0, \|u\| < +\infty\},$$

$$K_2 = \{v(x) = \int e^{-F(y,x)} b(y) \mu(dy); b(y) \geq 0, \|v\| < +\infty\}.$$

Let K be the closure of $K_1 \times K_2$. We remark that K is a cone in L , i.e., K is closed and convex, $tK \subset K$ if $t \geq 0$, and (u,v) and $(-u,-v) \in K$ implies $(u,v) = 0$. It is clear that $A(K) \subset K$.

Lemma 6. We assume (A,5) and $\sup_{x,y} F(x,y) < +\infty$. Then, there exists

a positive constant c such that $u(x) \geq c\|u\|$ and $v(x) \geq c\|v\|$ for all $(u,v) \in K$ and for almost all $x \in X$.

Proof. Let $u(x) = \int e^{-F(x,y)} a(y) \mu(dy) \in K_1$. We have

$$u(x) \geq e^{-\sup_{x,y} F(x,y)} \int a(y) \mu(dy).$$

On the other hand, by Hölder's inequality

$$u(x) \leq (\int a d\mu)^{\frac{n+s-1}{n+s}} \{ \int e^{-(n+s)F(x,y)} a(y) \mu(dy) \}^{\frac{1}{n+s}}.$$

Therefore,

$$\begin{aligned} \|u\|^{n+s} &\leq (\int a d\mu)^{n+s-1} \int \int e^{-(n+s)F(x,y)} a(y) \mu(dx) \mu(dy) \\ &\leq (\int a d\mu)^{n+s} \sup_y \int e^{-(n+s)F(x,y)} \mu(dx). \end{aligned}$$

Consequently,

$$\begin{aligned} u(x) &\geq e^{-\sup_{x,y} F(x,y)} \int a d\mu \\ &\geq e^{-\sup_{x,y} F(x,y)} \left\{ \sup_y \int e^{-(n+s)F(x,y)} \mu(dx) \right\}^{-\frac{1}{n+s}} \|u\|. \end{aligned}$$

Thus, there is a constant $c > 0$ such that $u(x) \geq c\|u\|$ and $v(x) \geq c\|v\|$ for $(u,v) \in K_1 \times K_2$. Take any $(u,v) \in K$. There exists a sequence $(u_n, v_n) \in K_1 \times K_2$ such that $\|(u_n, v_n) - (u, v)\| \rightarrow 0$, i.e., $\|u_n - u\|$ and $\|v_n - v\| \rightarrow 0$. We can find a subsequence $\{n_j\}$ such that $u_{n_j}(x) \rightarrow u(x)$ and $v_{n_j}(x) \rightarrow v(x)$ for almost all $x \in X$. Since $\|u_{n_j}\| \rightarrow \|u\|$ and $\|v_{n_j}\| \rightarrow \|v\|$, we have $u(x) \geq c\|u\|$ and $v(x) \geq c\|v\|$.

Lemma 7. (Rothe [10], Krasnosel'skii [6]) Let $A = (A_1, A_2)$ be a completely continuous mapping from a cone $K \subset L$ into itself.

Assume $\inf_{\substack{(u,v) \in K \\ \|u\|=\|v\|=1}} \|A_1(u,v)\| > 0$ and $\inf_{\substack{(u,v) \in K \\ \|u\|=\|v\|=1}} \|A_2(u,v)\| > 0$. Then

there exists $(u_0, v_0) \in K$ such that $\|u_0\| = \|v_0\| = 1$ and

$$(u_0, v_0) = \left(\frac{A_1(u_0, v_0)}{\|A_1(u_0, v_0)\|}, \frac{A_2(u_0, v_0)}{\|A_2(u_0, v_0)\|} \right).$$

Proof. Fix any $(\hat{u}_0, \hat{v}_0) \in K$ with $\hat{u}_0 \neq 0$ and $\hat{v}_0 \neq 0$. Put

$$\hat{A}_1(u, v) = A_1(u, v) + (1 - \|u\| \cdot \|v\|) \hat{u}_0,$$

$$\hat{A}_2(u, v) = A_2(u, v) + (1 - \|u\| \cdot \|v\|) \hat{v}_0.$$

Let $\hat{K} = \{(u, v) \in K; \|u\| \leq 1, \|v\| \leq 1\}$, which is bounded, closed and conex. Our assumption implies $\inf_{(u,v) \in \hat{K}} \|A_1(u, v)\| > 0$ and

$\inf_{(u,v) \in \hat{K}} \|A_2(u, v)\| > 0$. Put again

$$B_1(u, v) = \frac{\hat{A}_1(u, v)}{\|\hat{A}_1(u, v)\|}, \quad B_2(u, v) = \frac{\hat{A}_2(u, v)}{\|\hat{A}_2(u, v)\|}.$$

$B = (B_1, B_2)$ is a completely continuous mapping from $\hat{K}$ into $\hat{K}$.

By Schauder's fixed point theorem, there exists $(u_0, v_0) \in \hat{K}$ such

$$\text{that } (u_0, v_0) = B(u_0, v_0), \text{ i.e., } u_0 = \frac{\hat{A}_1(u_0, v_0)}{\|\hat{A}_1(u_0, v_0)\|} \text{ and } v_0 = \frac{\hat{A}_2(u_0, v_0)}{\|\hat{A}_2(u_0, v_0)\|}$$

Since $\|u_0\| = \|v_0\| = 1$, $\hat{A}_1(u_0, v_0) = A_1(u_0, v_0)$ and $\hat{A}_2(u_0, v_0) = A_2(u_0, v_0)$

Proof of Theorem 2 under the assumptions (A,5) and $\sup_{x,y} F(x,y) < +\infty$.

By Lemma 6, we see that for $(u, v) \in K$

$$A_1(u, v)(x) \geq c^{s+n-1} \|u\|^s \|v\|^{n-1} \int e^{-F(x,y)} \mu(dy),$$

$$A_2(u, v)(x) \geq c^{s+n-1} \|u\|^{s-1} \|v\|^n \int e^{-F(y,x)} \mu(dy).$$

Hence, $\inf_{\substack{(u,v) \in K \\ \|u\|=\|v\|=1}} \|A_1(u,v)\| > 0$ and $\inf_{\substack{(u,v) \in K \\ \|u\|=\|v\|=1}} \|A_2(u,v)\| > 0$. By Lemma 7,

there exists $(u_0, v_0) \in K$ with $\|u_0\| = \|v_0\| = 1$ satisfying

$$u_0 = \|A_1(u_0, v_0)\|^{-1} A_1(u_0, v_0),$$

$$v_0 = \|A_2(u_0, v_0)\|^{-1} A_2(u_0, v_0).$$

Positivity of u_0 and v_0 follows from $(u_0, v_0) \in K$.

