

Local Langlands correspondence for covering groups of tori, and the packet-indexing groups

Yuki Nakata

ABSTRACT. This paper determines the finite group $\mathcal{S}$ that indexes the representations within a packet in the local Langlands correspondence for a Brylinski-Deligne covering group of an algebraic torus. Weissman formulated this correspondence for functorial covers of a torus defined by Brylinski and Deligne. Inspired by Weissman's framework, we extend his result to include ramified cases, under the assumption that the ramification index of a fixed splitting field is relatively prime to the degree of the cover. Our proof relies on an orthogonality relation in the local Tate duality.

CONTENTS

1. Introduction	1
2. Sieves, sites and sheaves	5
3. Torsors and their addition	6
4. Picard categories	8
5. Central extensions and multiplicative torsors	11
6. Small vs. big sites	13
7. Simplicial construction of a 2-cocycle	14
8. Generic cocycles	18
9. Brylinski-Deligne covering groups of a torus	21
10. Local Langlands correspondence for a covering group	23
11. Description of the image $Z^\dagger$ of the center	26
12. Main theorem	33
13. Application to the global correspondence	39
Appendix A. 2-cocycles on the Chevalley group SL_2	44
Acknowledgements	49
References	49

1. INTRODUCTION

This paper aims to determine the finite group $\mathcal{S} = \mathcal{S}_{\tilde{T}}$ whose dual parametrizes a packet in the local Langlands correspondence for a covering group $\tilde{T} \rightarrow T$ of an algebraic torus. As in the case of reductive algebraic groups, their covering groups have also attracted arithmetic

Date: 2026.1.20, revised.

interest. For instance, Weil studied the Segal-Shale-Weil representations of the non-trivial double cover $\mathrm{Mp}_{2r} \rightarrow \mathrm{Sp}_{2r}$, in order to prove an identity now known as the Siegel-Weil formula [16]. Another notable example is an Eisenstein series on a double cover of GL_r , which yields a Rankin-Selberg integral representing the symmetric square L-function of a cuspidal representation of GL_r [3]. These examples motivate us to pursue the Langlands program for covering groups.

In the well-known case of a reductive algebraic group G over a non-archimedean local field F , we have the notions of the complex dual group $\widehat{G}$ of the group G and a Langlands parameter $\phi: W_F \times \mathrm{SL}_2(\mathbb{C}) \rightarrow {}^L G = \widehat{G} \rtimes W_F$. The centralizer $S_\phi = \mathrm{Cent}_{\widehat{G}}(\mathrm{Im} \phi)$ is conjectured to index the representations in the L-packet Π_ϕ , through its finite quotient

$$\mathcal{S}_\phi = S_\phi / S_\phi^0 Z(\widehat{G})^{W_F}.$$

Here, S_ϕ^0 is the connected component of the identity in the complex reductive group S_ϕ , and $Z(\widehat{G})^{W_F}$ is the subgroup consisting of fixed points in the center $Z(\widehat{G}) \subset \widehat{G}$ under the action of the Weil group W_F of the base field F . In this paper, we investigate an analogue of the group $\mathcal{S}_\phi$ for covering groups.

Although a local Langlands correspondence for a general covering group has not yet been satisfactorily formulated, Weissman [17] proposed one for a covering $\widetilde{T} \rightarrow T$ of a torus. In his framework, a certain finite abelian group $\mathcal{S} = \mathcal{S}_{\widetilde{T}}$ parametrizes the representations in a packet (see Section 10), which is analogous to $\mathcal{S}_\phi$ in the non-covering case. He also explicitly determined the group $\mathcal{S}_{\widetilde{T}}$ for a cover of an unramified torus T , that is, a torus splitting over an unramified extension of the base field.

To describe the group $\mathcal{S}_{\widetilde{T}}$, Weissman introduced certain subgroups $Y^\#$ and $Y^{\Gamma\#}$ of the cocharacter lattice Y of T . In his formulation of the local Langlands correspondence, the cover $\mu_n \rightarrow \widetilde{T} \rightarrow T$ is assumed to be a functorial one defined by Brylinski and Deligne [2], which yields the groups $Y^\#$ and $Y^{\Gamma\#}$ as follows (see Section 9). Such a cover $\widetilde{T}$ is known to be associated with a bilinear form $B_{\widetilde{T}}: Y \times Y \rightarrow \mathbb{Z}$. For any subgroup $Y' \subset Y$, the subgroup $Y'^\#$ of Y is defined by

$$Y'^\# = \{y \in Y \mid B_{\widetilde{T}}(y, Y') \subset n\mathbb{Z}\}.$$

In the present case, this subgroup Y' is Y itself or the subgroup Y^Γ of fixed points under the action of the absolute Galois group $\Gamma = \Gamma_F = \mathrm{Gal}(\overline{F}/F)$ of F .

For a subgroup Y' of Y , let ι denote the inclusion map $Y' \hookrightarrow Y$ and the induced ones, e.g. $\iota: (Y' \otimes \overline{\mathbf{f}}^\times)^\Gamma \rightarrow (Y \otimes \overline{\mathbf{f}}^\times)^\Gamma$, where $\overline{\mathbf{f}}^\times$ is the multiplicative group of an algebraic closure of the residue field $\mathbf{f}$ of the valuation ring in F . With this notation, Weissman's theorem can be stated as follows.

Theorem 1.1 (Weissman [17]). Let F be a non-archimedean local field of characteristic zero, $\mathbf{f}$ the residue field of the valuation ring in F , and $\mu_n \subset \mathbf{f}^\times$ the cyclic subgroup of order n . Suppose that an algebraic torus T defined over F splits over an unramified extension L/F , and that $\mu_n \rightarrow \tilde{T} \rightarrow T$ is a Brylinski-Deligne covering group.

Then the group $\mathcal{S} = \mathcal{S}_{\tilde{T}}$ parametrizing a packet of representations of $\tilde{T}$ is written as the quotient

$$\mathcal{S}_{\tilde{T}} = \iota((Y^{\Gamma\#} \otimes_{\mathbb{Z}} \bar{\mathbf{f}}^\times)^\Gamma) / \iota((Y^\# \otimes_{\mathbb{Z}} \bar{\mathbf{f}}^\times)^\Gamma).$$

of subgroups of $(Y \otimes_{\mathbb{Z}} \bar{\mathbf{f}}^\times)^\Gamma$.

While Weissman's framework assumes the torus to be unramified, our result removes this restriction. We assume only that the ramification index e of a fixed splitting field L/F is relatively prime to the degree n of the cover. Precisely:

Main Theorem (12.5). Let F be a non-archimedean local field of characteristic zero, $\mathbf{f}$ the residue field of the valuation ring of F , and $\mu_n \subset \mathbf{f}^\times$ the cyclic subgroup of order n . Suppose that an algebraic torus T defined over F splits over a finite Galois extension L/F whose ramification index e is relatively prime to n . Let $\mu_n \rightarrow \tilde{T} \rightarrow T$ be a Brylinski-Deligne covering group.

Then the group $\mathcal{S} = \mathcal{S}_{\tilde{T}}$ parametrizing a packet of representations of $\tilde{T}$ is written as the quotient

$$\mathcal{S}_{\tilde{T}} = \iota((Y^{\Gamma\#} \otimes_{\mathbb{Z}} \bar{\mathbf{f}}^\times)^\Gamma) / \iota((Y^\# \otimes_{\mathbb{Z}} \bar{\mathbf{f}}^\times)^\Gamma).$$

of subgroups of $(Y \otimes_{\mathbb{Z}} \bar{\mathbf{f}}^\times)^\Gamma$.

In particular, this theorem recovers Weissman's formula in the unramified case. Our main theorem extends his result by allowing ramified splitting fields L/F , and partially realizes his hope [17, Remark 5.18] to parametrize a packet in the correspondence for a cover of a ramified torus.

Most of this paper is devoted to a survey of the Brylinski-Deligne theory on covering groups. A Brylinski-Deligne covering group is formulated as a sheaf on the big Zariski site of the base field. In Section 2, we review basic notions related to sheaves on a general site, which is a category endowed with a Grothendieck topology. A covering group, or a central extension, is regarded as a torsor. We introduce torsors on a site in Section 3. Central extensions form a groupoid, which further bears an additive structure called a Picard category. We define it in Section 4.

In Section 5, we construct an equivalence between the Picard categories of central extensions and that of torsors with some multiplicative structure. We initially treat torsors on a big Zariski site, but it is often

useful to restrict it to a small Zariski site. As we explain in Section 6, this restriction defines an equivalence of categories. In Section 7, we review an abstract method to construct a 2-cocycle of a central extension of sheaves on a site. Since this method is useless for explicit calculation, we give an alternative way in Section 8.

We explain the definition and the classifying theorem of a Brylinski-Deligne covering group of a torus in Section 9. We formulate the local Langlands correspondence for a Brylinski-Deligne cover of a torus, and define the finite group $\mathcal{S}_{\tilde{T}}$ as a quotient of the image $Z^\dagger \subset T$ of the center $Z(\tilde{T}) \subset \tilde{T}$ in Section 10.

To prove the main theorem, we describe the subgroup $Z^\dagger \subset T$ by applying the Galois cohomology theory in Section 11. We overcome a critical part by reducing it to a certain orthogonality in the local Tate duality and prove the main theorem in Section 12. As an application of the local result, we prove the finiteness of a certain abelian group appearing in the global Langlands correspondence for a Brylinski-Deligne covering group of a torus in Section 13. In the appendix, we explain another way to construct a 2-cocycle on the Chevalley group SL_2 .

Notation. Let $\mathbb{Z}_{>0}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{C}$ respectively denote the set of positive rational integers, the ring of rational integers, the field of rational numbers, and the field of complex numbers. The symbol $\otimes$ always denotes the tensor product over $\mathbb{Z}$. For a map $f: A \rightarrow B$ and a subset $A' \subset A$, let $f|_{A'}$ denote the restriction, as usual. For a group G , let $Z(G) \subset G$ denote the center. If a group G acts on an abelian group A , then $A^G = H^0(G, A)$ is the subgroup consisting of fixed points. For a ring R , let $R^\times$ denote the group of units.

We write F for a non-archimedean local field of characteristic zero, $\mathcal{O} = \mathcal{O}_F$ for its valuation ring, and $\mathbf{f}$ for the residue field. We fix an algebraic closure $\overline{F}/F$, and write $\overline{\mathcal{O}} \subset \overline{F}$ for its valuation ring and $\overline{\mathbf{f}}$ for the residue field of $\overline{\mathcal{O}}$. Note that $\overline{\mathbf{f}}$ is an algebraic closure of the field $\mathbf{f}$. For a field k , let $\Gamma_k = \mathrm{Gal}(\overline{k}/k)$ denote its absolute Galois group. In the case of the local field F , we often omit the subscript and simply write $\Gamma = \Gamma_F$.

In this paper, we treat an algebraic torus $\mathbb{T}$ defined over the local field F , and fix a finite Galois extension L/F where $\mathbb{T}$ splits over L . Let $Y = \mathrm{Hom}(\mathbb{G}_m, \mathbb{T})$ be the cocharacter lattice, $T = \mathbb{T}(F) = (Y \otimes_{\mathbb{Z}} \overline{F}^\times)^\Gamma$ the group of rational points, and $e \in \mathbb{Z}_{>0}$ the ramification index of the extension L/F . We write $\mathrm{ord}: F^\times \rightarrow \mathbb{Z}$ for the map of additive valuation, and extend it to $\overline{F}^\times \rightarrow \mathbb{Q}$. We continue to write ord for the induced map

$$T = (Y \otimes \overline{F}^\times)^\Gamma \rightarrow (Y \otimes \mathbb{Q})^\Gamma = Y^\Gamma \otimes \mathbb{Q}.$$

Throughout the paper, the symbol ι denotes various morphisms between lattices or tori induced by inclusion, depending on the context.

For instance, $\iota: (Y' \otimes \overline{F}^\times)^\Gamma \rightarrow (Y \otimes \overline{F}^\times)^\Gamma$ is the morphism between tori induced by the inclusion $Y' \hookrightarrow Y$ of a subgroup.

2. SIEVES, SITES AND SHEAVES

A Brylinski-Deligne covering group is formulated as a sheaf on the big Zariski site of the base field. Here we review basic notions related to sheaves on a general site, which is a category endowed with a Grothendieck topology.

