

γ OPERATIONS IN K-THEORY AND EXISTENCE OF SINGULAR MAPS

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ABSTRACT. We introduce new obstructions to the existence of fold maps with orientable cokernel bundle by relating K-theory and the γ operation of Grothendieck [Ati61] to the h-principle of Ando [And04]. We compute these obstructions for fold maps of the projective spaces.

1. INTRODUCTION

For $n > k \geq 0$, let M^n and Q^{n-k} be a smooth closed n -dimensional and a smooth $(n-k)$ -dimensional manifold, respectively. We call a smooth map from M to Q a *corank 1 map* if the rank of its differential is not less than $n-k-1$ at any point of M . For a corank 1 map $f: M \rightarrow Q$ let Σ denote the set of singular points in M .

A basic example of a corank 1 map is a smooth map $M \rightarrow Q$ with only Morse type singularities, that is a *fold map*. Note that the restriction $f|_{\Sigma}$ is an immersion if f is a fold map.

Ando's h-principle [And04] states that there exists a fold map $f: M \rightarrow Q$ such that the immersion $f|_{\Sigma}$ is coorientable if and only if there exists a fiberwise epimorphism $TM \oplus \varepsilon^1 \rightarrow TQ$, also see [Sae92]. Note that in the case of even k the immersion $f|_{\Sigma}$ is always coorientable.

We call a corank 1 map $f: M \rightarrow Q$ *tame* if the 1-dimensional cokernel bundle $\text{coker } df|_{\Sigma}$ of the restriction $df|_{\Sigma}: TM|_{\Sigma} \rightarrow f^*TQ$ is trivial. For example, every fold map is tame for $k \equiv 0 \pmod{2}$ [And04] and it is easy to construct not tame fold maps for odd $k \leq n-3$, even between orientable manifolds.

Ando's h-principle [And04] enables us to reduce the problem of the existence of tame fold maps (and more generally tame corank 1 maps) to the existence of $n-k$ linearly independent sections of $TM \oplus \varepsilon^1$ if Q is stably parallelizable.

If such a partial framing exists, then clearly the Stiefel-Whitney classes $w_i(TM)$ vanish for $i \geq k+2$. Hence we obtain the following easy

Proposition 1.1. *Let $n+1 = 2^D m$, where $m > 1$ is odd. There is no tame corank 1 map from $\mathbb{R}P^n$ to any stably parallelizable Q^{n-k} if $2^D(m-1) \geq k+2$.*

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However, the tangential Stiefel-Whitney and Pontryagin classes of $\mathbb{R}P^{2^n-1}$ vanish, thus in order to obtain obstructions in this case, we need something else. By applying K-theory and following [Ati61], we obtain

Proposition 1.2. *If M^n admits a tame corank 1 map into a stably parallelizable Q^{n-k} , then $\gamma^i([TM] - [\varepsilon^n]) = 0$ for $i \geq k + 2$.*

Here γ denotes the γ operation in real K-theory, see [Ati61]. Proposition 1.2 can be useful if the higher tangential Stiefel-Whitney and Pontryagin classes of M vanish. For example, we obtain

Corollary 1.3. *Let $n \geq 4$, $2^{n-1} - 2^{\lfloor \log_2 n \rfloor} \geq k + 2$ and Q stably parallelizable.*

- (1) $\mathbb{R}P^{2^n-1}$ admits no fold map into Q^{2^n-1-k} if k is even,
- (2) $\mathbb{R}P^{2^n-1}$ admits no fold map with orientable singular set into Q^{2^n-1-k} if k is odd.

However, by using much sophisticated and deeper results of Atiyah, Bott and Shapiro [ABS64] and Steer [Ste67], which determine the geometric dimensions of the tangent bundles of the projective spaces, we have the stronger

Proposition 1.4. *There exists a tame corank 1 map from the projective space $\mathbb{F}P^n$ into an $(n-k)$ -dimensional stably parallelizable manifold if and only if $(n+1)d(\mathbb{F}) - q(n+1, \mathbb{F}) \leq k + 1$, where $d(\mathbb{F})$ denotes the dimension over $\mathbb{R}$ of the (skew) field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and $q(n+1, \mathbb{F})$ denotes the Radon-Hurwitz number associated to $n+1$ and $\mathbb{F}$.*

2. RESULTS

Let M^n and Q^{n-k} be a closed n -manifold and an $(n-k)$ -manifold, respectively. For a finite CW-complex X , $\tilde{K}_{\mathbb{R}}(X)$ and $K_{\mathbb{R}}(X)$ denote the reduced and unreduced real K-rings of X , respectively, with $\tilde{K}_{\mathbb{R}}(X) \subseteq K_{\mathbb{R}}(X)$. Recall that for a finite CW-complex X the geometric dimension $g.\dim(x)$ of an element $x \in \tilde{K}_{\mathbb{R}}(X)$ is the least integer k such that $x + k$ is a class of a genuine vector bundle over X (see e.g. [Ati61]).