On the other hand, we have

$$\begin{aligned} \int u_0^s v_0^n d\mu &= \int u_0(x)^{s-1} v_0(x)^n u_0(x) \mu(dx) \\ &= \|A_1(u_0, v_0)\|^{-1} \int u_0(x)^{s-1} v_0(x)^n A_1(u_0, v_0)(x) \mu(dx) \\ &= \|A_1(u_0, v_0)\|^{-1} \int \int u_0(x)^{s-1} v_0(x)^n e^{-F(x,y)} u_0(y)^s v_0(y)^{n-1} \mu(dx) \mu(dy) \\ \int u_0^s v_0^n d\mu &= \\ &= \|A_2(u_0, v_0)\|^{-1} \int \int u_0(y)^{s-1} v_0(y)^n e^{-F(y,x)} u_0(x)^s v_0(x)^{n-1} \mu(dx) \mu(dy) \end{aligned}$$

Integrals above are finite, since

$$\int u_0^s v_0^n d\mu \leq (\int u_0^{n+s} d\mu)^{\frac{s}{n+s}} (\int v_0^{n+s} d\mu)^{\frac{n}{n+s}} < +\infty.$$

Consequently, $\|A_1(u_0, v_0)\| = \|A_2(u_0, v_0)\|$. Put

$$u(x) = \left\{ \|A_1(u_0, v_0)\|^{-1} \left(\frac{\int u_0^{s-1} v_0^n d\mu}{\int u_0^s v_0^{n-1} d\mu} \right)^{n-1} \right\}^{\frac{1}{n+s-2}} u_0(x),$$

$$v(x) = \left\{ \|A_2(u_0, v_0)\|^{-1} \left(\frac{\int u_0^s v_0^{n-1} d\mu}{\int u_0^{s-1} v_0^n d\mu} \right)^{s-1} \right\}^{\frac{1}{n+s-2}} v_0(x).$$

It is easy to see that (u, v) is a positive solution of $(*)'$.

Proof of Theorem 2 under the assumptions (A,5) and (A,6). Let

$F_k(x, y) = \min \{F(x, y), k\}$ for $k = 1, 2, \dots$. Let (u_k, v_k) be a positive solution of $(*)'$ with the potential F_k . We have

Lemma 8. Under the assumptions (A,5) and (A,6), there exist positive constants c_1 and c_2 such that $c_1 \leq u_k(x)$, $v_k(x) \leq c_2$ for all k and almost all $x \in X$.

Proof. Remark that

$$\sup_{k, x} \{ \int e^{-(n+s)F_k(x, y)} \mu(dy), \int e^{-(n+s)F_k(y, x)} \mu(dy) \} < +\infty,$$

$$\sup_{k, x} \{ \int e^{(n+s)(n+s-2)F_k(x, y)} \mu(dy), \int e^{(n+s)(n+s-2)F_k(y, x)} \mu(dy) \} < +\infty.$$

The proof of Lemma 8 is essentially the same as that of Lemma 12.

Since u_k 's and v_k 's are bounded, we can extract a subsequence $\{k_j\}$ such that u_{k_j} , v_{k_j} , $u_{k_j}^s v_{k_j}^{n-1}$ and $u_{k_j}^{s-1} v_{k_j}^n$ are weakly convergent in L_2 as $j \rightarrow \infty$. Put $u = w\text{-}\lim u_{k_j}$, $v = w\text{-}\lim v_{k_j}$, and $\hat{u} = w\text{-}\lim u_{k_j}^s v_{k_j}^{n-1}$. Remark $c_1 \leq u(x)$, $v(x) \leq c_2$ for almost all $x \in X$. Take an arbitrary bounded measurable function f on X . We have

$$\begin{aligned} \int f(x) u_{k_j}(x) \mu(dx) &= \iint f(x) e^{-F_{k_j}(x,y)} u_{k_j}(y)^s v_{k_j}(y)^{n-1} \mu(dx) \mu(dy) \\ &= \iint f(x) e^{-F(x,y)} u_{k_j}(y)^s v_{k_j}(y)^{n-1} \mu(dx) \mu(dy) \\ &\quad + \iint f(x) \{e^{-F_{k_j}(x,y)} - e^{-F(x,y)}\} u_{k_j}(y)^s v_{k_j}(y)^{n-1} \mu(dx) \mu(dy). \end{aligned}$$

Since $g(y) = \int f(x) e^{-F(x,y)} \mu(dx)$ is a bounded function of y , the first term of the right-hand side converges to

$$\int g(y) \hat{u}(y) \mu(dy) = \iint f(x) e^{-F(x,y)} \hat{u}(y) \mu(dx) \mu(dy).$$

As for the second term, we have

$$\begin{aligned} &| \iint f(x) \{e^{-F_{k_j}(x,y)} - e^{-F(x,y)}\} u_{k_j}(y)^s v_{k_j}(y)^{n-1} \mu(dx) \mu(dy) | \\ &\leq \|f\|_{\infty} c_2^{s+n-1} \iint \{e^{-F_{k_j}(x,y)} - e^{-F(x,y)}\} \mu(dx) \mu(dy). \end{aligned}$$

The right-hand side converges to 0 as $j \rightarrow \infty$, since $0 \leq e^{-F_{k_j}} - e^{-F} \leq e^{-k_j}$. Therefore, we have

$$\begin{aligned} \int f(x) u(x) \mu(dx) &= \lim_{j \rightarrow \infty} \int f(x) u_{k_j}(x) \mu(dx) \\ &= \iint f(x) e^{-F(x,y)} \hat{u}(y) \mu(dx) \mu(dy), \end{aligned}$$

from which it follows

$$u(x) = \int e^{-F(x,y)} \hat{u}(y) \mu(dy) \quad \text{a.e. } x.$$

Therefore,

$$\begin{aligned} u_{k_j}(x) - u(x) &= \int e^{-F_{k_j}(x,y)} u_{k_j}(y) s_{v_{k_j}}(y)^{n-1} \mu(dy) - \int e^{-F(x,y)} \hat{u}(y) \mu(dy) \\ &= \int \{e^{-F_{k_j}(x,y)} - e^{-F(x,y)}\} u_{k_j}(y) s_{v_{k_j}}(y)^{n-1} \mu(dy) \\ &\quad + \int e^{-F(x,y)} \{u_{k_j}(y) s_{v_{k_j}}(y)^{n-1} - \hat{u}(y)\} \mu(dy). \end{aligned}$$

The first integral converges to 0 as $j \rightarrow \infty$ for all x . The second integral also converges to 0, because $e^{-F(x,y)}$ belongs to

$L_{(n+s)} \subset L_2 = L_2^*$ as a function of y by the assumption (A,5).

Consequently, $\lim_{j \rightarrow \infty} u_{k_j}(x) = u(x)$ for almost all x . By the same

argument, we have $\lim_{j \rightarrow \infty} v_{k_j}(x) = v(x)$. Letting $j \rightarrow \infty$ in

$$u_{k_j}(x) = \int e^{-F_{k_j}(x,y)} u_{k_j}(y) s_{v_{k_j}}(y)^{n-1} \mu(dy),$$

$$v_{k_j}(x) = \int e^{-F_{k_j}(y,x)} u_{k_j}(y) s_{v_{k_j}}(y)^{n-1} \mu(dy),$$

we conclude by Lebesgue's convergence theorem that

$$u(x) = \int e^{-F(x,y)} u(y) s_v(y)^{n-1} \mu(dy),$$

$$v(x) = \int e^{-F(y,x)} u(y) s_v(y)^{n-1} \mu(dy).$$

4. Reversibility of Markov chains. We say that $p = p(x,y)$ is reversible if $p = \hat{p}$, which means $h(x)p(x,y) = h(y)p(y,x)$. We prove the following

Theorem 3. 1) If there exists a reversible chain in $M(F)$, the potential F is symmetrizable.

2) Let F be a symmetric potential. Assume (A,3) in Lemma 4 and assume

$$(A,5) \quad \sup_x \int e^{-(n+s)F(x,y)} \mu(dy) < +\infty.$$

Then, all chains in $M(F)$ are reversible.