Let $\mathcal{C}$ be a category, and X an object of $\mathcal{C}$. A *sieve* on X is a subset $\Sigma \subset \text{Ob}(\mathcal{C}_{/X})$ such that for any $(A \xrightarrow{\sigma} X) \in \Sigma$ and $(B \xrightarrow{\beta} A) \in \text{Ob}(\mathcal{C}_{/A})$, one has $(B \xrightarrow{\sigma \circ \beta} X) \in \Sigma$.

For a morphism $f: Y \rightarrow X$ in $\mathcal{C}$, the pullback of a sieve $\Sigma \subset \text{Ob}(\mathcal{C}_{/X})$ is defined as $f^*\Sigma = \{(A \xrightarrow{\alpha} Y) \in \text{Ob}(\mathcal{C}_{/Y}) \mid (A \xrightarrow{f \circ \alpha} X) \in \Sigma\}$. This is indeed a sieve by definition. If $\Sigma \subset \Sigma'$ are sieves on X , then $f^*\Sigma \subset f^*\Sigma'$.

Definition 2.1. A *site* is a category S equipped with a map

$$\mathcal{U}: \text{Ob } S \ni X \mapsto \mathcal{U}_X \subset 2^{\text{Ob}(S_{/X})}$$

satisfying the following properties for any $X \in \text{Ob } S$:

- (1) Any $\Sigma \in \mathcal{U}_X$ is a sieve on X .
- (2) (non-emptiness) $\text{Ob}(S_{/X}) \in \mathcal{U}_X$.
- (3) (pullback) For any $(Y \xrightarrow{f} X) \in S_{/X}$ and $\Sigma \in \mathcal{U}_X$, we have $f^*\Sigma \in \mathcal{U}_Y$.
- (4) (localization) Let $\Sigma' \subset \text{Ob}(S_{/X})$ be a sieve, and suppose that for some $\Sigma \in \mathcal{U}_X$, any $(U \xrightarrow{\eta} X) \in \Sigma$ gives $\eta^*\Sigma' \in \mathcal{U}_U$. Then $\Sigma' \in \mathcal{U}_X$.

Let $(S, \mathcal{U})$ be a site, and $X \in \text{Ob } S$. A *covering sieve* on X is just an element $\Sigma \in \mathcal{U}_X$. For $(Y_\varepsilon \xrightarrow{\varepsilon} X) \in S_{/X}$, let $(\mathcal{U}_{/X})_{(Y_\varepsilon \xrightarrow{\varepsilon} X)} = \mathcal{U}_{Y_\varepsilon}$. Then $(S_{/X}, \mathcal{U}_{/X})$ is a site, called a localized site

Proposition 2.2. Let $(S, \mathcal{U})$ be a site, and X an object of S .

- (1) Any sieve $\Sigma' \subset \text{Ob}(S_{/X})$ containing a covering sieve $\Sigma \in \mathcal{U}_X$ is also a covering sieve.
- (2) For any two covering sieves $\Sigma, \Sigma' \in \mathcal{U}_X$, the intersection $\Sigma \cap \Sigma'$ is also a covering sieve.

Proof. Let $(U \xrightarrow{\eta} X) \in \Sigma$. (1) By definition, $\eta^*\Sigma' = \text{Ob}(S_{/X}) \in \mathcal{U}_U$. (2) Similarly, $\eta^*(\Sigma \cap \Sigma') = \eta^*\Sigma' \in \mathcal{U}_U$. $\square$

Definition 2.3. A functor $A: S^{\text{op}} \rightarrow \mathbf{Set}$ is called a *sheaf* if the following statement holds. Let $X \in \text{Ob } S$, $\Sigma \in \mathcal{U}_X$, and

$$(s_\eta)_\eta \in \prod_{(U_\eta \xrightarrow{\eta} X) \in \Sigma} \Gamma(U_\eta, A),$$

and suppose that for any $(U_\eta \xrightarrow{\eta} X) \in \Sigma$ and $(V \xrightarrow{\nu} U_\eta) \in \text{Ob}(S/U_\eta)$, one has $s_\eta|_V = s_{\eta \circ \nu}$; then there exists a unique $s \in \Gamma(X, A)$ such that for any $(U_\eta \xrightarrow{\eta} X) \in \Sigma$, one has $s|_{U_\eta} = s_\eta$.

For a sheaf $P: S^{\text{op}} \rightarrow \mathbf{Set}$ and $(Y_\varepsilon \xrightarrow{\varepsilon} X) \in S/X$, let $P|_X(Y_\varepsilon \xrightarrow{\varepsilon} X) := P(Y_\varepsilon)$. Then the restriction $P|_X: S/X^{\text{op}} \rightarrow \mathbf{Set}$ is also a sheaf.

Recall that a presheaf on S is just a functor $S^{\text{op}} \rightarrow \mathbf{Set}$. The sheafification functor

$$\mathbf{a}: \text{PSh}(S, \mathbf{Set}) \rightarrow \text{Sh}(S, \mathbf{Set})$$

is the left adjoint to the inclusion functor

$$\text{Sh}(S, \mathbf{Set}) \rightarrow \text{PSh}(S, \mathbf{Set}).$$

Remark 2.4 ([8]). (1) The sheafification functor $\mathbf{a}$ exists.

(2) Sheafification commutes with restriction. Precisely, let $(S, \mathcal{U})$ be a site, and $X \in \text{Ob } S$. Then the following diagram commutes:

$$\begin{array}{ccc} \text{PSh}(S, \mathbf{Set}) & \xrightarrow{\mathbf{a}} & \text{Sh}(S, \mathbf{Set}) \\ (\cdot)|_X \downarrow & & \downarrow (\cdot)|_X \\ \text{PSh}(S/X, \mathbf{Set}) & \xrightarrow{\mathbf{a}} & \text{Sh}(S/X, \mathbf{Set}) \end{array}$$

(3) We can similarly discuss the sheaves of groups on a site.

3. TORSORS AND THEIR ADDITION

A torsor is an object sometimes called a principal homogeneous space. We review the definition and basic properties of a torsor on a site. Let $(S, \mathcal{U})$ be a site, and $A: S^{\text{op}} \rightarrow \mathbf{Grp}$ a sheaf of groups.

Definition 3.1. A sheaf P of right A -sets on S is called an A -torsor if for any $X \in \text{Ob } S$, there exists some $\Sigma \in \mathcal{U}_X$ such that the restrictions of P and A to any $(U_\eta \xrightarrow{\eta} X) \in \Sigma$ are isomorphic as right $A|_{U_\eta}$ -sheaves:

$$P|_{U_\eta} \cong A|_{U_\eta}.$$

As the simplest example, let S be the space consisting of one point, i.e., the sheaves on S are just sets. Then for a non-empty right A -set P , the following conditions are equivalent:

- There is an isomorphism $P \cong A$ as right A -sets.
- The action $P \curvearrowright A$ is simply transitive.
- The map $P \times A \rightarrow P \times P, (p, a) \mapsto (p, pa)$ is bijective.

On any site $(S, \mathcal{U})$, a morphism $f: P \rightarrow Q$ of A -torsors is an isomorphism. Indeed, let X be an object of S . Since $P \curvearrowright A$ is a torsor, we find a covering sieve $\Sigma \in \mathcal{U}_X$ such that for any $(U_\eta \xrightarrow{\eta} X) \in \Sigma$, the restrictions $P|_{U_\eta}$ and $A|_{U_\eta}$ are isomorphic as right $A|_{U_\eta}$ -sheaves. For any $(U_\eta \xrightarrow{\eta} X) \in \Sigma$, we see that $f_{U_\eta}: P(U_\eta) \rightarrow Q(U_\eta)$ is an isomorphism of $A(U_\eta)$ -sets, since $P(U_\eta) \neq \emptyset$.

Proposition 3.2 (Product). For $i = 1, 2$, let $A_i: S^{\text{op}} \rightarrow \mathbf{Grp}$ be a sheaf, and $P_i \curvearrowright A_i$ a torsor. Then $P_1 \times P_2 \curvearrowright A_1 \times A_2$ is a torsor.

Proof. Let X be an object of S . For some covering sieve $\Sigma_i \in \mathcal{U}_X$, the restrictions to any $(U_\eta \xrightarrow{\eta} X) \in \Sigma_i$ gives an isomorphism $P_i|_{U_\eta} \cong A_i|_{U_\eta}$ as right $A_i|_{U_\eta}$ -sheaves, by definition. Then $\Sigma_1 \cap \Sigma_2 \in \mathcal{U}_X$ by 2.2. For any $(U_\eta \xrightarrow{\eta} X) \in \Sigma_1 \cap \Sigma_2$, we get

$$(P_1 \times P_2)|_{U_\eta} \cong (A_1 \times A_2)|_{U_\eta}$$

as right $A_1 \times A_2|_{U_\eta}$ -sheaves. $\square$

Let $f: A \rightarrow B$ be a morphism of sheaves of groups, and $P \curvearrowright A$ a torsor. A push-out of P by f is a torsor $Q = f(P) \curvearrowright B$ equipped with a map $f: P \rightarrow Q$ of sheaves such that for any object X in the site S and sections $p \in P(X)$ and $a \in A(X)$, one has $f(pa) = f(p)f(a)$.

Proposition 3.3 (Universality of push-out). Let $f: A \rightarrow B$ be a morphism of sheaves of groups, $P \curvearrowright A$ a torsor, and $(Q_1, f_1), (Q_2, f_2)$ two push-outs of P by f . Then there exists a unique morphism $\chi: Q_1 \rightarrow Q_2$ of B -torsors such that $\chi \circ f_1 = f_2$.

Proof. Let X be an object in the site S , and $q \in Q_1(X)$. For some $\Sigma \in \mathcal{U}_X$, the restrictions to any $(U_\eta \xrightarrow{\eta} X) \in \Sigma$ gives an isomorphism $P|_{U_\eta} \cong A|_{U_\eta}$ as right $A|_{U_\eta}$ -sheaves, by definition. Especially, for any $(U_\eta \xrightarrow{\eta} X) \in \Sigma$, we have a section $p_\eta \in P(U_\eta)$. Since $Q_i \curvearrowright B$ is a torsor, there is a unique section $b_\eta \in B(U_\eta)$ such that

$$q|_{U_\eta} = f_1(p_\eta)b_\eta.$$

First, we show the existence of χ . For $b' \in B(U_\eta)$, let

$$\chi(q|_{U_\eta}b') := f_2(p_\eta)b_\eta b'.$$

This gives a well-defined map $\chi: Q_1(U_\eta) \rightarrow Q_2(U_\eta)$. Indeed, for $a \in A(U_\eta)$, we have $q|_{U_\eta} = f_1(p_\eta a)f(a^{-1})b_\eta$, and $f_2(p_\eta a)f(a^{-1})b_\eta b' = f_2(p_\eta)b_\eta b'$. By definition, we see that $\chi \circ f_1 = f_2$, and that χ commutes with the right action of B and defines a morphism of B -torsors $Q_1 \rightarrow Q_2$.

To show the uniqueness of χ , let $\chi': Q_1 \rightarrow Q_2$ be another morphism of B -torsors such that $\chi' \circ f_1 = f_2$. Then for any $b' \in B(U_\eta)$, we get

$$\chi'(q|_{U_\eta}b') = \chi'(f_1(p_\eta))b_\eta b' = f_2(p_\eta)b_\eta b' = \chi(q|_{U_\eta}b').$$

Thus $\chi' = \chi$ as morphisms of torsors. $\square$

Proposition 3.4 (Construction of the push-out). Let $f: A \rightarrow B$ be a morphism of sheaves of groups, and $P \curvearrowright A$ a torsor. We define an action $A \curvearrowright P \times B$ by

$$a \cdot (p, b) := (pa^{-1}, f(a)b).$$

Then the push-out of P is the quotient $f(P) := A \backslash (P \times B)$ as a sheaf, i.e. the sheaf associated with the quotient $A \backslash_{\text{PSh}}(P \times B)$ as a presheaf.

Proof. (1) The right action $A \backslash_{\text{PSh}}(P \times B) \circ B$ is simply transitive, over each object X in the site. Indeed, let $[p, b], [p', b'] \in (A \backslash_{\text{PSh}}(P \times B))(X)$. Since $P \circ A$ is a torsor, we get some $a \in A(X)$ such that $pa = p'$ in $P(X)$. Then

$$[p', b'] = [pa, b'] = [p, f(a)b'] = [p, b]b^{-1}f(a)b'$$

in $A(X) \backslash (P \times Q)(X)$, i.e. the action is transitive. We see the action is free by definition.