Similarly to [And04], we have

Proposition 2.1. *The following are equivalent:*

- (1) M admits a tame corank 1 map into Q ,
- (2) there is a fiberwise epimorphism $TM \oplus \varepsilon^1 \rightarrow TQ$.

If Q is stably parallelizable, then (1) and (2) hold if and only if $g.\dim([TM] - [\varepsilon^n]) \leq k + 1$.

For a finite CW-complex X , let $\lambda_t = \sum_{i=0}^{\infty} \lambda^i t^i$, where λ^i are the exterior power operators (for details, see [Ati61]). Define $\gamma_t = \sum_{i=0}^{\infty} \gamma^i t^i$ to be the homomorphism $\lambda_{t/1-t}$ of $K_{\mathbb{R}}(X)$ into the multiplicative group of formal power series in t with coefficients in $K_{\mathbb{R}}(X)$ and constant term 1. By the above proposition and [Ati61, Proposition 2.3], we immediately have

Corollary 2.2. ¹ If M admits a tame corank 1 map into a stably parallelizable Q , then

- (1) $w_i(TM) = 0$ for $i \geq k + 2$,
- (2) $p_i(TM) = 0$ for $2i > k + 1$,
- (3) $\gamma^i([TM] - [\varepsilon^n]) = 0$ for $i \geq k + 2$.

Remark 2.3. Note that the conditions (1) and (2) may not give strong results in general: for example, all the positive degree Stiefel-Whitney and Pontryagin classes of $\mathbb{R}P^{2^n-1}$ vanish², and if $k + 1 \geq n/2$, then condition (2) is satisfied trivially for any M . In particular cases, though, condition (1) can still give strong results, e.g. all Stiefel-Whitney classes of $\mathbb{R}P^{2^n-2}$ of degree up to $2^n - 2$ are non-zero.

For an integer s let $2^{R(s)}$ be the maximal power of 2 that divides s , and define $\kappa(n) = \max\{0 < s < 2^{n-1} : s - R(s) < 2^{n-1} - n\}$. By using Corollary 2.2 (3) and following a similar argument to [Ati61], we obtain the following:

Proposition 2.4. For $n \geq 4$, $\mathbb{R}P^{2^n-1}$ does not admit tame corank 1 map into any stably parallelizable Q^{2^n-1-k} for $k \leq \kappa(n) - 2$.

Remark 2.5. Obviously $s_0 = 2^{n-1} - 2^{\min\{r:r+2^r>n\}}$ satisfies $s_0 + n - R(s_0) < 2^{n-1}$, thus $s_0 \leq \kappa(n)$ and we obtain that $\mathbb{R}P^{2^n-1}$ admits no fold map with orientable singular set into $\mathbb{R}^{2^{n-1}+2^{\min\{r:r+2^r>n\}}+j}$ for $n \geq 4$ and $j \geq 1$. Also, since $\min\{r : r + 2^r > n\} \leq \lceil \log_2 n \rceil$, the same conclusion holds in the case of the target $\mathbb{R}^{2^{n-1}+2^{\lceil \log_2 n \rceil}+j}$ for $n \geq 4$ and $j \geq 1$. For example, there exists neither a fold map from $\mathbb{R}P^{31}$ to $\mathbb{R}^{21+2j}$ for $0 \leq j \leq 5$ nor a fold map with orientable singular set from $\mathbb{R}P^{31}$ to $\mathbb{R}^{22+2j}$ for $0 \leq j \leq 4$.

Remark 2.6. However, we have stronger results about maps of the projective spaces that follow immediately from Proposition 2.1 and [Ste67], which determines the geometric dimensions of the tangent bundles of projective spaces in terms of Radon-Hurwitz numbers.

Proof of Proposition 2.1. (2) $\implies$ (1): By [And04], if there is a $TM \oplus \varepsilon^1 \rightarrow TQ$ epimorphism, then there is a fold map $M \rightarrow Q$ with orientable singular set. (1) $\implies$ (2): Assume that we have a tame corank 1 map $f: M \rightarrow Q$. The bundle $\text{coker } df|_\Sigma = (f^*TQ/f^*df(TM))|_\Sigma$ is considered as a subbundle of f^*TQ and it is trivial. Similarly to [And04, Proof of Lemma 3.1], let $L: \varepsilon^1 \rightarrow TQ$ be an extension of the bundle monomorphism $\text{coker } df|_\Sigma \rightarrow f^*TQ \rightarrow TQ$ as a bundle homomorphism covering f . Then $df + L$ is an epimorphism $TM \oplus \varepsilon^1 \rightarrow TQ$.