Proof. 1) Let p be a reversible chain in $M(F)$. By Theorem 1,

we have $p(x,y) = \lambda(s,n) u(x)^{-1} u(y)^s v(y)^{n-1} e^{-F(x,y)}$ and $h(x) = c u(x)^s v(x)^n$. From $h(x)p(x,y) = h(y)p(y,x)$, it follows

$v(x)u(x)^{-1} e^{-F(x,y)} = v(y)u(y)^{-1} e^{-F(y,x)}$, which means

$F(x,y) - F(y,x) = \log v(x)u(x)^{-1} - \log v(y)u(y)^{-1}$. By Lemma 2, F is symmetrizable.

2) Let $p = (u,v) \in M(F)$. Put $K(x,y) = e^{-F(x,y)} u(y)^{s-1} v(y)^{n-1}$. We have, by Theorem 1,

$$u(x) = \lambda(s,n) \int K(x,y) u(y) \mu(dy),$$

$$v(x) = \lambda(s,n) \int K(x,y) v(y) \mu(dy).$$

Since $\sup_x u(x) < +\infty$ and $\sup_x v(x) < +\infty$ as will be shown in the following Lemma 9, we have

$$\begin{aligned} & \int \int K(x,y)^2 \mu(dx) \mu(dy) \\ & \leq \|u\|_\infty^{2(s-1)} \|v\|_\infty^{2(n-1)} \int \int e^{-2F(x,y)} \mu(dx) \mu(dy) \\ & \leq \|u\|_\infty^{2(s-1)} \|v\|_\infty^{2(n-1)} \int \mu(dx) \left\{ \int e^{-(n+s)F(x,y)} \mu(dy) \right\}^{\frac{2}{n+s}} \mu(X)^{\frac{n+s-2}{n+s}} \\ & \leq \|u\|_\infty^{2(s-1)} \|v\|_\infty^{2(n-1)} \left\{ \sup_x \int e^{-(n+s)F(x,y)} \mu(dy) \right\}^{\frac{2}{n+s}} \mu(X)^{\frac{2(n+s-1)}{n+s}} < +\infty. \end{aligned}$$

The kernel $K(x,y)$ being square-integrable, positive eigenfunctions in L_2 are unique up to a multiple of constants [7]. Consequently, there is a constant c_1 such that $u(x) = c_1 v(x)$. From the equality $\int u s \mu = \int v d\mu = 1$ in case $s = n = 1$, or from $\int u^s v^{n-1} d\mu = \int u^{s-1} v^n d\mu$ in case $s+n > 2$, it follows $c_1 = 1$, i.e., $u = v$. Therefore we have $p(x,y) = \lambda(s,n) u(x)^{-1} u(y)^{s+n-1} e^{-F(x,y)}$ and $h(x) = c u(x)^{s+n}$, which implies $h(x)p(x,y) = h(y)p(y,x)$.

Corollary. Assume that a symmetric potential F satisfies (A,3) and (A,5). Then, a transition density $p = p(x,y)$ belongs to $M(F)$, if and only if $p(x,y)$ has the expression:

$$p(x,y) = \lambda(s,n) u(x)^{-1} u(y)^{s+n-1} e^{-F(x,y)},$$

where u is a positive measurable function satisfying

$$(**) \begin{cases} u(x) = \lambda(s,n) \int e^{-F(x,y)} u(y)^{s+n-1} \mu(dy), \\ \int u(x) \mu(dx) = 1, \text{ if } s = n = 1, \\ \int u(x)^{s+n} \mu(dx) < +\infty. \end{cases}$$

The invariant probability density $h(x)$ has the form:

$$h(x) = c u(x)^{s+n},$$

where c is a normalizing constant. The expression is unique.

Lemma 9. We assume (A,3) and (A,5). Then, $\sup_x u(x) < +\infty$ and $\sup_x v(x) < +\infty$ for each $(u,v) \in M(F)$.

Proof. Put $\sigma = \int u^s v^{n-1} d\mu = \int u^{s-1} v^n d\mu < +\infty$. We have by Hölder's inequality

$$\begin{aligned} u(x) &= \int e^{-F(x,y)} u(y) s v(y)^{n-1} \mu(dy) \\ &\leq \sigma^{\frac{n+s-1}{n+s}} \left\{ \int e^{-(n+s)F(x,y)} u(y) s v(y)^{n-1} \mu(dy) \right\}^{\frac{1}{n+s}}. \end{aligned}$$

Consequently,

$$\begin{aligned} \int u^{s+n} d\mu &\leq \sigma^{n+s-1} \int \int e^{-(n+s)F(x,y)} u(y) s v(y)^{n-1} \mu(dy) \mu(dx) \\ &\leq \sigma^{n+s} \sup_x \int e^{(n+s)F(x,y)} \mu(dy) < +\infty. \end{aligned}$$

By the same argument, we have

$$\int v^{s+n} d\mu \leq \sigma^{n+s} \sup_x \int e^{(n+s)F(y,x)} \mu(dy) < +\infty.$$

We have, by Hölder's inequality again,

$$\begin{aligned} u(x) &\leq \left\{ \int e^{-(n+s)F(x,y)} \mu(dy) \right\}^{\frac{1}{n+s}} \left\{ \int u(y)^{n+s} \mu(dy) \right\}^{\frac{s}{n+s}} \left\{ \int v(y)^{n+s} \mu(dy) \right\}^{\frac{n-1}{n+s}} \\ &\leq \left\{ \sup_x \int e^{(n+s)F(x,y)} \mu(dy) \right\}^{\frac{1}{n+s}} \left(\int u^{n+s} d\mu \right)^{\frac{s}{n+s}} \left(\int v^{n+s} d\mu \right)^{\frac{n-1}{n+s}}. \end{aligned}$$

As for reversibility of chains in $M(F)$ with a symmetrizable potential F , we have the following

Theorem 3'. We assume (A,3) and

$$(A,5) \quad \sup_x \left\{ \int e^{-(n+s)F(x,y)} \mu(dy), \int e^{-(n+s)F(y,x)} \mu(dy) \right\} < +\infty,$$

$$(A,6)' \quad \sup_x \left\{ \int e^{(n+s)(n+s-2)'F(x,y)} \mu(dy), \int e^{(n+s)(n+s-2)'F(y,x)} \mu(dy) \right\} < \infty$$

where $(n+s)(n+s-2)' = \max \{ (n+s)(n+s-2), 1 \}$. Then the following three

statements are equivalent to each other.

- 1) A potential F is uniformly symmetrizable.
- 2) There exists a reversible chain in $M(F)$.
- 3) All chains in $M(F)$ are reversible.

To prove this, we need the the following

Lemma 10. We assume $(A,3)$ and

$$(A,6)'' \quad \sup_x \{ \int e^{F(x,y)} \mu(dy), \int e^{F(y,x)} \mu(dy) \} < +\infty.$$

Then, $\inf_x u(x) > 0$ and $\inf_x v(x) > 0$ for each $(u,v) \in M(F)$.

Proof. We have by Hölder's inequality

$$\begin{aligned} \int (u^s v^n)^{\frac{n+s-1}{2n+s}} d\mu &\leq \{ \int e^{-F(x,y)} u(y)^s v(y)^{n-1} \mu(dy) \}^{\frac{n}{2n+s}} \times \\ &\times \{ \int u^{s-1} v^n d\mu \}^{\frac{s}{2n+s}} \{ \int e^{F(x,y)} \mu(dy) \}^{\frac{n}{2n+s}} \\ &\leq u(x)^{\frac{n}{2n+s}} \{ \int u^{s-1} v^n d\mu \}^{\frac{s}{2n+s}} \{ \sup_x \int e^{F(x,y)} \mu(dy) \}^{\frac{n}{2n+s}}, \end{aligned}$$

from which follows $\inf_x u(x) > 0$.