(2) The associated sheaf $A \backslash (P \times Q) \circ B$ is a torsor. More generally, let $R: S^{\text{op}} \rightarrow \mathbf{Set}$ be a presheaf, and $R \circ B$ a simply transitive action. Then the associated sheaf $\mathbf{a}P \circ B$ is a torsor. Indeed, let X be an object of S . For some $\Sigma \in \mathcal{U}_X$, the restriction to any $(U_\eta \xrightarrow{\eta} X) \in \Sigma$ gives an isomorphism $P|_{U_\eta} \cong A|_{U_\eta}$ as right $A|_{U_\eta}$ -presheaves, by assumption. Then for $(U_\eta \xrightarrow{\eta} X) \in \Sigma$, we have

$$A|_{U_\eta} \cong P|_{U_\eta} = \mathbf{a}(P|_{U_\eta}) = (\mathbf{a}P)|_{U_\eta}$$

as right $A|_{U_\eta}$ -sheaves.

(3) It remains to show that the composite

$$P \xrightarrow{f' := (\text{id}, 1)} A \backslash_{\text{PSh}}(P \times B) \rightarrow A \backslash (P \times Q)$$

is compatible with the morphism $f: A \rightarrow B$. The map f' is compatible with f . Indeed, let X be an object in the site, and $p \in P(X)$ and $a \in A(X)$ two sections. Then

$$f'(pa) = [pa, 1] = [p, f(a)] = [p, 1]f(a) = f'(p)f(a).$$

The canonical map $A \backslash_{\text{PSh}}(P \times B) \rightarrow A \backslash (P \times Q)$ is B -equivariant, i.e., the diagram

$$\begin{array}{ccc} A \backslash_{\text{PSh}}(P \times B) \times B & \longrightarrow & A \backslash_{\text{PSh}}(P \times B) \\ \downarrow & & \downarrow \\ A \backslash (P \times B) \times B & \longrightarrow & A \backslash (P \times B) \end{array}$$

commutes, since the sheafification commutes with the product. $\square$

Definition 3.5. For a sheaf $A: S^{\text{op}} \rightarrow \mathbf{Ab}$ of abelian groups and torsors $P_1, P_2 \circ A$, we have the product torsor $P_1 \times P_2 \circ A \times A$. The *addition* $P_1 + P_2 \circ A$ is the push-out of $P_1 \times P_2$ by the multiplication map $A \times A \rightarrow A$.

4. PICARD CATEGORIES

The category of central extensions is a groupoid. Further, it has a finer structure called a Picard category. In our terminology, a semi-group may not have the identity element. For example, the empty set $\emptyset$ is a trivial semigroup. Let $S(A)$ denote the free abelian semigroup generated by A , and $e: A \rightarrow S(A)$ the canonical injection.

Definition 4.1. Let $\mathcal{C}$ be a category. An *associative and strictly commutative functor* on $\mathcal{C}$ is a functor $F: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ equipped with isomorphisms

$$\begin{aligned}\sigma_{X,Y,Z}: F(F(X,Y), Z) &\xrightarrow{\sim} F(X, F(Y, Z)) \\ \tau_{X,Y}: F(X, Y) &\xrightarrow{\sim} F(Y, X)\end{aligned}$$

between the functors that admit the existence of the following:

- (1) a map $\mathbf{F}: S(\text{Ob } \mathcal{C}) \rightarrow \text{Ob } \mathcal{C}$;
- (2) for each $X \in \text{Ob } \mathcal{C}$, an isomorphism $a_X: \mathbf{F}(e(X)) \xrightarrow{\sim} X$; and
- (3) for each $g, h \in S(\text{Ob } \mathcal{C})$, an isomorphism $a_{g,h}: \mathbf{F}(gh) \xrightarrow{\sim} F(\mathbf{F}(g), \mathbf{F}(h))$

such that the following two diagrams commute for any $f, g, h \in S(\text{Ob } \mathcal{C})$:

$$\begin{array}{ccccc} \mathbf{F}(f(gh)) & \xrightarrow[\sim]{a_{f,gh}} & F(\mathbf{F}(f), \mathbf{F}(gh)) & \xrightarrow[\sim]{F(\text{id}_{\mathbf{F}(f)}, a_{g,h})} & F(\mathbf{F}(f), F(\mathbf{F}(g), \mathbf{F}(h))) \\ \parallel & & & & \uparrow \wr \sigma_{\mathbf{F}(f), \mathbf{F}(g), \mathbf{F}(h)} \\ \mathbf{F}((fg)h) & \xrightarrow[\sim]{a_{fg,h}} & F(\mathbf{F}(fg), \mathbf{F}(h)) & \xrightarrow[\sim]{F(a_{f,g}, \text{id}_{\mathbf{F}(h)})} & F(F(\mathbf{F}(f), \mathbf{F}(g)), \mathbf{F}(h)), \end{array}$$

and

$$\begin{array}{ccc} \mathbf{F}(gh) & \xrightarrow[\sim]{a_{g,h}} & F(\mathbf{F}(g), \mathbf{F}(h)) \\ \parallel & & \downarrow \wr \tau_{\mathbf{F}(g), \mathbf{F}(h)} \\ \mathbf{F}(hg) & \xrightarrow[\sim]{a_{h,g}} & F(\mathbf{F}(h), \mathbf{F}(g)). \end{array}$$

The notion of an associative and strictly commutative functor enables us to define a Picard category. Precisely:

Definition 4.2 ([1, XVIII Définition 1.4.2]). A *strictly commutative Picard category* is a non-empty category $\mathcal{P}$ equipped with a functor $+: \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P}$ and isomorphisms

$$\begin{aligned}\sigma_{X,Y,Z}: (X + Y) + Z &\xrightarrow{\sim} X + (Y + Z) \\ \tau_{X,Y}: X + Y &\xrightarrow{\sim} Y + X\end{aligned}$$

between the functors such that

- (1) the functor $+$ is associative and strictly commutative with respect to σ and τ ; and
- (2) for any $X \in \text{Ob } \mathcal{P}$, the functor $Y \mapsto X + Y$ is an equivalence of the category.

Then we should also define a functor of strictly commutative Picard categories.

Definition 4.3. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two strictly commutative Picard categories.

- (1) An *additive functor* $F: \mathcal{P}_1 \rightarrow \mathcal{P}_2$ between Picard categories is a functor equipped with an isomorphism

$$F(X + Y) \xrightarrow{\sim} F(X) + F(Y)$$

of functors such that the following two diagrams commute:

$$\begin{array}{ccc} F(X + Y) & \longrightarrow & F(X) + F(Y) \\ \downarrow F(\tau) & & \downarrow \tau \\ F(Y + X) & \longrightarrow & F(Y) + F(X), \end{array}$$

and

$$\begin{array}{ccccc} F((X + Y) + Z) & \longrightarrow & F(X + Y) + F(Z) & \longrightarrow & (F(X) + F(Y)) + F(Z) \\ \downarrow F(\sigma) & & & & \downarrow \sigma \\ F(X + (Y + Z)) & \longrightarrow & F(X) + F(Y + Z) & \longrightarrow & F(X) + (F(Y) + F(Z)). \end{array}$$

- (2) Let F and G be two additive functors between two Picard categories $\mathcal{P}_1$ and $\mathcal{P}_2$. Then a morphism $F \rightarrow G$ of additive functors is a morphism u of functors making the diagram

$$\begin{array}{ccc} F(X + Y) & \xrightarrow{u_{X+Y}} & G(X + Y) \\ \downarrow & & \downarrow \\ F(X) + F(Y) & \xrightarrow{u_X + u_Y} & G(X) + G(Y) \end{array}$$

commute.

These definitions provide the appropriate definition of an equivalence of strict commutative Picard categories. If two strict commutative Picard categories are said to be equivalent, this usually means that they are equivalent as strict commutative Picard categories, not just as categories.

As a typical example, the central extensions of an algebraic group G over a scheme S by a sheaf A of abelian groups on the big Zariski site S_{ZAR} form a strictly commutative Picard category with the Baer sum. Another important example is derived from a cochain complex $K = \{K^n\}_n$. As usual, let $Z^n(K)$ be the group of n -cocycles and $B^n(K)$ the group of n -coboundaries. We make the set $Z^n(K)$ into a category $\mathcal{C}_n(K)$ by defining a morphism between two objects $z_0, z_1 \in Z^n(K)$ as an element $k \in K^{n-1}/B^{n-1}(K)$ such that

$$z_1 - z_0 = dk.$$

For three objects z_0, z_1, z_2 in the category $\mathcal{C}_n(K)$ and two morphisms $k_0: z_0 \rightarrow z_1$ and $k_1: z_1 \rightarrow z_2$, the composite $k_1 \circ k_0: z_0 \rightarrow z_2$ is just the sum $k_1 + k_0 \in K^{n-1}/B^{n-1}(K)$. The addition of two objects in the category $\mathcal{C}_n(K)$ is defined as the addition of n -cocycles. Thus $\mathcal{C}_n(K)$ is a strictly commutative Picard category. Note that the group of isomorphism classes of objects in the category $\mathcal{C}_n(K)$ is the n -th cohomology

group $H^n(K)$. The automorphism group of any object in $\mathcal{C}_n(K)$ is the $(n-1)$ -th cohomology group $H^{n-1}(K)$.

Example 4.4. We may realize the category of torsors as a category constructed from a complex. Let A be a sheaf of abelian groups on a topological space S , and

$$A \rightarrow C^0 \xrightarrow{d} C^1 \rightarrow C^2 \rightarrow \dots$$

a flabby resolution on S . Then the category $\mathcal{C}_1(C^*)$ is equivalent to the category of A -torsors on the space S . The equivalence is constructed as follows. Let $c \in C^1(S)$ be a 1-cocycle of the complex. For an open subset $U \subset S$, let $P_c(U) \subset C^0(U)$ be the inverse image $d^{-1}(c|_U)$ of the restriction of c . Then P_c is an A -torsor on the space S . Further, any 0-cochain $f \in C^0(S)$ defines an isomorphism $P_c \rightarrow P_{c+df}$ of A -torsors. On an open subset $U \subset S$, the map $P_c(U) \rightarrow P_{c+df}(U)$ is defined by $s \mapsto s + f|_U$.

5. CENTRAL EXTENSIONS AND MULTIPLICATIVE TORSORS

Central extensions can be described as torsors equipped with some multiplicative structure. Let S be a scheme, $G \rightarrow S$ an algebraic group defined over S , and A a sheaf of abelian groups on the big Zariski site S_{ZAR} of S . A central extension $A \rightarrow \tilde{G} \rightarrow G$ as a sheaf of groups on the site S_{ZAR} is regarded as a sheaf on the big Zariski site G_{ZAR} of G . On an object $U \xrightarrow{\eta} G$ in the site G_{ZAR} , the sections

$$\tilde{G}(U \rightarrow G) = \text{Hom}_G(U, \tilde{G})$$

form the inverse image of $\eta \in G(U)$ in the set $\tilde{G}(U)$ of sections.

Lemma 5.1. Let $A \rightarrow \tilde{G} \rightarrow G$ be a central extension as a sheaf of groups on the site S_{ZAR} , and regard it as a sheaf on the big Zariski site G_{ZAR} of G . Then the sheaf $\tilde{G}$ is an $A|_G$ -torsor on G_{ZAR} .

Proof. Any object $U \xrightarrow{\eta} G$ in G_{ZAR} has a covering sieve $\Sigma \subset \text{Ob } U_{\text{ZAR}}$ such that for any scheme $V \xrightarrow{\epsilon} U$ belonging to Σ , the map $\tilde{G}(V) \rightarrow G(V)$ on the sections is surjective. Then, in the exact sequence

$$1 \rightarrow A(V) \rightarrow \tilde{G}(V) \rightarrow G(V) \rightarrow 1,$$

the inverse image $\tilde{G}(V \xrightarrow{\eta \circ \epsilon} G)$ of $\eta \circ \epsilon \in G(V)$ is an $A(V)$ -torsor. Hence the sheaf $\tilde{G}$ on the site G_{ZAR} is an $A|_G$ -torsor. $\square$

On an object $U \rightarrow S$ in the big Zariski site S_{ZAR} , the product map $\tilde{G}(U) \times \tilde{G}(U) \rightarrow \tilde{G}(U)$ can be described fiber-wisely. Precisely, the multiplication on the fibers over sections $u, v \in G(U)$ defines a map

$$m_{u,v}: \tilde{G}(U \xrightarrow{u} G) \times \tilde{G}(U \xrightarrow{v} G) \rightarrow \tilde{G}(U \xrightarrow{uv} G).$$

To see it as a morphism of torsors, we take the pull-backs of the sheaf $\tilde{G}$ on the site G_{ZAR} by several maps $\partial: G \times G \rightarrow G$. The pull-backs

$\partial^* \tilde{G}$ are $A|_{G \times G}$ -torsors on the site $(G \times G)_{\text{ZAR}}$. In our case, we take the pull-backs by the multiplication $\mu: G \times G \rightarrow G$ and the projection map pr_i to the i -th component for $i = 1, 2$.