Finally, if (1) or (2) holds and Q is stably parallelizable, then by the above, we have $TM \oplus \varepsilon^1 \oplus \varepsilon^N \cong \zeta \oplus f^*TQ \oplus \varepsilon^N \cong \zeta \oplus \varepsilon^{N+n-k}$ for some $N \gg 0$ and a $(k+1)$ -dimensional bundle ζ . Thus $g.\dim([TM] - [\varepsilon^n]) \leq k + 1$.

¹Compare with [Ati61, Proposition 3.2].

²We have $w(T\mathbb{R}P^{2^n-1}) = (1+x)^{2^n} = 1 \in \mathbb{Z}_2[x]/x^{2^n} = H^*(\mathbb{R}P^{2^n-1}; \mathbb{Z}_2)$, where x denotes the generator of $H^1(\mathbb{R}P^{2^n-1}; \mathbb{Z}_2)$. The natural homomorphism $H^s(\mathbb{R}P^{2^n-1}; \mathbb{Z}) \rightarrow H^s(\mathbb{R}P^{2^n-1}; \mathbb{Z}_2)$ is an isomorphism for all positive even s . Our claim follows by applying the fact that $p_i \equiv w_{2i}^2 \pmod{2}$.

If Q is stably parallelizable and $g.\dim([TM] - [\varepsilon^n]) \leq k + 1$, then $TM \oplus \varepsilon^N \cong \zeta^{k+1} \oplus \varepsilon^{N+n-k-1} \cong \zeta^{k+1} \oplus TQ \oplus \varepsilon^{N-1}$ for some $N \gg 0$, and thus $TM \oplus \varepsilon^1 \cong \zeta^{k+1} \oplus TQ$, which proves (2). $\square$

Proof of Proposition 2.4. Let $\varphi(n)$ denote the cardinality of the set $\{0 < s \leq n : s \equiv 0, 1, 2, 4 \pmod{8}\}$. By [Ati61, §5], $[T\mathbb{R}P^n] - [\varepsilon^n] = (n+1)x$ and $\gamma^i([T\mathbb{R}P^n] - [\varepsilon^n]) = 2^{i-1} \binom{n+1}{i} x$, $i \geq 1$, where x denotes the generator of $\tilde{K}_{\mathbb{R}}(\mathbb{R}P^n) = \mathbb{Z}_{2^{\varphi(n)}}$. Therefore $\gamma^i([T\mathbb{R}P^n] - [\varepsilon^n]) = 0$ if and only if $2^{\varphi(n)}$ divides $2^{i-1} \binom{n+1}{i}$. Let $r(n)$ denote the greatest integer s for which $2^{s-1} \binom{n+1}{s}$ is not divisible by $2^{\varphi(n)}$. Then by Proposition 2.1 there is no tame corank 1 map of $\mathbb{R}P^{2^n-1}$ into $\mathbb{R}^{2^n-1-k}$ for $k \leq r(2^n-1) - 2$. It is easy to see that $\varphi(2^n-1) = 2^{n-1} - 1$ if $n \geq 3$. By a classical result of E. Kummer, the highest power $c(s)$ of 2 which divides $\binom{2^n}{s}$ can be obtained by counting the number of carries when s and $2^n - s$ are added in base 2. For $s \leq 2^{n-1} - 1$, we claim that $c(s) = n - R(s)$, where $2^{R(s)}$ is the maximal power of 2 which divides s . Indeed, $2^n - 1 - s$ is obtained by negating the binary form of s bitwise, hence $2^n - s$ is obtained by negating the binary form of s bitwise from the $(n-1)$ st to the $R(s)$ th binary position, where both of s and $2^n - s$ have the digit 1, and after that position both have digits 0. Therefore when we add s and $2^n - s$ in base 2, we have $n - R(s)$ carries. By the definition of $r(n)$ it follows that $r(2^n-1)$ is the largest integer s for which $s + n - R(s) < 2^{n-1}$. $\square$

When n is not a power of 2, we have the following easy results for $\mathbb{R}P^{n-1}$.

Proposition 2.7. *Let $n = 2^D m$, where $m > 1$ is odd. Then $\binom{n}{2^D}$ is odd. Hence $w_{2^D(m-1)}(T\mathbb{R}P^{n-1}) \neq 0$.*

Proof. It is obvious from [Gla99], details are left to the reader. $\square$

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