Proof of Theorem 3'. $2) \Rightarrow 1)$. Let $(u,v) \in M(F)$. By the proof of Theorem 3, $F(x,y) - F(y,x) = \log v(x)u(x)^{-1} - \log v(y)u(y)^{-1}$. By Lemmas 9 and 10, the function $\log v(x)u(x)^{-1}$ is bounded, hence, F is uniformly symmetrizable by Lemma 2.

$1) \Rightarrow 3)$. Let F be a uniformly symmetrizable potential which satisfies $(A,3)$ and $(A,5)$. Then, the uniform symmetrization $\hat{F}$ of F also satisfies $(A,3)$ and $(A,5)$. Therefore, by Theorem 3, all chains in $M(F) = M(\hat{F})$ are reversible.

3) $\Rightarrow$ 2) is trivial, since $M(F) \neq \emptyset$ by Theorem 2.

We present an example in which $M(F)$ contains infinitely many chains. Let X be the unit circle S^1 which we identify with the interval $[0,1)$, and let μ be the Lebesgue measure on S^1 . Let $s+n = 3$. Let a_0, a_1 and a_2 be positive numbers. Put, for $k = 0,1,2$,

$$\gamma_k = \frac{a_k}{\sum_{j=-2}^2 a |k-j| a |j|},$$

and put

$$\begin{aligned} u(x) &= \sum_{k=-2}^2 a |k| e^{2\pi i k x} \\ &= a_0 + 2a_1 \cos 2\pi x + 2a_2 \cos 4\pi x, \end{aligned}$$

$$\begin{aligned} \Gamma(x) &= \sum_{k=-2}^2 \gamma |k| e^{2\pi i k x} \\ &= \gamma_0 + 2\gamma_1 \cos 2\pi x + 2\gamma_2 \cos 4\pi x. \end{aligned}$$

It is clear by the definition of γ_k that $u(x) = \int_0^1 \Gamma(x-y) u(y)^2 dy$.

If $\gamma_1 - 4\gamma_2 > 0$, then $\min_x \Gamma(x) = \Gamma(x)|_{\cos 2\pi x = -1} = \gamma_0 - 2\gamma_1 + 2\gamma_2$, since

$$\Gamma(x) = 4\gamma_2 \left(\cos 2\pi x + \frac{\gamma_1}{4\gamma_2} \right)^2 + \gamma_0 - 2\gamma_2 - \frac{\gamma_1^2}{4\gamma_2}. \quad \text{We can see}$$

$$\gamma_1 - 4\gamma_2 = \frac{a_1^2 - 6a_0 a_2 - 8a_2^2}{2(a_0 + a_2)(a_1^2 + 2a_0 a_2)},$$

$$\gamma_0 - 2\gamma_1 + 2\gamma_2 = \frac{a_1^2 a_2 (a_0 + 2a_2) + 4a_2^2 (a_0^2 + a_2^2) + 2(a_0^3 a_2 - a_1^4)}{(a_0^2 + 2a_1^2 + 2a_2^2)(a_0 + a_2)(a_1^2 + 2a_0 a_2)}.$$

Let $a_1^2 > 8a_2(a_0 + a_2)$, $a_1^4 \leq a_0^3 a_2$ and let a_1 and a_2 be sufficiently small in comparison with a_0 . Then, functions u and Γ are positive.

Put

$$F(x, y) = -\log \Gamma(x - y),$$

$$u_\alpha(x) = u(x + \alpha) \quad (\alpha \in [0, 1]),$$

then u'_α ($0 \leq \alpha < 1$) are positive solutions of (**) in Corollary to Theorem 3, that are distinguished from each other.

Dobrushin and Shlosman [3] show that all Gibbs distributions in Z^2 with the state space S^1 , whose potential is of finite range, of C^2 -class and invariant under rotation of S^1 , are also rotation-invariant. On the contrary, Spitzer's Markov chains determined by u_α are not rotation-invariant. But, $M(F)$ contains also a rotation-invariant chain, which is determined by a constant solution $\hat{u} = (\int \Gamma(x) dx)^{-1}$ of (**).

5. Uniqueness of Markov chains at high temperature. In the following we consider potentials with the form βF , where $\beta > 0$ is the reciprocal temperature. We prove

Theorem 4. Assume (A,3), as in Lemma 4, and assume

$$(A,7) \quad \sup_x \{ \int e^{|F(x,y)|} \mu(dy), \int e^{|F(y,x)|} \mu(dy) \} < +\infty.$$

If β is sufficiently small, then $M(\beta F)$ consists of one chain.

Proof. If β is sufficiently small, the potential βF satisfies (A,5) and (A,6). Therefore $M(\beta F) \neq \emptyset$ by Theorem 2. In case $s = n = 1$, (*)' in Theorem 1' takes the form

$$(*)' \begin{cases} u(x) = \lambda \int e^{-\beta F(x,y)} u(y) \mu(dy), \\ v(x) = \lambda \int e^{-\beta F(y,x)} v(y) \mu(dy), \\ \int u(x) \mu(dx) = \int v(x) \mu(dx) = 1, \\ \int u(x) v(x) \mu(dx) < +\infty. \end{cases}$$

As is shown in Lemma 8, solutions u and v of $(*)'$ are bounded from above if $\beta < \frac{1}{2}$, since (A,5) is satisfied by βF . Since the kernel $e^{-\beta F(x,y)}$ is square-integrable if $\beta < \frac{1}{2}$, the normalized positive solutions of the Perron-Frobenius equation $(*)'$ are unique ([7]).

To prove in case $s+n > 2$, we need several lemmas.

Lemma 11. Assume (A,7). Put

$$c_1(\beta) = \sup_x \{ | \int e^{\pm \beta F(x,y)} \mu(dy) - \mu(X) |, | \int e^{\pm \beta F(y,x)} \mu(dy) - \mu(X) | \}.$$

Then, we have $\lim_{\beta \rightarrow 0} c_1(\beta) = 0$.

Proof. By Hölder's inequality, we have

$$\begin{aligned} \int e^{\pm \beta F(x,y)} \mu(dy) &\leq \{ \int e^{\pm F(x,y)} \mu(dy) \}^\beta \mu(X)^{1-\beta} \\ &\leq \{ \sup_x \int e^{|F(x,y)|} \mu(dy) \}^\beta \mu(X)^{1-\beta}. \end{aligned}$$

The right-hand side converges to $\mu(X)$ as $\beta \rightarrow 0$. By Hölder's inequality again, we have

$$\begin{aligned} \mu(X)^2 &= \{ \int e^{\pm \frac{\beta}{2} F(x,y)} e^{\mp \frac{\beta}{2} F(x,y)} \mu(dy) \}^2 \\ &\leq \{ \int e^{\pm \beta F(x,y)} \mu(dy) \} \{ \int e^{\mp \beta F(x,y)} \mu(dy) \} \\ &\leq \{ \int e^{\pm \beta F(x,y)} \mu(dy) \} \{ \sup_x \int e^{|F(x,y)|} \mu(dy) \}^\beta \mu(X)^{1-\beta}. \end{aligned}$$

Consequently,

$$\int e^{\pm \beta F(x,y)} \mu(dy) \geq \left\{ \sup_x \int e^{|F(x,y)|} \mu(dy) \right\}^{-\beta} \mu(X)^{1+\beta},$$

the right-hand side of which converges to $\mu(X)$ as $\beta \rightarrow 0$.