The multiplication map $m_{u,v}$ is then interpreted as

$$(\text{pr}_1^*(\tilde{G}) \times \text{pr}_2^*(\tilde{G}))(U \xrightarrow{u,v} G \times G) \rightarrow (\mu^* \tilde{G})(U \xrightarrow{u,v} G \times G),$$

and defines a map $m': \text{pr}_1^*(\tilde{G}) \times \text{pr}_2^*(\tilde{G}) \rightarrow \mu^* \tilde{G}$ of sheaves on the site $(G \times G)_{\text{ZAR}}$. By definition, the torsor $\mu^* \tilde{G}$ is a push-out of $\text{pr}_1^*(\tilde{G}) \times \text{pr}_2^*(\tilde{G})$ by the multiplication $A \times A \rightarrow A$. Thus, the map m' factors through a morphism

$$m: \text{pr}_1^* \tilde{G} + \text{pr}_2^* \tilde{G} \rightarrow \mu^* \tilde{G}$$

of $A|_{G \times G}$ -torsors on the site $(G \times G)_{\text{ZAR}}$.

To describe the associativity of the multiplication on $\tilde{G}$, we further employ the triple product map $\mu_{123}: G \times G \times G \rightarrow G$, and the projections $\text{pr}_i: G \times G \times G \rightarrow G$ and $\text{pr}_{ij}: G \times G \times G \rightarrow G \times G$ for $i, j \in \{1, 2, 3\}$. Then the associativity of multiplication means the commutativity of the diagram

$$\begin{array}{ccc} \text{pr}_1^* \tilde{G} + \text{pr}_2^* \tilde{G} + \text{pr}_3^* \tilde{G} & \longrightarrow & \text{pr}_{12}^* \mu^* \tilde{G} + \text{pr}_3^* \tilde{G} \\ \downarrow & & \downarrow \\ \text{pr}_1^* \tilde{G} + \text{pr}_{23}^* \mu^* \tilde{G} & \longrightarrow & \mu_{123}^* \tilde{G} \end{array}$$

of morphisms induced by m between $A|_{G \times G \times G}$ -torsors on the site $(G \times G \times G)_{\text{ZAR}}$. We have thus interpreted the multiplicative structure in terms of torsors:

Definition 5.2. Let $G \rightarrow S$ be an algebraic group defined over a scheme S , and A a sheaf of abelian groups on the big Zariski site S_{ZAR} of S . Then, a *multiplicative A -torsor* on the site G_{ZAR} is an $A|_G$ -torsor E equipped with a morphism $\text{pr}_1^* E + \text{pr}_2^* E \rightarrow \mu^* E$ of $A|_{G \times G}$ -torsors on $(G \times G)_{\text{ZAR}}$ such that the induced diagram

$$\begin{array}{ccc} \text{pr}_1^* E + \text{pr}_2^* E + \text{pr}_3^* E & \longrightarrow & \text{pr}_{12}^* \mu^* E + \text{pr}_3^* E \\ \downarrow & & \downarrow \\ \text{pr}_1^* E + \text{pr}_{23}^* \mu^* E & \longrightarrow & \mu_{123}^* E \end{array}$$

commutes.

As above, a central extension $A \rightarrow \tilde{G} \rightarrow G$ as a sheaf of groups on the big Zariski site S_{ZAR} is regarded as a multiplicative A -torsor on the site G_{ZAR} . Conversely, let E be a multiplicative A -torsor on G_{ZAR} , equipped with map $m: \text{pr}_1^* E + \text{pr}_2^* E \rightarrow \mu^* E$. Then the torsor E can be seen as a sheaf on the site S_{ZAR} . On an object $X \rightarrow S$ in S_{ZAR} , the

sections $E(X)$ decompose into the disjoint union

$$\bigsqcup_{g: X \rightarrow G} E(X \xrightarrow{g} G) = \bigsqcup_{g: X \rightarrow G} (g^* E)(X),$$

which naturally gives a map $E(X) \rightarrow G(X)$. Then the multiplication on the sections $E(X)$ is again described fiber-wise. For sections $g, h \in G(X)$, the product map $E(X \xrightarrow{g} G) \times E(X \xrightarrow{h} G) \rightarrow E(X \xrightarrow{gh} G)$ is just the map

$$m: (\mathrm{pr}_1^* E \times \mathrm{pr}_2^* E)(X \xrightarrow{g,h} G \times G) \rightarrow (\mu^* E)(X \xrightarrow{g,h} G \times G).$$

The commutative diagram for the map m ensures that the multiplication is associative. The identity element of $E(X)$ is locally constructed in the fiber $E(X \xrightarrow{1} G)$ over the identity $1 \in G(X)$, and defines an isomorphism $A(X) \xrightarrow{\sim} E(X \xrightarrow{1} G)$ of groups. Thus $A(X)$ is the kernel of the map $E(X) \rightarrow G(X)$. We see that $A \rightarrow E \rightarrow G$ is a central extension as a sheaf of groups on the big Zariski site S_{ZAR} , and further:

Proposition 5.3. Let $G \rightarrow S$ be an algebraic group defined over a scheme S , and A a sheaf of abelian groups on the big Zariski site S_{ZAR} of S . Then the category of central extensions of G by A as sheaves of groups on the site S_{ZAR} is equivalent to the category of multiplicative A -torsors on the site G_{ZAR} .

6. SMALL VS. BIG SITES

Let S be a scheme, and A a sheaf of abelian groups on the big Zariski site S_{ZAR} of S . For an object $X \rightarrow S$ in the site S_{ZAR} , the restriction A_X of A to the small Zariski site X_{zar} of X is of course a sheaf. Then the sheaf A on the big site S_{ZAR} is regarded as the collection $(A_X)_{X \rightarrow S}$ of restrictions to small sites. A morphism $f: X \rightarrow Y$ in the site S_{ZAR} further gives a map

$$\phi(f): f^* A_Y \rightarrow A_X$$

of sheaves on X with the following two conditions. First, if f is an open embedding, then the map $\phi(f)$ is an isomorphism. Secondly, the maps $\{\phi(f)\}_f$ are compatible with composition.

An A -torsor P on the big Zariski site S_{ZAR} is similarly regarded as a collection $(P_X)_{X \rightarrow S}$ of A_X -torsors on the small sites X with some compatibility.

Proposition 6.1. Let S be a scheme. Then, the category of A -torsors on the big Zariski site S_{ZAR} is equivalent to that of A_S -torsors on the small Zariski site S .

Proof. In the direction from the big to the small site, we have the restriction functor. To observe the opposite direction, let P be an A_S -torsor on the small site S , and $X \xrightarrow{\xi} S$ an object in the big site S_{ZAR} .

Then the pull-back ξ^*P as a sheaf is a ξ^*A_S -torsor on X . We thus obtain an A_S -torsor

$$P_X = \phi(f)\xi^*P$$

on X as the push-out by the map $\phi(f)$. We see that this construction gives an equivalence of the categories. $\square$

We may also restrict a multiplicative torsor to the small Zariski site. Let G be an algebraic group defined over the scheme S , and E a multiplicative A -torsor on the big Zariski site G_{ZAR} with map

$$\text{pr}_1^*E + \text{pr}_2^*E \rightarrow \mu^*E$$

of $A|_{G \times G}$ -torsors on $(G \times G)_{\text{ZAR}}$. Then E is interpreted as an A_G -torsor on the small site G with a multiplicative structure map $\text{pr}_1^*E + \text{pr}_2^*E \rightarrow \mu^*E$ of $A_{G \times G}$ -torsors on $G \times G$. For an A_G -torsor E on G and a map $\partial: G \times G \rightarrow G$, we write ∂^*E for the pull-back as a torsor, i.e., the push-out by $\partial^*A_G \rightarrow A_{G \times G}$ of the pull-back as a sheaf.

7. SIMPLICIAL CONSTRUCTION OF A 2-COCYCLE

We review the strict simplicial classifying space BG of a group scheme G defined over a field k . For an integer $n \geq 0$, let Δ_n be the totally ordered finite set $\{0, 1, \dots, n\}$. For an index $i = 0, 1, \dots, n+1$, let

$$\delta_i: \Delta_n \xrightarrow{\sim} \Delta_{n+1} \setminus \{i\} \subset \Delta_{n+1}$$

be the increasing injective map. Note that any increasing injective map $\Delta_n \rightarrow \Delta_{n+1}$ is written in this form. We regard the system $\{\Delta_n\}_{n \geq 0}$ as a category, whose morphisms are increasing injective maps.

Definition 7.1. Let G be a group scheme defined over a field k .

- (1) The *strict simplicial classifying space* BG of G is the contravariant functor from the category $\{\Delta_n\}_n$ of totally ordered finite sets to the big Zariski site k_{ZAR} that sends Δ_n to the quotient G^{Δ_n}/G . Here the diagonal action $G^{\Delta_n} \curvearrowright G$ is given by right translations.
- (2) For $i = 0, 1, \dots, n+1$, let $\partial_i = \delta_i^*$ be the i -th *face map* $G^{\Delta_{n+1}}/G \rightarrow G^{\Delta_n}/G$ corresponding to the injection δ_i . The map ∂_i just removes the i -th component.

We usually identify the quotient G^{Δ_n}/G with the direct product G^n by

$$(g_0, g_1, \dots, g_n) \mapsto (g_0g_1^{-1}, g_1g_2^{-1}, \dots, g_{n-1}g_n^{-1}).$$

Then the face maps $\partial_i: G^{n+1} \rightarrow G^n$ appear in the terms of the differential in the inhomogeneous bar resolution defining the group cohomology. For example, the three face maps $\partial_i: G^2 \rightarrow G$ for $i = 0, 1, 2$ are respectively pr_2 , μ , and pr_1 .

Let A be a sheaf of abelian groups on the big Zariski site k_{ZAR} of $\text{Spec } k$, and $A \rightarrow C^0 \rightarrow C^1 \rightarrow C^2 \rightarrow \dots$ a flabby resolution on k_{ZAR} .

Then a 2-cocycle $(c^0, c, c') \in C^0(G \times G) \oplus C^1(G) \oplus C^2(k)$ in the double complex

$$\begin{array}{ccccccc}
 & & C^0(G \times G \times G) & & & & \\
 \partial_0^* - \partial_1^* + \partial_2^* - \partial_3^* \uparrow & & & & & & \\
 & C^0(G \times G) & \longrightarrow & C^1(G \times G) & \longrightarrow & \cdots & \\
 \partial_0^* - \partial_1^* + \partial_2^* \uparrow & & & \uparrow & & & \\
 & C^0(G) & \xrightarrow{d} & C^1(G) & \longrightarrow & C^2(G) & \\
 \partial_0^* - \partial_1^* \uparrow & & & \uparrow & & \uparrow & \\
 & C^0(k) & \longrightarrow & C^1(k) & \longrightarrow & C^2(k) & \longrightarrow C^3(k)
 \end{array}$$

gives the following central extension $1 \rightarrow A(k) \rightarrow \widetilde{G(k)} \rightarrow G(k) \rightarrow 1$. For a rational point $g \in G(k)$, let $E_g = d^{-1}(g^*c)$ be the inverse image of the section $g^*c \in C^1(k)$ in $C^0(k)$. Since the sequence

$$1 \rightarrow A(k) \rightarrow C^0(k) \rightarrow C^1(k) \rightarrow C^2(k)$$

of the sections over $\text{Spec } k$ is exact, the fiber $E_g \subset C^0(k)$ is an $A(k)$ -torsor, especially non-empty. Thus, we obtain a surjection $\widetilde{G(k)} := \bigsqcup_{g \in G(k)} E_g \rightarrow G(k)$.