Lemma 12. Assume (A,3) and (A,7). Put

$$c_2(\beta) = \sup_{(u,v) \in M(\beta F)} \left\{ \|u - \mu(X)^{-\frac{1}{n+s-2}}\|_{\infty}, \|v - \mu(X)^{-\frac{1}{n+s-2}}\|_{\infty} \right\},$$

$$c_2'(\beta) = \sup_{(u,v) \in M(\beta F)} \left\{ \|u^{s-1} v^{n-1} - \mu(X)^{-1}\|_{\infty}, \|u^s v^{n-2} - \mu(X)^{-1}\|_{\infty}, \|u^{s-2} v^n - \mu(X)^{-1}\|_{\infty} \right\}$$

where $\|f\|_{\infty} = \sup_{x \in X} |f(x)|$. Then, we have $\lim_{\beta \rightarrow 0} c_2(\beta) = \lim_{\beta \rightarrow 0} c_2'(\beta) = 0$.

Proof. Take any $(u,v) \in M(\beta F)$. Put $\sigma = \int u^s v^{n-1} d\mu = \int u^{s-1} v^n d\mu$.

1°. $\int u^{s+n} d\mu, \int v^{s+n} d\mu \leq \sigma^{s+n} \{\mu(X) + c_1(\beta(s+n))\}$.

In fact, we have

$$\begin{aligned} u(x) &= \int e^{-\beta F(x,y)} u(y) s_v(y)^{n-1} \mu(dy) \\ &\leq \sigma^{\frac{s+n-1}{s+n}} \left\{ \int e^{-\beta(s+n)F(x,y)} u(y) s_v(y)^{n-1} \mu(dy) \right\}^{\frac{1}{n+s}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int u^{s+n} d\mu &\leq \sigma^{s+n-1} \int \int e^{-\beta(s+n)F(x,y)} u(y) s_v(y)^{n-1} \mu(dx) \mu(dy) \\ &\leq \sigma^{s+n} \sup_x \int e^{-\beta(s+n)F(x,y)} \mu(dy) \\ &\leq \sigma^{s+n} \{\mu(X) + c_1(\beta(s+n))\}. \end{aligned}$$

2°. Put $c_3(\beta) = \{\mu(X) + c_1(\beta(s+n))\}^{\frac{s+n-1}{s+n}} \{\mu(X) + c_1(\beta(s+n)(s+n-2))\}^{\frac{1}{s+n}} - \mu(X)$

Then, we have $u(x), v(x) \geq \{\mu(X) + c_3(\beta)\}^{-\frac{1}{s+n-2}}$ and $\lim_{\beta \rightarrow 0} c_3(\beta) = 0$.

To show this, put $p_1 = \frac{s+n-1}{s+n-2}$, $p_2 = (s+n)(s+n-1)$, $p_3 = s^{-1}p_2$,
and $p_4 = (n-1)^{-1}p_2$. Remark that $\sum_{i=1}^4 p_i^{-1} = 1$ and $p_3^{-1} + p_4^{-1} =$
 $(s+n)^{-1}$. We have

$$\begin{aligned}\sigma &= \int u^s v^{n-1} d\mu \\ &\leq \{ \int e^{-\beta F(x,y)} u(y)^s v(y)^{n-1} \mu(dy) \}^{\frac{1}{p_1}} \{ \int e^{\frac{\beta p_2}{p_1} F(x,y)} \mu(dy) \}^{\frac{1}{p_2}} \times \\ &\quad \times \left(\int u^{s+n} d\mu \right)^{\frac{1}{p_3}} \left(\int v^{s+n} d\mu \right)^{\frac{1}{p_4}} \\ &\leq u(x)^{\frac{1}{p_1}} \{ \mu(X) + c_1 \left(\frac{\beta p_2}{p_1} \right) \}^{\frac{1}{p_2}} \sigma^{(s+n)(p_3^{-1} + p_4^{-1})} \{ \mu(X) + c_1(\beta(s+n)) \}^{p_3^{-1} + p_4^{-1}}\end{aligned}$$

Hence,

$$\begin{aligned}u(x) &\geq \{ \mu(X) + c_1 \left(\frac{\beta p_2}{p_1} \right) \}^{-\frac{p_1}{p_2}} \{ \mu(X) + c_1(\beta(s+n)) \}^{-\frac{p_1}{s+n}} \\ &= \{ \mu(X) + c_3(\beta) \}^{-\frac{1}{s+n-2}}.\end{aligned}$$

3 . Put $c_4(\beta) = \mu(X) - \mu(X)^{-(s+n-2)} \{ \mu(X) + c_3(\beta) \}^{-(n+s-3)} \{ \mu(X) - c_1(\beta) \}^{2(s+n-2)}$

Then, we have $\sigma = \int u^s v^{n-1} d\mu = \int u^{s-1} v^n d\mu \leq \{ \mu(X) - c_4(\beta) \}^{-\frac{1}{s+n-2}}$
and $\lim_{\beta \rightarrow 0} c_4(\beta) = 0$.

In fact, we have by 2 ,

$$\{ \mu(X) + c_3(\beta) \}^{-\frac{s+n-3}{2(s+n-2)}} \leq u(x)^{\frac{s}{2}-1} v(x)^{\frac{n-1}{2}}.$$

Therefore,

$$\begin{aligned}\{ \mu(X) + c_3(\beta) \}^{-\frac{s+n-3}{2(s+n-2)}} u(x) &\leq \{ u(x)^s v(x)^{n-1} \}^{\frac{1}{2}}, \\ \{ \mu(X) + c_3(\beta) \}^{-\frac{s+n-3}{2(s+n-2)}} \int u d\mu &\leq \int (u^s v^{n-1})^{\frac{1}{2}} d\mu \\ &\leq \sigma^{\frac{1}{2}} \mu(X)^{\frac{1}{2}}.\end{aligned}$$

On the other hand by Lemma 11,

$$\begin{aligned} \int u d\mu &= \int f e^{-\beta F(x,y)} u(y) s v(y)^{n-1} \mu(dx) \mu(dy) \\ &\geq \{\mu(X) - c_1(\beta)\} \sigma, \end{aligned}$$

hence,

$$\{\mu(X) + c_3(\beta)\}^{-\frac{s+n-3}{2(s+n-2)}} \{\mu(X) - c_1(\beta)\} \sigma \leq \sigma^{\frac{1}{2}} \mu(X)^{\frac{1}{2}}.$$

Thus, we have

$$\begin{aligned} \sigma &\leq \mu(X) \{\mu(X) + c_3(\beta)\}^{\frac{s+n-3}{s+n-2}} \{\mu(X) - c_1(\beta)\}^{-2} \\ &= \{\mu(X) - c_4(\beta)\}^{-\frac{1}{s+n-2}}. \end{aligned}$$

4°. We have $u(x), v(x) \leq \{\mu(X) - c_4(\beta)\}^{-\frac{s+n-1}{s+n-2}} \{\mu(X) + c_1(\beta(s+n))\}$.

In fact, we have by Lemma 11, 1° and 3°,

$$\begin{aligned} u(x) &= \int e^{-\beta F(x,y)} u(y) s v(y)^{n-1} \mu(dy) \\ &\leq \{\int e^{-\beta(n+s)F(x,y)} \mu(dy)\}^{\frac{1}{n+s}} (\int u^{s+n} d\mu)^{\frac{s}{n+s}} (\int v^{s+n} d\mu)^{\frac{n-1}{s+n}} \\ &\leq \{\mu(X) + c_1(\beta(s+n))\} \sigma^{s+n-1} \\ &\leq \{\mu(X) + c_1(\beta(s+n))\} \{\mu(X) - c_4(\beta)\}^{-\frac{s+n-1}{s+n-2}}. \end{aligned}$$

The assertions in Lemma 12 follow from 2° and 4°.

Lemma 13. 1) Put

$$R_1(x) \equiv R_1(u_1, v_1; u_2, v_2; x) = u_2^s v_2^{n-1} - \{u_1^s v_1^{n-1} + s u_1^{s-1} v_1^{n-1} w_1 + (n-1) u_1^s v_1^{n-2} w_2\},$$

$$R_2(x) \equiv R_2(u_1, v_1; u_2, v_2; x) = u_2^{s-1} v_2^n - \{u_1^{s-1} v_1^n + (s-1) u_1^{s-2} v_1^n w_1 + n u_1^{s-1} v_1^{n-1} w_2\},$$

where $w_1 = u_2 - u_1$ and $w_2 = v_2 - v_1$. Then, there exists a constant

$c > 0$ such that

$$\|R_1\|_\infty, \|R_2\|_\infty \leq c \cdot c_2(\beta) \cdot \max(\|u_2 - u_1\|_\infty, \|v_2 - v_1\|_\infty)$$

for all $0 < \beta \leq 1$ and for all (u_1, v_1) and $(u_2, v_2) \in M(\beta F)$.

2) There exists a function $c_5(\beta)$ with $\lim_{\beta \rightarrow 0} c_5(\beta) = 0$ such that

$$|\int (u_2 - u_1) d\mu - \int (v_2 - v_1) d\mu| \leq c_5(\beta) \max(\|u_2 - u_1\|_\infty, \|v_2 - v_1\|_\infty)$$

for all (u_1, v_1) and $(u_2, v_2) \in M(\beta F)$.

Proof. 1) The assertion is clear, since

$$\begin{aligned} R_1 &= (u_1 + w_1)^s (v_1 + w_2)^{n-1} \{u_1^s v_1^{n-1} + s u_1^{s-1} v_1^{n-1} w_1 + (n-1) u_1^s v_1^{n-2} w_2\} \\ &= \sum_{\substack{j+k \geq 2 \\ j \leq s, k \leq n-1}} \binom{s}{j} \binom{n-1}{k} u_1^{s-j} v_1^{n-1-k} w_1^j w_2^k \end{aligned}$$

and since $\sup\{\|u\|_\infty, \|v\|_\infty; (u, v) \in M(\beta F), 0 < \beta \leq 1\} < +\infty$ and $\|w_1\|_\infty, \|w_2\|_\infty \leq 2c_2(\beta)$ by Lemma 12.

2) We have $\mu(X)^{-1} \int (w_1 - w_2) d\mu =$

$$\begin{aligned} &= \int [s\{\mu(X)^{-1} - u_1^{s-1} v_1^{n-1}\} w_1 + (n-1)\{\mu(X)^{-1} - u_1^s v_1^{n-2}\} w_2] d\mu \\ &+ \int [(s-1)\{u_1^{s-2} v_1^{n-1} - \mu(X)^{-1}\} w_1 + n\{u_1^{s-1} v_1^{n-1} - \mu(X)^{-1}\} w_2] d\mu \\ &+ \int [\{s u_1^{s-1} v_1^{n-1} w_1 + (n-1) u_1^s v_1^{n-2} w_2\} - \{(s-1) u_1^{s-2} v_1^{n-1} w_1 + n u_1^{s-1} v_1^{n-1} w_2\}] d\mu. \end{aligned}$$

The first integral in the right-hand side is bounded in the absolute value by

$$\{s\|\mu(X)^{-1} - u_1^{s-1} v_1^{n-1}\|_\infty \cdot \|w_1\|_\infty + (n-1)\|\mu(X)^{-1} - u_1^s v_1^{n-2}\|_\infty \cdot \|w_2\|_\infty\} \mu(X),$$

which is not less than $(s+n-1)c_2'(\beta)\mu(X)\max(\|w_1\|_\infty, \|w_2\|_\infty)$ by Lemma 12.

The second integral is also bounded in the absolute value by $(s+n-1)c_2'(\beta)\mu(X)\max(\|w_1\|_\infty, \|w_2\|_\infty)$. The third integral is equal to $\int \{(u_2^s v_2^{n-1} - u_1^s v_1^{n-1} - R_1) - (u_2^{s-1} v_2^n - u_1^{s-1} v_1^{n-1} - R_2)\} d\mu = \int (R_2 - R_1) d\mu$, since $\int u_i^s v_i^{n-1} d\mu = \int u_i^{s-1} v_i^n d\mu$ ($i=1,2$). The absolute value of the right-hand side is not less than $(\|R_1\|_\infty + \|R_2\|_\infty)\mu(X) \leq 2\mu(X) \cdot c \cdot c_2(\beta)\max(\|w_1\|_\infty, \|w_2\|_\infty)$. Therefore, we have

$$|\int (w_1 - w_2) d\mu| \leq 2\{(s+n-1)c_2'(\beta) + c \cdot c_2(\beta)\}\mu(X)\max(\|w_1\|_\infty, \|w_2\|_\infty).$$

Proof of Theorem 4 in case $s+n > 2$. Take arbitrary (u_1, v_1) and $(u_2, v_2) \in M(\beta F)$. Put $w_1 = u_2 - u_1$ and $w_2 = v_2 - v_1$. From $u_i(x) = \int e^{-\beta F(x,y)} u_i(y) s_{v_i}(y)^{n-1} \mu(dy)$ ($i=1,2$), it follows that

$$\begin{aligned} w_1(x) &= \\ &= \int e^{-\beta F(x,y)} \{s u_1(y) s^{-1} v_1(y)^{n-1} w_1(y) + (n-1) u_1(y) s v_1(y)^{n-2} w_2(y) + R_1(y)\} \mu(dy) \\ &= (s+n-1)\mu(X)^{-1} \int w_1 d\mu + (n-1)\mu(X)^{-1} \int (w_2 - w_1) d\mu \\ &\quad + s\mu(X)^{-1} \int (e^{-\beta F(x,y)} - 1) w_1(y) \mu(dy) + (n-1)\mu(X)^{-1} \int (e^{-\beta F(x,y)} - 1) w_2(y) \mu(dy) \\ &\quad + s \int e^{-\beta F(x,y)} \{u_1(y) s^{-1} v_1(y)^{n-1} - \mu(X)^{-1}\} w_2(y) \mu(dy) \\ &\quad + (n-1) \int e^{-\beta F(x,y)} \{u_1(y) s v_1(y)^{n-2} - \mu(X)^{-1}\} w_2(y) \mu(dy) \\ &\quad + \int e^{-\beta F(x,y)} R_1(y) \mu(dy). \end{aligned}$$