For rational points $g, h \in G(k)$, the 0-th component $c^0 \in C^0(G \times G)$ of our 2-cocycle in the double complex gives the product map $E_g \times E_h \rightarrow E_{gh}$ on the fibers as follows. A part of the cocycle condition in the double complex $\{C^m(G^n)\}_{m,n}$ requires the equality $dc^0 = \text{pr}_1^* c + \text{pr}_2^* c - \mu^* c$ of sections in $C^1(G \times G)$. Then $c^0 \in C^0(G \times G)$ defines a map

$$d^{-1}(\text{pr}_1^* c + \text{pr}_2^* c) \rightarrow d^{-1}(\mu^* c)$$

between the inverse images in $C^0(G \times G)$. The pull-back by the pair $(g, h): \text{Spec } k \rightarrow G \times G$ gives an isomorphism

$$E_g + E_h = (g, h)^* d^{-1}(\text{pr}_1^* c + \text{pr}_2^* c) \xrightarrow{(g, h)^* c^0} (g, h)^* d^{-1}(\mu^* c) = E_{gh}$$

of $A(k)$ -torsors contained in $C^0(k)$. This defines the product map $E_g \times E_h \xrightarrow{+} E_g + E_h \rightarrow E_{gh}$, sending a pair (e_g, e_h) to the sum

$$e_g + e_h + (g, h)^* c^0$$

calculated in the group $C^0(k)$.

Lemma 7.2. Let $(c^0, c, c') \in C^0(G \times G) \oplus C^1(G) \oplus C^2(k)$ be a 2-cocycle in the double complex $\{C^m(G^n)\}_{m,n}$. Then the product map $\widetilde{G(k)} \times \widetilde{G(k)} \rightarrow \widetilde{G(k)}$ defined fiber-wise is indeed associative. Thus the union $\widetilde{G(k)} = \bigsqcup_{g \in G(k)} E_g$ is a group with this product map.

Proof. We apply the cocycle condition $0 = (\partial_0^* - \partial_1^* + \partial_2^* - \partial_3^*)c^0$ as sections in $C^0(G \times G \times G)$. For a triplet $(g_1, g_2, g_3): \text{Spec } k \rightarrow G \times G \times G$, the identity

$$\begin{aligned} 0 &= (g_1, g_2, g_3)^*(\partial_0^* - \partial_1^* + \partial_2^* - \partial_3^*)c^0 \\ &= (g_2, g_3)^*c^0 - (g_1g_2, g_3)^*c^0 + (g_1, g_2g_3)^*c^0 - (g_1, g_2)^*c^0 \end{aligned}$$

implies that the diagram

$$\begin{array}{ccc} E_{g_1} \times E_{g_2} \times E_{g_3} & \longrightarrow & E_{g_1g_2} \times E_{g_3} \\ \downarrow & & \downarrow \\ E_{g_1} \times E_{g_2g_3} & \longrightarrow & E_{g_1g_2g_3} \end{array}$$

of product maps commutes. $\square$

The fiber $E_1 \subset \widetilde{G(k)}$ is a subgroup and isomorphic to $A(k)$, since E_1 is also an $A(k)$ -torsor. Hence, we obtain a central extension

$$1 \rightarrow A(k) \rightarrow \widetilde{G(k)} \rightarrow G(k) \rightarrow 1.$$

In some cases, this construction naturally determines a 2-cocycle of the central extension:

Proposition 7.3. Let G be an algebraic group defined over a field k . Let $A \rightarrow C^0 \rightarrow C^1 \rightarrow C^2 \rightarrow \dots$ be a flabby resolution of a sheaf A of abelian groups on the big Zariski site k_{ZAR} of $\text{Spec } k$, and assume the identity $A(k) = C^0(k)$. Then the central extension $A(k) \rightarrow \widetilde{G(k)} \rightarrow G(k)$ determined by a 2-cocycle $(c^0, c, c') \in C^0(G \times G) \oplus C^1(G) \oplus C^2(k)$ in the double complex $\{C^m(G^n)\}_{m,n}$ is given by the 2-cocycle

$$G(k) \times G(k) \rightarrow A(k), (g, h) \mapsto (g, h)^*c^0.$$

In the local Langlands correspondence for covering groups, we focus on the central extensions by Milnor's group K_2 from algebraic K-theory. The groups K_2 are generalized by Quillen [12] to be defined over schemes, and form a presheaf on the big Zariski site of the base field k . Let $\mathbb{K}_2$ be the Zariski sheaf associated with the presheaf K_2 .

Let G be an algebraic group defined over the field k . To construct a flabby resolution of the sheaf $(\mathbb{K}_2)_G$ on the small Zariski site G , we prepare a spectral sequence made of Quillen's higher K-groups K_n . For a point x in the underlying space of G , let $k(x)$ denote the residue field of x on G .

Proposition 7.4. For an integer $i \geq 0$, let $G^{(i)} \subset G$ be the set of points of codimension i . Then we have a convergent fourth-quadrant cohomological spectral sequence

$$E_1^{p,q} = \bigoplus_{x \in G^{(i)}} K_{-p-q}(k(x)) \Rightarrow K_{-p-q}(G).$$

Proof. This is deduced from the localization exact sequence [15, Proposition V.9.5]. $\square$

For a point x in the underlying space of G , let $i_x: \{x\} \hookrightarrow G$ be the inclusion map. Then the (-2) -th row in the lattice E_1 induces a cochain complex

$$0 \rightarrow (\mathbb{K}_2)_G \rightarrow \bigoplus_{x \in G^{(0)}} (i_x)_* K_2(k(x)) \rightarrow \bigoplus_{x \in G^{(1)}} (i_x)_* k(x)^\times \rightarrow \bigoplus_{x \in G^{(2)}} (i_x)_* \mathbb{Z} \rightarrow 0$$

of sheaves on the small Zariski site G , which is called the Gersten-Quillen complex.

Proposition 7.5 (Gersten conjecture). For an algebraic group G defined over a field k , the Gersten-Quillen complex defined above is a flabby resolution of the sheaf $(\mathbb{K}_2)_G$ on the small Zariski site G .

Proof. This is proved by Quillen in more general context [12, §7 Theorem 5.11]. $\square$

Unfortunately, Gersten-Quillen complexes are not functorial. For example, let k be a field, and $\phi: \operatorname{Spec} k \rightarrow \mathbb{G}_a = \operatorname{Spec}(k[t])$ the closed immersion corresponding to the map $k[t] \rightarrow k, f(t) \mapsto f(0)$. Then the morphism ϕ does not induce a map

$$K_2(k(t)) \rightarrow K_2(k).$$

Therefore, Gersten-Quillen complexes do not define a resolution on the big Zariski site k_{ZAR} .

Although Gersten-Quillen complexes cannot construct a 2-cocycle in the previous manner, sometimes they can in a modified way. As a typical example, let $G = \mathbb{G}_m$ be the multiplicative group defined over a field k . For an integer $n \geq 0$, we express the coordinate ring of the product $\mathbb{G}_m^n$ as $k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. Then the sections

$$\left\{ \bigoplus_{x \in (\mathbb{G}_m^n)^{(m)}} K_{2-m}(k(x)) \right\}_{m,n}$$

of Gersten-Quillen resolutions on the products $\mathbb{G}_m^n$ form a double complex, whose vertical differentials are the alternating sums of face maps ∂_i^* . At the spot $(0, 2)$ in the double complex, the element

$$[t_1 \otimes t_2] \in K_2(k(t_1, t_2))$$

is mapped to the identity by both the horizontal and the vertical differentials. Thus $[t_1 \otimes t_2] \in K_2(k(t_1, t_2))$ yields a 2-cocycle in the total complex of our double complex, by letting the other two components be trivial. For a pair $(a, b): \operatorname{Spec} k \rightarrow \mathbb{G}_m \times \mathbb{G}_m$, the pull back $(a, b)^*$ does not define a map $K_2(k(t_1, t_2)) \rightarrow K_2(k)$, but define the value

$$(a, b)^*[t_1 \otimes t_2] = [a \otimes b]$$

in the group $K_2(k)$. The map $\mathbb{G}_m(k) \times \mathbb{G}_m(k) \rightarrow K_2(k)$, $(a, b) \mapsto [a \otimes b]$ is bilinear, and especially a 2-cocycle defining a cover

$$K_2(k) \rightarrow \widetilde{\mathbb{G}_m(k) \times \mathbb{G}_m(k)} \rightarrow \mathbb{G}_m(k) \times \mathbb{G}_m(k).$$

This is an elementary example of a generic cocycle.

8. GENERIC COCYCLES

The construction of a 2-cocycle explained in Section 7 is useless for explicit calculation. Hence, we give an alternative way in this section.

Let k be an infinite field. For an algebraic group G defined over k , let $k(G)$ denote the field of rational functions. As above, the face operators in the strict simplicial classifying space $BG = \{G^n\}_n$ yields homomorphisms

$$k \rightrightarrows k(G) \rightrightarrows k(G \times G) \rightrightarrows k(G \times G \times G) \cdots$$

of fields. The K -groups $\{K_2 k(G^n)\}_n$ also form cosimplicial systems, which gives a cochain complex

$$K_2 k \xrightarrow{\partial_0^* - \partial_1^*} K_2 k(G) \xrightarrow{\partial_0^* - \partial_1^* + \partial_2^*} K_2 k(G \times G) \xrightarrow{\partial_0^* - \partial_1^* + \partial_2^* - \partial_3^*} K_2 k(G \times G \times G) \rightarrow \cdots$$

whose differentials are the alternating sums of coface maps.

Definition 8.1. A *generic cocycle* is a 2-cocycle of the complex $\{K_2 k(G^n)\}_n$ associated with the cosimplicial system of K_2 -groups of fields.

A generic cocycle $c \in K_2 k(G \times G)$ yields a 2-cocycle on an open dense subset of $G \times G$, as follows. By definition, we have an integer $r > 0$ and rational functions $f_i, g_i \in k(G \times G)^\times$ for $i = 1, \dots, r$ such that the cocycle c is written as the finite sum $\sum_i [f_i \otimes g_i]$. Then there is an open dense subset $U \subset G \times G$ such that for any $i = 1, \dots, r$, the functions f_i and g_i are regular and invertible on the set U . Thus the map

$$\gamma: U(k) \rightarrow K_2(k), (x, y) \mapsto \sum_i [f_i(x, y) \otimes g_i(x, y)]$$

is well-defined. Since the element $c \in K_2 k(G \times G)$ is a 2-cocycle in the cochain complex of K -groups, the map γ also meets the cocycle condition. That is, points (x, y, z) in some open dense subset of $G(k) \times G(k) \times G(k)$ satisfy the identity

$$\gamma(y, z) - \gamma(xy, z) + \gamma(x, yz) - \gamma(x, y) = 0$$

. Hence the map γ on the set $U(k)$ almost determines a covering group $K_2(k) \rightarrow \widetilde{G(k)} \rightarrow G(k)$. Moreover:

Proposition 8.2 (Brylinski-Deligne [2, 8.2]). Assume that k is an infinite field, and that an algebraic group G over k is connected and reductive. Then the strict commutative Picard category $\mathcal{C}_2(K_2 k(G^*))$

constructed by the complex of K-groups is equivalent to the strict commutative Picard category of central extensions of the algebraic group G by the sheaf $\mathbb{K}_2$ on the big Zariski site k_{ZAR} of the base field.

If G is the multiplicative group $\mathbb{G}_m$ defined over a field k , then the generic cocycle $[t_1 \otimes t_2]$ in $K_2 k(t_1, t_2) = K_2 k(\mathbb{G}_m \times \mathbb{G}_m)$ defines a 2-cocycle on the whole group $\mathbb{G}_m(k) \times \mathbb{G}_m(k)$, as already observed. If G is the special linear group SL_2 defined over k , then the Kubota 2-cocycle is obtained in the open Bruhat cell from a generic cocycle in the following way.

For an integer $n > 0$, we write the coordinate ring of the product SL_2^n as $\bigotimes_{i=1}^n k[a_i, b_i, c_i, d_i]/(a_i d_i - b_i c_i - 1)$. Note the equality

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + b_1 c_2 & * \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix}.$$

For an index $i > 0$, we write $x_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ for brevity. We show that

$$\gamma(x_1, x_2) = \left[\frac{c_1 a_2 + d_1 c_2}{c_1} \otimes \frac{c_1 a_2 + d_1 c_2}{c_2} \right]$$

is a generic cocycle, that is, a 2-cocycle in the complex $\{K_2 k(\text{SL}_2^n)\}_n$.

From now, we shorten the symbol $[- \otimes -]$ to $[-, -]$. The following lemma is useful for computing the K_2 -symbols.

Lemma 8.3. Any non-zero rational functions $f_1, f_2, h \in k(\text{SL}_2^n)^\times$ satisfy the equality $[f_2 h, f_1 h] = [f_2, f_1][h, -f_1 f_2^{-1}]$.