We have

$$\begin{aligned} |\int (w_2 - w_1) d\mu| &\leq c_5(\beta)\max(\|w_1\|_\infty, \|w_2\|_\infty) \quad (\text{by Lemma 13}), \\ |\int e^{-\beta F(x,y)} \{u_1(y) s^{-1} v_1(y)^{n-1} - \mu(X)^{-1}\} w_1(y) \mu(dy)| \end{aligned}$$

$$\leq \{\mu(X) + c_1(\beta)\} \|u_1^{s-1} v_1^{n-1} \mu(X)^{-1}\|_\infty \cdot \|w_1\|_\infty \quad (\text{by Lemma 11})$$

$$\leq \{\mu(X) + c_1(\beta)\} c_2'(\beta) \max(\|w_1\|_\infty, \|w_2\|_\infty) \quad (\text{by Lemma 12}),$$

$$|\int e^{-\beta F(x,y)} R_1(y) \mu(dy)| \leq \{\mu(X) + c_1(\beta)\} \|R_1\|_\infty \quad (\text{by Lemma 11})$$

$$\leq \{\mu(X) + c_1(\beta)\} c \cdot c_2(\beta) \max(\|w_1\|_\infty, \|w_2\|_\infty) \quad (\text{by Lemma 13}).$$

As for $\int (e^{-\beta F} - 1) w_1 d\mu$, we have

$$\begin{aligned} & |\int \{e^{-\beta F(x,y)} - 1\} w_1(y) \mu(dy)| \\ & \leq \{\int (e^{-\beta F(x,y)} - 1)^2 \mu(dy)\}^{\frac{1}{2}} (\int w_1^2 d\mu)^{\frac{1}{2}} \\ & \leq \|w_1\|_\infty \cdot \mu(X)^{\frac{1}{2}} \{\int (e^{-2\beta F(x,y)} - 2e^{-\beta F(x,y)} + 1) \mu(dy)\}^{\frac{1}{2}}. \end{aligned}$$

The last integral converges to 0 uniformly in x as $\beta \rightarrow 0$ by Lemma 11.

Consequently, $w_1(x) = (s+n-1)\mu(X)^{-1} \int w_1 d\mu + R_3(x)$, where $\|R_3\|_\infty \leq$

$c_6(\beta) \max(\|w_1\|_\infty, \|w_2\|_\infty)$ with $\lim_{\beta \rightarrow 0} c_6(\beta) = 0$. Hence, we have

$$\int w_1 d\mu = -\frac{1}{s+n-2} \int R_3 d\mu,$$

$$|\int w_1 d\mu| \leq \frac{\mu(X)}{s+n-2} \|R_3\|_\infty,$$

$$\|w_1\|_\infty \leq (s+n-1)\mu(X)^{-1} |\int w_1 d\mu| + \|R_3\|_\infty$$

$$\leq \left(\frac{s+n-1}{s+n-2} + 1\right) c_6(\beta) \max(\|w_1\|_\infty, \|w_2\|_\infty).$$

By the same argument as above, we have

$$\|w_2\|_\infty \leq \left(\frac{s+n-1}{s+n-2} + 1\right) c_6(\beta) \max(\|w_1\|_\infty, \|w_2\|_\infty),$$

from which it follows

$$\max (\|w_1\|_\infty, \|w_2\|_\infty) \leq \left(\frac{s+n-1}{s+n-2}+1\right)c_6(\beta)\max(\|w_1\|_\infty, \|w_2\|_\infty).$$

If β is so small that $\left(\frac{s+n-1}{s+n-2}+1\right)c_6(\beta) < 1$, then $\max(\|w_1\|_\infty, \|w_2\|_\infty) = 0$, which means $u_1 = u_2$ and $v_1 = v_2$.

6. The number of chains at low temperature. An example. We present an example, in which the number of chains in $M(\beta F)$ is exactly calculated for sufficiently large β . Let X be a finite set and let $\mu_i = \mu(\{i\}) > 0$ for all $i \in X$. We prove

Theorem 5. Let F be a symmetric potential on X satisfying

$$(A,8) \quad F(i,j) > F(j,j) + \frac{1}{n+s-1} |F(i,i) - F(j,j)|$$

for all $i \neq j \in X$. Then, the number of chains in $M(\beta F)$ is equal to $2^{\#X} - 1$ for sufficiently large β , if $n+s > 2$.

Proof. We look for positive solutions of

$$(**) \quad u_i = \sum_{j \in X} e^{-\beta F(i,j)} u_j^{s+n-1} \mu_j \quad (i \in X).$$

For simplicity we put $p = s+n-1$. If we put

$$x_i = \{e^{-\beta F(i,i)} \mu_i\}^{\frac{1}{p-1}} u_i,$$

the equation $(**)$ is transformed into

$$(**)' \quad x_i = x_i^p + \sum_{j:j \neq i} a_{ij} x_j^p \quad (i \in X),$$

where $a_{ij} = \mu_i^{\frac{1}{p-1}} \mu_j^{-\frac{1}{p-1}} \exp[-\beta\{F(i,j) - F(j,j) - \frac{1}{p-1}(F(j,j) - F(i,i))\}]$.

Under the assumption $(A,8)$, we have $\lim_{\beta \rightarrow 0} a_{ij} = 0$. Therefore,

Theorem 5 is a corollary to the following

Lemma 14. The number of non-trivial solutions of the equation

$$(***) \quad x_i = |x_i|^p + \sum_{\substack{1 \leq j \leq N \\ j \neq i}} a_{ij} |x_j|^p \quad (1 \leq i \leq N)$$

is equal to $2^N - 1$, if $p > 1$ and positive coefficients a_{ij} ($1 \leq i \neq j \leq N$) are sufficiently small.

Proof. Put, for $\underline{x} = (x_1, x_2, \dots, x_N)$ and $\underline{a} = (a_{ij} : 1 \leq i \neq j \leq N)$,

$$F_i(\underline{x}, \underline{a}) = |x_i|^p x_i + \sum_{\substack{1 \leq j \leq N \\ j \neq i}} a_{ij} |x_j|^p \quad (1 \leq i \leq N),$$

$$J(\underline{x}, \underline{a}) = \det \left(\frac{\partial F_i}{\partial x_j}(\underline{x}, \underline{a}) \right)_{1 \leq i, j \leq N},$$

where

$$\frac{\partial F_i}{\partial x_j}(\underline{x}, \underline{a}) = p \delta_{ij} |x_i|^{p-1} - \delta_{ij} + p(1 - \delta_{ij}) a_{ij} |x_j|^{p-1}.$$

1°. The number of non-trivial solutions of (***) is not less than $2^N - 1$, if a_{ij} 's are sufficiently small.

In fact, let $\hat{\underline{x}} = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_N) \neq \underline{0}$ with $\hat{x}_i = 0$ or 1.