Proof. We see the equalities

$$\begin{aligned} [f_2 h, f_1 h] &= [f_2, f_1 h][h, f_1 h] = [f_2, f_1][f_2, h][h, -f_1] \\ &= [f_2, f_1][h, -f_1 f_2^{-1}]. \end{aligned}$$

□

In particular,

$$\gamma(x_1, x_2) = \left[c_2 \left(\frac{d_1}{c_1} + \frac{a_2}{c_2} \right), c_1 \left(\frac{d_1}{c_1} + \frac{a_2}{c_2} \right) \right] = [c_2, c_1] \left[\frac{d_1}{c_1} + \frac{a_2}{c_2}, -c_1 c_2^{-1} \right].$$

To compute $\gamma(x_1 x_2, x_3)$, note that

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 c_2 \left(\frac{d_1}{c_1} + \frac{a_2}{c_2} - \frac{1}{a_1 c_1} \right) & * \\ c_1 c_2 \left(\frac{d_1}{c_1} + \frac{a_2}{c_2} \right) & c_1 d_2 \left(\frac{d_1}{c_1} + \frac{a_2}{c_2} - \frac{1}{c_2 d_2} \right) \end{pmatrix},$$

and that the quotient of the $(2, 2)$ -entry by the $(2, 1)$ -entry in this matrix is $\frac{d_2}{c_2} - c_2^{-2}(\frac{d_1}{c_1} + \frac{a_2}{c_2})^{-1}$. Then, we obtain

$$\begin{aligned}
& \gamma(x_1 x_2, x_3) \\
&= \left[c_3 \left(\frac{d_2}{c_2} + \frac{a_3}{c_3} - c_2^{-2} \left(\frac{d_1}{c_1} + \frac{a_2}{c_2} \right)^{-1} \right), c_1 c_2 \left(\frac{d_1}{c_1} + \frac{a_2}{c_2} \right) \left(\frac{d_2}{c_2} + \frac{a_3}{c_3} - c_2^{-2} \left(\frac{d_1}{c_1} + \frac{a_2}{c_2} \right)^{-1} \right) \right] \\
&= \left[-c_2^{-2} c_3 \left(\frac{d_1}{c_1} + \frac{a_2}{c_2} \right)^{-1} \left(1 - c_2^2 \left(\frac{d_1}{c_1} + \frac{a_2}{c_2} \right) \left(\frac{d_2}{c_2} + \frac{a_3}{c_3} \right) \right), \right. \\
&\quad \left. -c_1 c_2^{-1} \left(1 - c_2^2 \left(\frac{d_1}{c_1} + \frac{a_2}{c_2} \right) \left(\frac{d_2}{c_2} + \frac{a_3}{c_3} \right) \right) \right] \\
&= \left[-c_2^{-2} c_3 \left(\frac{d_1}{c_1} + \frac{a_2}{c_2} \right)^{-1}, -c_1 c_2^{-1} \right] \left[1 - c_2^2 \left(\frac{d_1}{c_1} + \frac{a_2}{c_2} \right) \left(\frac{d_2}{c_2} + \frac{a_3}{c_3} \right), -c_1 c_2 c_3^{-1} \left(\frac{d_1}{c_1} + \frac{a_2}{c_2} \right) \right].
\end{aligned}$$

Let A denote the last factor. Then,

$$\begin{aligned}
& \gamma(x_1, x_2) \gamma(x_1 x_2, x_3) \\
&= [c_2, c_1] \left[\frac{d_1}{c_1} + \frac{a_2}{c_2}, -c_1 c_2^{-1} \right] \left[-c_2^{-2} c_3 \left(\frac{d_1}{c_1} + \frac{a_2}{c_2} \right)^{-1}, -c_1 c_2^{-1} \right] A \\
&= [c_2, c_1] [-c_2^{-2} c_3, -c_1 c_2^{-1}] A \\
&= [c_2, c_1] [-c_2^{-2} c_3, c_1] [-c_2^{-2} c_3, -c_2^{-1}] A \\
&= [c_3, c_1] [-c_2^{-1}, c_1] [-c_2^{-1}, -c_3^{-1}] A \\
&= [c_3, c_1] [-c_2^{-1}, -c_1 c_3^{-1}] A.
\end{aligned}$$

To compute another factor $\gamma(x_1 x_2, x_3)$ in the cocycle condition, note that the quotient of the $(1, 1)$ -entry by the $(2, 1)$ -entry in the matrix $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ is $\frac{a_1 a_2 + b_1 c_2}{c_1 a_2 + d_1 c_2} = \frac{a_1}{c_1} - c_1^{-2}(\frac{d_1}{c_1} + \frac{a_2}{c_2})^{-1}$. Then we see the

equalities

$$\begin{aligned}
& \gamma(x_1, x_2 x_3) \\
&= \left[c_2 c_3 \left(\frac{d_2}{c_2} + \frac{a_3}{c_3} \right) \left(\frac{d_1}{c_1} + \frac{a_2}{c_2} - c_2^{-2} \left(\frac{d_2}{c_2} + \frac{a_3}{c_3} \right)^{-1} \right), c_1 \left(\frac{d_1}{c_1} + \frac{a_2}{c_2} - c_2^{-2} \left(\frac{d_2}{c_2} + \frac{a_3}{c_3} \right)^{-1} \right) \right] \\
&= \left[-c_2^{-1} c_3 \left(1 - c_2^2 \left(\frac{d_1}{c_1} + \frac{a_2}{c_2} \right) \left(\frac{d_2}{c_2} + \frac{a_3}{c_3} \right) \right), \right. \\
&\quad \left. -c_1 c_2^{-2} \left(\frac{d_2}{c_2} + \frac{a_3}{c_3} \right)^{-1} \left(1 - c_2^2 \left(\frac{d_1}{c_1} + \frac{a_2}{c_2} \right) \left(\frac{d_2}{c_2} + \frac{a_3}{c_3} \right) \right) \right] \\
&= \left[-c_2^{-1} c_3, -c_1 c_2^{-2} \left(\frac{d_2}{c_2} + \frac{a_3}{c_3} \right)^{-1} \right] \\
&\quad \cdot \left[1 - c_2^2 \left(\frac{d_1}{c_1} + \frac{a_2}{c_2} \right) \left(\frac{d_2}{c_2} + \frac{a_3}{c_3} \right), -c_1 c_2^{-1} c_3^{-1} \left(\frac{d_2}{c_2} + \frac{a_3}{c_3} \right)^{-1} \right] \\
&= [-c_2^{-1} c_3, -c_1 c_2^{-2}] \left[-c_2^{-1} c_3, \left(\frac{d_2}{c_2} + \frac{a_3}{c_3} \right)^{-1} \right] \\
&\quad \cdot \left[1 - c_2^2 \left(\frac{d_1}{c_1} + \frac{a_2}{c_2} \right) \left(\frac{d_2}{c_2} + \frac{a_3}{c_3} \right), -c_1 c_2 c_3^{-1} \left(\frac{d_1}{c_1} + \frac{a_2}{c_2} \right)^{-1} \right] \\
&= [-c_2^{-1}, -c_1 c_2^{-2}] [c_3, -c_1 c_2^{-2}] \left[\frac{d_2}{c_2} + \frac{a_3}{c_3}, -c_2^{-1} c_3 \right] A.
\end{aligned}$$

Hence,

$$\begin{aligned}
& \gamma(x_2, x_3) \gamma(x_1, x_2 x_3) \\
&= [c_3, c_2] \left[\frac{d_2}{c_2} + \frac{a_3}{c_3}, -c_2 c_3^{-1} \right] [-c_2^{-1}, -c_1 c_2^{-2}] [c_3, -c_1 c_2^{-2}] \left[\frac{d_2}{c_2} + \frac{a_3}{c_3}, -c_2^{-1} c_3 \right] A \\
&= [c_3, c_1] [c_3, -c_2^{-1}] [-c_2^{-1}, -c_1] A \\
&= [c_3, c_1] [-c_2^{-1}, -c_1 c_3^{-1}] A \\
&= \gamma(x_1, x_2) \gamma(x_1 x_2, x_3).
\end{aligned}$$

Therefore $\gamma(x_1, x_2) \in K_2 k(\mathrm{SL}_2^2)$ is a generic cocycle for the special linear group SL_2 , and defines Kubota's 2-cocycle in the open Bruhat cell.

9. BRYLINSKI-DELIGNE COVERING GROUPS OF A TORUS

In the literature on Langlands correspondence for general covering groups [5, 19], the authors often focus on certain covering groups defined by Brylinski and Deligne [2]. In this section, we review the definition and the classification theorem for Brylinski-Deligne covering groups of an algebraic torus $\mathbb{T}$ defined over a local field F .

Roughly speaking, a Brylinski-Deligne covering group is a central extension by Milnor's group K_2 in algebraic K-theory, or its variation. The groups K_2 are generalized by Quillen [12] to be defined over schemes, and form a presheaf on the big Zariski site of the base field F . Let $\mathbb{K}_2$ be the Zariski sheaf associated with the presheaf K_2 .

To define a Brylinski-Deligne covering group, let $\mu_n \subset F^\times$ denote the cyclic subgroup of order n .

Definition 9.1 (Brylinski-Deligne [2]). Let $\mathbb{T} \rightarrow \text{Spec } F$ be an algebraic torus. Depending on the context, a *Brylinski-Deligne covering group* of $\mathbb{T}$, or of $T = \mathbb{T}(F)$, refers to one of the following:

- (1) a central extension $1 \rightarrow \mathbb{K}_2 \rightarrow \tilde{\mathbb{T}} \rightarrow \mathbb{T} \rightarrow 1$ as a sheaf of groups on the big Zariski site of $\text{Spec } F$,
- (2) the sections $1 \rightarrow K_2(F) \rightarrow \tilde{\mathbb{T}}(F) \rightarrow T \rightarrow 1$ of (1) over $\text{Spec } F$,
or
- (3) the push-out $1 \rightarrow \mu_n \rightarrow \tilde{T} \rightarrow T \rightarrow 1$ of (2) via the Hilbert symbol $K_2(F) \rightarrow \mu_n$.

A Brylinski-Deligne covering group defined in 9.1 (3) is indeed a topological covering group [2, Construction 10.3]. Further, Brylinski and Deligne generally defined their covering groups for any algebraic group.

For a split torus T , a 2-cocycle of a Brylinski-Deligne cover $\mu_n \rightarrow \tilde{T} \rightarrow T$ is written as a product of n -th Hilbert symbols $(\cdot, \cdot)_n: F^\times \times F^\times \rightarrow \mu_n$ [5, 3.3]. For instance, let $T = F^\times$ be the one-dimensional split torus, that is, just the multiplicative group. Then for an integer a , the 2-cocycle $(\cdot, \cdot)_n^a: F^\times \times F^\times \rightarrow \mu_n$ determines a Brylinski-Deligne cover $\mu_n \rightarrow \tilde{T} \rightarrow T$. As another example, let $T = F^\times \times F^\times$ be the two-dimensional split torus. Then for four integers $(a_{ij})_{i,j \in \{1,2\}}$, the 2-cocycle

$$(F^\times \times F^\times) \times (F^\times \times F^\times) \rightarrow \mu_n, ((s_1, s_2), (t_1, t_2)) \mapsto \prod_{i,j \in \{1,2\}} (s_i, t_j)_n^{a_{ij}}$$

gives a Brylinski-Deligne cover $\mu_n \rightarrow \tilde{T} \rightarrow T$. Especially if $n \geq 2$ and $(a_{ij})_{i,j} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then the covering group $\tilde{T}$ is non-abelian.

For any torus T , one has a finite Galois extension L/F of the base field such that T splits over L . Then the following classification theorem gives a bilinear form

$$Y \times Y \rightarrow \mathbb{Z}$$

on the cocharacter lattice, which is a key ingredient to describe our main theorem.

Theorem 9.2 (Classification Theorem of covers [2]). The following two Picard categories are equivalent:

- (1) the category of Brylinski-Deligne covers $\mathbb{K}_2 \rightarrow \tilde{\mathbb{T}} \rightarrow \mathbb{T}$, and

(2) the category of the following data $(Q, \mathcal{E})$:

- $Q: Y \rightarrow \mathbb{Z}$ is a Galois-invariant quadratic form on the cocharacter lattice, which gives a bilinear form $B: Y \times Y \rightarrow \mathbb{Z}$ by

$$(y, y') \mapsto Q(y + y') - Q(y) - Q(y').$$

- $1 \rightarrow L^\times \rightarrow \mathcal{E} \xrightarrow{\overline{(\cdot)}} Y \rightarrow 1$ is a Galois-equivariant central extension such that for any $\epsilon, \zeta \in \mathcal{E}$,

$$\epsilon \zeta \epsilon^{-1} \zeta^{-1} = (-1)^{B(\overline{\epsilon}, \overline{\zeta})}$$

in $L^\times$.

- For any data $(Q, \mathcal{E})$ and $(Q', \mathcal{E}')$,

$$\text{Hom}((Q, \mathcal{E}), (Q', \mathcal{E}')) = \begin{cases} \emptyset & \text{if } Q \neq Q' \\ \{\text{morphisms } \mathcal{E} \rightarrow \mathcal{E}' \text{ of extensions}\} & \text{if } Q = Q'. \end{cases}$$

Brylinski and Deligne generally proved this type of classification theorem for covering groups of reductive algebraic groups [2].

Definition 9.3. Let $\mathbb{K}_2 \rightarrow \tilde{\mathbb{T}} \rightarrow \mathbb{T}$ be a Brylinski-Deligne cover, and $Q_{\tilde{\mathbb{T}}}: Y \rightarrow \mathbb{Z}$ the Galois-invariant quadratic form corresponding to $\tilde{\mathbb{T}}$ by Theorem 9.2. Then we write $B_{\tilde{\mathbb{T}}}: Y \times Y \rightarrow \mathbb{Z}$ for the associated bilinear form

$$(y, y') \mapsto Q(y + y') - Q(y) - Q(y').$$

If $\mu_n \rightarrow \tilde{T} \rightarrow T$ is the cover induced from $\tilde{\mathbb{T}}$ (Definition 9.1), then we write $B_{\tilde{T}} = B_{\tilde{\mathbb{T}}}$ by abuse of notation.

10. LOCAL LANGLANDS CORRESPONDENCE FOR A COVERING GROUP

In this section, we introduce a finite abelian group $\mathcal{S} = \mathcal{S}_{\tilde{T}}$, which is the main object of this work (Definition 10.3). Then we shall explain how $\mathcal{S}_{\tilde{T}}$ parametrizes a packet in the local Langlands correspondence for a Brylinski-Deligne covering group

$$\mu_n \rightarrow \tilde{T} \rightarrow T$$

of a torus. Our formulation is essentially due to Weissman [17].

The local Langlands correspondence for a covering group $\tilde{T} \rightarrow T$ of a torus aims to parametrize genuine irreducible representations of $\tilde{T}$. To treat genuine representations, we fix an embedding

$$j: \mu_n \rightarrow \mathbb{C}^\times$$

of the cyclic group of order n . Recall that a representation of a covering group $\mu_n \rightarrow \tilde{G} \rightarrow G$ of a topological group is *genuine* if $\mu_n \subset \tilde{G}$ acts

as multiplication by scalars in $\mathbb{C}$ via the embedding j . That is, for any $\zeta \in \mu_n$ and any vector v in the representation space, one has

$$\zeta \cdot v = j(\zeta)v.$$

A *genuine character* $\chi: \tilde{G} \rightarrow \mathbb{C}^\times$ is a character that is genuine as a representation, which means $\chi|_{\mu_n} = j$.

Example 10.1 (central character). (1) Let $\mu_n \rightarrow \tilde{T} \rightarrow T$ be a covering group of a torus, and ρ an irreducible genuine representation of $\tilde{T}$. Let $Z^\dagger \subset T$ denote the image of the center $Z(\tilde{T}) \subset \tilde{T}$, and note that

$$\mu_n \rightarrow Z(\tilde{T}) \rightarrow Z^\dagger$$

is a covering group. Then the central character $\rho|_{Z(\tilde{T})}: Z(\tilde{T}) \rightarrow \mathbb{C}^\times$ is a genuine character.

(2) Let $\mu_n \rightarrow \tilde{T} \rightarrow T$ be a Brylinski-Deligne covering of a torus. Then the correspondence from the genuine irreducible representations of the covering group $\tilde{T}$ to their central characters is bijective onto the set of genuine characters $Z(\tilde{T}) \rightarrow \mathbb{C}^\times$ [17, Theorem 3.1].

To establish the local Langlands correspondence for a Brylinski-Deligne covering group $\tilde{T} \rightarrow T$, it suffices to parametrize the genuine characters of the center $Z(\tilde{T})$, by Example 10.1 (2). To estimate $Z(\tilde{T}) \subset \tilde{T}$, we choose the following isogeny

$$T^\# \xrightarrow{\iota} T$$

of tori, which approximates the image $Z^\dagger \subset T$ of $Z(\tilde{T})$.

Definition 10.2. Let $Y = \text{Hom}(\mathbb{G}_m, \mathbb{T})$ be the cocharacter group of the torus $T = \mathbb{T}(F)$. Let $B_{\tilde{T}}: Y \times Y \rightarrow \mathbb{Z}$ be the bilinear form attached to the covering $\mu_n \rightarrow \tilde{T} \rightarrow T$, precisely to the Brylinski-Deligne cover $\mathbb{K}_2 \rightarrow \tilde{\mathbb{T}} \rightarrow \mathbb{T}$ defining $\tilde{T}$ (Definition 9.3).

(1) We define the subgroup $Y^\# \subset Y$ by

$$Y^\# = \{y \in Y \mid B_{\tilde{T}}(y, Y) \subset n\mathbb{Z}\}.$$

(2) We write $T^\# = (Y^\# \otimes \bar{F}^\times)^\Gamma$ for the torus defined by $Y^\#$. Then the inclusion $Y^\# \hookrightarrow Y$ induces an isogeny $\iota: T^\# \rightarrow T$.

It is known that $\iota(T^\#) \subset Z^\dagger$, and that this index is finite [18, Theorem 1.3].

Definition 10.3. For the Brylinski-Deligne covering group $\tilde{T} \rightarrow T$, we define the finite abelian group $\mathcal{S} = \mathcal{S}_{\tilde{T}}$ as the quotient

$$\mathcal{S}_{\tilde{T}} = Z^\dagger / \iota(T^\#).$$

This may be called the *packet group* [17].

In the rest of this section, we explain the role of this group $\mathcal{S}_{\tilde{T}}$. First, $\mathcal{S}_{\tilde{T}}$ and the other groups defined above form the exact sequences

$$\begin{array}{ccccccc}
 & & 1 & & & & \\
 & & \uparrow & & & & \\
 & & \mathcal{S}_{\tilde{T}} & & & & \\
 & & \uparrow & & & & \\
 1 & \leftarrow & Z^\dagger & \leftarrow & Z(\tilde{T}) & \leftarrow & \mu_n \leftarrow 1 \\
 & & \uparrow & & & & \\
 1 & \leftarrow & \iota(T^\#) & \xleftarrow{\iota} & T^\# & & \\
 & & \uparrow & & & & \\
 & & 1. & & & &
 \end{array}$$

To apply the functor $\text{Hom}(_, \mathbb{C}^\times)$ to this diagram, we note the following facts.

- Fact 10.4.** (1) The multiplicative group $\mathbb{C}^\times$ is an injective object in the category of locally compact abelian groups [4, Théorème 5]. That is, for any locally compact abelian group A and its closed subgroup $C \subset A$, any continuous character $C \rightarrow \mathbb{C}^\times$ extends to a continuous character $A \rightarrow \mathbb{C}^\times$.
- (2) Let W_F be the Weil group of the local field F , and $\widehat{T^\#} = \text{Hom}(Y^\#, \mathbb{C}^\times)$ the Langlands dual of the torus $T^\#$. Then the Langlands correspondence for $T^\#$ [7, Théorème 6.2] asserts an isomorphism

$$\text{LLC}_{T^\#} : \text{Hom}(T^\#, \mathbb{C}^\times) \cong H^1(W_F, \widehat{T^\#}).$$

These facts ensure the following frame of the local Langlands correspondence for the covering group $\tilde{T} \rightarrow T$.

Proposition 10.5. The objects introduced above fit in the diagram

$$\begin{array}{ccccccc}
 1 & & \{\text{genuine irreducible representations of } \tilde{T}\} & & & & \\
 \downarrow & & \downarrow \scriptstyle{(\cdot)|_{Z(\tilde{T})}} \downarrow \scriptstyle{1:1} & & & & \\
 \text{Hom}(\mathcal{S}_{\tilde{T}}, \mathbb{C}^\times) & & \{\text{genuine characters}\} & \rightarrow & \{j\} & & \\
 \downarrow & & \cap & & \cap & & \\
 1 \rightarrow \text{Hom}(Z^\dagger, \mathbb{C}^\times) & \rightarrow & \text{Hom}(Z(\tilde{T}), \mathbb{C}^\times) & \rightarrow & \text{Hom}(\mu_n, \mathbb{C}^\times) \rightarrow 1 & & \\
 \downarrow \scriptstyle{(\cdot)|_{\iota(T^\#)}} & & & & & & \\
 \text{Hom}(\iota(T^\#), \mathbb{C}^\times) & \xrightarrow{\iota^*} & \text{Hom}(T^\#, \mathbb{C}^\times) & & & & \\
 \downarrow & & \parallel \scriptstyle{\text{LLC}_{T^\#}} & & & & \\
 1 & & H^1(W_F, \widehat{T^\#}), & & & &
 \end{array}$$

where the middle horizontal sequence and the left vertical one are exact.

By definition, the group $\text{Hom}(Z^\dagger, \mathbb{C}^\times)$ acts simply transitive on the set of genuine characters $Z(\tilde{T}) \rightarrow \mathbb{C}^\times$. Thus, fixing a genuine character χ of $Z(\tilde{T})$ as a base point defines a bijection

$$b_\chi : \{\text{genuine characters of } Z(\tilde{T})\} \rightarrow \text{Hom}(Z^\dagger, \mathbb{C}^\times).$$

By Proposition 10.5, the composed map

$$\mathrm{LLC}_{T^\#} \circ \iota^* \circ (\)|_{\iota(T^\#)} : \mathrm{Hom}(Z^\dagger, \mathbb{C}^\times) \rightarrow H^1(W_F, \widehat{T^\#})$$

has kernel $\mathrm{Hom}(\mathcal{S}_{\tilde{T}}, \mathbb{C}^\times)$, which is a finite group.

Definition 10.6. Fix a genuine character $\chi: Z(\tilde{T}) \rightarrow \mathbb{C}$. Then the *local Langlands correspondence* for the Brylinski-Deligne cover $\tilde{T} \rightarrow T$ is the composition

$$\begin{array}{c} \{\text{genuine irreducible representations of } \tilde{T}\} \\ \downarrow b_\chi \circ (\)|_{Z(\tilde{T})} \quad 1:1 \\ \mathrm{Hom}(Z^\dagger, \mathbb{C}^\times) \\ \downarrow \mathrm{LLC}_{T^\#} \circ \iota^* \circ (\)|_{\iota(T^\#)} \\ H^1(W_F, \widehat{T^\#}) \end{array}$$

As remarked above, this is a finite-to-one correspondence. Note that some packets, or fibers, may be empty. By definition, the non-empty packets are described by the group $\mathcal{S}_{\tilde{T}}$, as follows.

Proposition 10.7 (role of $\mathcal{S}_{\tilde{T}}$). In the local Langlands correspondence defined in 10.6, each non-empty packet Π is just an orbit under the group $\mathrm{Hom}(\mathcal{S}_{\tilde{T}}, \mathbb{C}^\times)$. Thus, Choosing a base point in the packet Π defines a one-to-one correspondence between Π and $\mathrm{Hom}(\mathcal{S}_{\tilde{T}}, \mathbb{C}^\times)$.

The correspondence between a packet Π and the group $\mathrm{Hom}(\mathcal{S}_{\tilde{T}}, \mathbb{C}^\times)$ stated in Proposition 10.7 is an analogue of the usual formulation to parametrize a packet in the local Langlands correspondence for a reductive algebraic group.

11. DESCRIPTION OF THE IMAGE $Z^\dagger$ OF THE CENTER

To determine the finite group $\mathcal{S}_{\tilde{T}} = Z^\dagger / \iota(T^\#)$ parametrizing a packet, we prepare exact sequences, and describe the subgroup $Z^\dagger \subset T$ via a non-degenerate bilinear form.

For $k \in \mathbb{Z}$ and a finite Γ -module M , let $M(k)$ denote the k -th Tate twist of M . For instance, $(\mathbb{Z}/m\mathbb{Z})(1) = \mu_m \subset \overline{F}^\times$ is the cyclic subgroup of order m , and $(\mathbb{Z}/m\mathbb{Z})(2) = \mu_m \otimes \mu_m$, for $m \in \mathbb{Z}_{>0}$.