We have $F_i(\hat{\underline{x}}, \underline{0}) = 0$ ($1 \leq i \leq N$) and $J(\hat{\underline{x}}, \underline{0}) \neq 0$, since

$$\frac{\partial F_i}{\partial x_i}(\hat{\underline{x}}, \underline{0}) = p \hat{x}_i^{p-1} \quad 1 \quad \text{and} \quad \frac{\partial F_i}{\partial x_j}(\hat{\underline{x}}, \underline{0}) = 0 \quad (i \neq j).$$

Consequently, there exist a constant A and an R^N -valued continuous function $\underline{f}^{\hat{\underline{x}}} = \underline{f}^{\hat{\underline{x}}}(\underline{a})$ defined for $\underline{a}$ with $\|\underline{a}\| = \max_{i \neq j} |a_{ij}| \leq A$, such that

$$\underline{f}^{\hat{\underline{x}}}(\underline{0}) = \hat{\underline{x}},$$

$$F_i(\underline{f}^{\hat{\underline{x}}}(\underline{a}), \underline{a}) = 0 \quad \text{for } \underline{a} \text{ with } \|\underline{a}\| \leq A \quad (1 \leq i \leq N).$$

Since $\underline{f}^{\hat{\underline{x}}}(\underline{a}) \neq \underline{0}$ if $\underline{a}$ is sufficiently small, it is a non-trivial solution of (***). Remark that if $\underline{x} \neq \underline{x}'$, $\underline{f}^{\hat{\underline{x}}}(\underline{a}) \neq \underline{f}^{\hat{\underline{x}'}}(\underline{a})$ for sufficiently small $\underline{a}$. The number of non-trivial solution of (***) is not less than $\#\{\hat{\underline{x}}; \hat{\underline{x}} \neq \underline{0}, \hat{x}_i = 0 \text{ or } 1 \ (1 \leq i \leq N)\} = 2^N - 1$.

2°. If $\underline{a}$ is sufficiently small, then $J(\underline{x}, \underline{a}) \neq 0$ for any solution $\underline{x} = (x_1, x_2, \dots, x_N)$ of (***) .

In fact, from $x_i - |x_i|^p = \sum_{j \neq i} a_{ij} |x_j|^p \geq 0$, it follows $0 \leq x_i \leq 1$.

From $0 \leq x_i - |x_i|^p = \sum_{j \neq i} a_{ij} |x_j|^p \leq \sum_{j \neq i} a_{ij} \leq (N-1) \|\underline{a}\|$, it follows

that x_i is close to 0 or 1 if $\|\underline{a}\|$ is small. Therefore, $|\frac{\partial F_i}{\partial x_i}(\underline{x}, \underline{a})| = |px_i^{p-1} - 1| \geq \frac{1}{2}$ for sufficiently small $\underline{a}$. On the other hand, for $i \neq j$

$$\frac{\partial F_i}{\partial x_j}(\underline{x}, \underline{a}) = pa_{ij} x_j^{p-1} \leq p \|\underline{a}\|.$$

Hence, $J(\underline{x}, \underline{a}) \neq 0$ if $\underline{a}$ is sufficiently small.

3°. Let $\underline{a}$ be sufficiently small and let $\underline{x} = (x_1, x_2, \dots, x_N)$ be a solution of (***) . There exist continuous functions $f_1(t), f_2(t), \dots, f_N(t)$ defined on $[0, 1]$ such that

$$f_i(1) = x_i \quad (1 \leq i \leq N),$$

$$f_i(t) = |f_i(t)|^p + \sum_{j \neq i} ta_{ij} |f_j(t)|^p \quad (1 \leq i \leq N, 0 \leq t \leq 1).$$

$$\text{In fact, put } \tilde{F}_i(\underline{x}; t) = |x_i|^p - x_i + \sum_{j \neq i} ta_{ij} |x_j|^p \quad (1 \leq i \leq N) \text{ and}$$

let A_0 be the infimum of A such that there exists a continuous function $\underline{f}(t) = (f_1(t), f_2(t), \dots, f_N(t))$ on $[A, 1]$ such that

$$\underline{f}(1) = \underline{x},$$

$$\tilde{F}_i(\underline{f}(t); t) = 0 \quad (1 \leq i \leq N, A \leq t \leq 1).$$

Put $\tilde{J}(\underline{x}, t) = \det \left(\frac{\partial \tilde{F}_i}{\partial x_j}(\underline{x}, t) \right)_{1 \leq i, j \leq N}$. Since $\tilde{J}(\underline{x}, 1) \neq 0$ by 2°, such a

function $\underline{f}(t)$ exists in a neighbourhood of 1. Therefore, $A_0 < 1$.

Suppose $A_0 \geq 0$. Then there exists a sequence $A_n \searrow A_0$ and continuous functions $\underline{f}^{(n)}(t)$ on $[A_n, 1]$ such that

$$\underline{f}^{(n)}(1) = \underline{x},$$

$$\tilde{F}_i(\underline{f}^{(n)}(t); t) = 0 \quad (1 \leq i \leq N, A_n \leq t \leq 1).$$

Since $\tilde{J}(\underline{f}^{(n)}(t); t) \neq 0$ by 2° , uniqueness of implicit functions implies $\underline{f}^{(n)}(t) = \underline{f}^{(m)}(t)$ for $m > n$ and $A_n \leq t \leq 1$. Put

$$\underline{f}(t) = \underline{f}^{(n)}(t) \quad \text{for } A_n \leq t \leq 1 \quad (n=1, 2, \dots).$$

The function $\underline{f}(t)$ satisfies

$$\underline{f}(1) = \underline{x},$$

$$\tilde{F}_i(\underline{f}(t); t) = 0 \quad (1 \leq i \leq N, A_0 < t \leq 1).$$

Remark that every component $f_i(t)$ of $\underline{f}(t)$ satisfies $0 \leq f_i(t) \leq 1$. Let $t_n \searrow A_0$. There exists a subsequence $\{t_{n_k}\}$ such that $\underline{f}(t_{n_k})$ converges as $k \rightarrow \infty$. Put $\underline{y} = \lim_{k \rightarrow \infty} \underline{f}(t_{n_k})$. We have

$$\tilde{F}_i(\underline{y}; A_0) = 0 \quad (1 \leq i \leq N),$$

hence, $\tilde{J}(\underline{y}; A_0) \neq 0$ by 2° . There exists a unique function $\tilde{\underline{f}}(t)$ in some neighbourhood $(A_0 - \varepsilon, A_0 + \varepsilon)$ of A_0 such that

$$\tilde{\underline{f}}(A_0) = \underline{y},$$

$$\tilde{F}_i(\tilde{\underline{f}}(t); t) = 0 \quad (1 \leq i \leq N, A_0 - \varepsilon < t < A_0 + \varepsilon).$$

By uniqueness of implicit functions, we have $\underline{f}(t) = \tilde{\underline{f}}(t)$ for

$t \in (A_0 - \varepsilon, A_0 + \varepsilon)$. Therefore, $A_0 - \varepsilon$ is not less than the infimum of A such that there exists a continuous function $\underline{f}(t)$ on $[A, 1]$ with

$\underline{f}(1) = \underline{x}$ and $\tilde{F}_i(\underline{f}(t):t) = 0$ ($1 \leq i \leq N, A \leq t \leq 1$), which we have put A_0 . This is a contradiction. Hence $A_0 < 0$.

4°. Let $\underline{a}$ be sufficiently small. There is a one-to-one correspondence between non-trivial solutions $\underline{x}$ of (***) and $\hat{\underline{x}} = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_N) \neq 0$ with $\hat{x}_i = 0$ or 1 .

In fact, let $\underline{x}$ be a non-trivial solution of (***). There is a continuous function $\underline{f}(t)$ on $[0,1]$ such that

$$\underline{f}(1) = \underline{x},$$

$$f_i(t) = |f_i(t)|^P + \sum_{j \neq i} t a_{ij} |f_j(t)|^P \quad (1 \leq i \leq N, 0 \leq t \leq 1).$$

Since $f_i(0) = |f_i(0)|^P$, we have $f_i(0) = 0$ or 1 . If $\underline{f}(0) = \underline{0}$, then $\underline{f}(t) = 0$ for all $0 \leq t \leq 1$ by uniqueness of implicit functions.

Institute of Mathematics
Yoshida College
Kyoto University

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