Lemma 11.1. Let A be a Γ -module, and $Y' \subset Y$ a Γ -submodule of the cocharacter lattice. Assume that for some $m \in \mathbb{Z}_{>0}$, we have the inclusion $mY \subset Y'$ and exact sequence

$$0 \rightarrow (\mathbb{Z}/m\mathbb{Z})(1) \rightarrow A \xrightarrow{m} A \rightarrow 0.$$

Then they induce a short exact sequence

$$0 \rightarrow (Y/Y')(1) \rightarrow Y' \otimes A \rightarrow Y \otimes A \rightarrow 0.$$

Proof. The sequence $0 \rightarrow Y' \rightarrow Y \rightarrow Y/Y' \rightarrow 0$ induces an exact sequence

$$0 \rightarrow \mathrm{Tor}_1(Y/Y', A) \rightarrow Y' \otimes A \rightarrow Y \otimes A \rightarrow 0.$$

It suffices to give an isomorphism $\mathrm{Tor}_1(Y/Y', A) \cong (Y/Y')(1)$. Indeed, the sequence $0 \rightarrow (\mathbb{Z}/m\mathbb{Z})(1) \rightarrow A \xrightarrow{m} A \rightarrow 0$ gives an exact sequence

$$\begin{array}{ccccccc} \mathrm{Tor}_1(Y/Y', A) & \xrightarrow{0} & \mathrm{Tor}_1(Y/Y', A) & \rightarrow & (Y/Y') \otimes (\mathbb{Z}/m\mathbb{Z})(1) & \rightarrow & (Y/Y') \otimes A \\ & & & & \parallel & & \parallel \\ & & & & (Y/Y')(1) & & 0. \end{array}$$

□

Example 11.2. (1) Let $A = \bar{\mathbf{f}}^\times$ be an algebraic closure of the residue field, and $Y' = Y^\# = \{y \in Y \mid B_{\hat{T}}(y, Y) \in n\mathbb{Z}\}$ as defined in 10.2. Suppose the inclusion $\mu_n \subset \mathbf{f}^\times$, so that the assumption in Lemma 11.1 holds for $m = n$. Then we have a short exact sequence

$$0 \rightarrow (Y/Y^\#)(1) \rightarrow Y^\# \otimes \bar{\mathbf{f}}^\times \rightarrow Y \otimes \bar{\mathbf{f}}^\times \rightarrow 0.$$

(2) Let $Y' \subset Y$ be a Γ -submodule of the same rank. That is, some $m \in \mathbb{Z}_{>0}$ satisfies $mY \subset Y'$. Then the pairs $(\bar{\mathcal{O}}^\times, Y')$ and $(\bar{F}^\times, Y')$ respectively satisfy the assumption in Lemma 11.1. Hence, the tensor products with the valuation sequence $1 \rightarrow \bar{\mathcal{O}}^\times \rightarrow \bar{F}^\times \xrightarrow{\mathrm{ord}} \mathbb{Q} \rightarrow 0$ gives the commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (Y/Y')(1) & \longrightarrow & Y' \otimes \bar{\mathcal{O}}^\times & \longrightarrow & Y \otimes \bar{\mathcal{O}}^\times \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & (Y/Y')(1) & \longrightarrow & Y' \otimes \bar{F}^\times & \longrightarrow & Y \otimes \bar{F}^\times \longrightarrow 0 \\ & & & & \downarrow \mathrm{ord} & & \downarrow \mathrm{ord} \\ & & & & Y' \otimes \mathbb{Q} & \xrightarrow{\sim} & Y \otimes \mathbb{Q} \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

of exact sequences.

In Example 11.2 (2), note that $(Y' \otimes \mathbb{Q})^\Gamma = Y'^\Gamma \otimes \mathbb{Q}$. Then, the Galois cohomology sequences are described as follows.

Proposition 11.3. Let $Y' \subset Y$ be a Γ -submodule of the same rank. Then the tensor products with the valuation sequence $1 \rightarrow \bar{\mathcal{O}}^\times \rightarrow$

$\overline{F}^\times \xrightarrow{\text{ord}} \mathbb{Q} \rightarrow 0$ gives the diagram

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (Y/Y')(1)^\Gamma & \longrightarrow & (Y' \otimes \overline{\mathcal{O}}^\times)^\Gamma & \xrightarrow{\iota} & (Y \otimes \overline{\mathcal{O}}^\times)^\Gamma \xrightarrow{\partial_{Y/Y'}} H^1(\Gamma, (Y/Y')(1)) \\
& & \parallel & & \downarrow & & \downarrow & & \parallel \\
0 & \longrightarrow & (Y/Y')(1)^\Gamma & \longrightarrow & (Y' \otimes \overline{F}^\times)^\Gamma & \xrightarrow{\iota} & (Y \otimes \overline{F}^\times)^\Gamma \xrightarrow{\partial_{Y/Y'}} H^1(\Gamma, (Y/Y')(1)) \\
& & & & \downarrow \text{ord} & & \downarrow \text{ord} & & \\
& & & & Y'^\Gamma \otimes \mathbb{Q} & \xrightarrow{\sim} & Y^\Gamma \otimes \mathbb{Q} & &
\end{array}$$

of exact sequences.

We write $\partial = \partial_{Y/Y'}$ for the connecting homomorphisms. Let $V_{Y'} = \text{ord}((Y' \otimes \overline{F}^\times)^\Gamma)$ be the image of ord in the diagram of Proposition 11.3.

Corollary 11.4. (1) The images of ∂ and ord in Proposition 11.3 gives the commutative diagram

$$\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 \rightarrow & (Y/Y')(1)^\Gamma & \rightarrow & (Y' \otimes \overline{\mathcal{O}}^\times)^\Gamma & \xrightarrow{\iota} & (Y \otimes \overline{\mathcal{O}}^\times)^\Gamma & \xrightarrow{\partial} & \partial((Y \otimes \overline{\mathcal{O}}^\times)^\Gamma) & \rightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow & & \\
0 \rightarrow & (Y/Y')(1)^\Gamma & \rightarrow & (Y' \otimes \overline{F}^\times)^\Gamma & \xrightarrow{\iota} & (Y \otimes \overline{F}^\times)^\Gamma & \xrightarrow{\partial} & \partial((Y \otimes \overline{F}^\times)^\Gamma) & \rightarrow 0 \\
& & & \downarrow & & \downarrow \text{ord} & & \downarrow & \\
& 0 & \rightarrow & V_{Y'} & \hookrightarrow & V_Y & \rightarrow & V_Y/V_{Y'} & \rightarrow 0 \\
& & & \downarrow & & \downarrow & & \downarrow & \\
& & & 0 & & 0 & & 0 &
\end{array}$$

of exact sequences.

(2) In the diagram (1), we have

$$\iota((Y' \otimes \overline{F}^\times)^\Gamma) \cap (Y \otimes \overline{\mathcal{O}}^\times)^\Gamma = \iota((Y' \otimes \overline{\mathcal{O}}^\times)^\Gamma).$$

(3) Suppose that $T' = (Y' \otimes \overline{F}^\times)^\Gamma$ is a split torus over F . That is, Γ acts trivially on Y' . Then $\partial((Y \otimes \overline{F}^\times)^\Gamma) = H^1(\Gamma, (Y/Y')(1))$.

Proof. (1) This follows from the snake lemma. (2) The diagram displays that the map

$$(Y \otimes \overline{\mathcal{O}}^\times)^\Gamma / \iota((Y' \otimes \overline{\mathcal{O}}^\times)^\Gamma) \rightarrow (Y \otimes \overline{F}^\times)^\Gamma / \iota((Y' \otimes \overline{F}^\times)^\Gamma)$$

is injective. (3) Hilbert's theorem 90 shows $H^1(\Gamma, Y' \otimes \overline{F}^\times) = 0$. $\square$

For the subgroup $Y' = nY$ of Y , the connecting homomorphism

$$\partial = \partial_{Y/nY}: T = (Y \otimes \overline{F}^\times)^\Gamma \rightarrow H^1(\Gamma, (Y/nY)(1))$$

is applied to express commutators of lifts of elements in T to $\tilde{T}$.

Fact 11.5 ([2, Proposition 9.9]). The composite map

$$T \times T \xrightarrow{\partial \times \partial} H^1(\Gamma, (Y/nY)(1))^{\oplus 2} \xrightarrow{\cup} H^2(\Gamma, (Y/nY)(1)^{\otimes 2}) \xrightarrow{B_{\tilde{T}}} H^2(\Gamma, \mathbb{Z}/n(2)) = \mu_n$$

gives the commutator of lifts of two elements in T to $\tilde{T}$. That is, for any $s, t \in T$ and their lift $\tilde{s}, \tilde{t} \in \tilde{T}$, we have

$$\text{comm}_{\tilde{T}}(s, t) := B_{\tilde{T}}(\partial s \cup \partial t) = \tilde{s}\tilde{t}\tilde{s}^{-1}\tilde{t}^{-1}.$$

Thus, $Z^\dagger = \{t \in T \mid \forall t' \in T, B_{\tilde{T}}(\partial t \cup \partial t') = 0\}$.

We now generalize the description of the commutator map $\text{comm}_{\tilde{T}}$ slightly, so that other connecting homomorphisms can be applied.

For $i = 1, 2$, let T_i be an algebraic torus with cocharacter lattice Y_i .

Definition 11.6. Let $B: Y_1 \times Y_2 \rightarrow \mathbb{Z}$ be a Γ -invariant bilinear form. Then we define the Γ -submodule $Y_2^\# \subset Y_1$ as the annihilator

$$Y_2^\# = \{y \in Y_1 \mid B(y, Y_2) \subset n\mathbb{Z}\}$$

of Y_2 for the bilinear form $Y_1 \times Y_2 \rightarrow \mathbb{Z}/n$. We similarly define the Γ -submodule $Y_1^\# \subset Y_2$ by $Y_1^\# = \{y \in Y_2 \mid B(Y_1, y) \subset n\mathbb{Z}\}$.

For $B = B_{\tilde{T}}: Y \times Y \rightarrow \mathbb{Z}$, Definition 11.6 agrees with the definition of the submodule $Y^\#$ in 10.2. In general, the modules $Y_2^\#$ and $Y_1^\#$ are used to define non-degenerate bilinear forms:

Proposition 11.7. Let $B: Y_1 \times Y_2 \rightarrow \mathbb{Z}$ be a Γ -invariant bilinear form. For $i = 1, 2$, let $Y'_i \subset Y_i$ be a submodule.

- (1) The form B induces a bilinear form $Y_1/Y'_1 \times Y_2/Y'_2 \rightarrow \mathbb{Z}/n$ if and only if $Y'_1 \subset Y_2^\#$ and $Y'_2 \subset Y_1^\#$.
- (2) The induced bilinear form $B: Y_1/Y_2^\# \times Y_2/Y_1^\# \rightarrow \mathbb{Z}/n$ for the subgroups $Y_2^\#$ and $Y_1^\#$ is non-degenerate.
- (3) The bilinear form $B: Y_1/Y_2^\# \times Y_2/Y_1^\# \rightarrow \mathbb{Z}/n$ induces a non-degenerate bilinear form

$$\begin{array}{ccc} H^1(\Gamma, (Y_1/Y_2^\#)(1)) \times H^1(\Gamma, (Y_2/Y_1^\#)(1)) & \rightarrow & H^2(\Gamma, \mathbb{Z}/n(2)) \\ \cup & & \cup \\ (h_1, h_2) & \mapsto & B(h_1 \cup h_2). \end{array}$$

Proof. (1) and (2) hold by definition.

(3) By definition, the bilinear form $B: (Y_1/Y_2^\#)(1) \times (Y_2/Y_1^\#)(1) \rightarrow \mathbb{Z}/n(2) \cong \mathbb{Z}/n(1)$ is non-degenerate, that is, gives a pairing of dual Galois modules. By local Tate duality [13, II.5.2], the cup product $H^1(\Gamma, (Y_1/Y_2^\#)(1)) \times H^1(\Gamma, (Y_2/Y_1^\#)(1)) \rightarrow H^2(\Gamma, \mathbb{Z}/n(2)) \cong \mathbb{Z}/n$ is a pairing of groups, that is, a non-degenerate form. $\square$

If two submodules $Y'_1 \subset Y_2^\#$ and $Y'_2 \subset Y_1^\#$ satisfy the equalities $\text{rank } Y'_i = \text{rank } Y_i$ for $i = 1, 2$, then the inclusion $Y'_i \subset Y_i$ induces the connecting homomorphism $\partial_{Y_i/Y'_i}: T_i = (Y_i \otimes \bar{F}^\times)^\Gamma \rightarrow H^1(\Gamma, (Y_i/Y'_i)(1))$.

Thus, we may compose the homomorphisms ∂_{Y_1/Y'_1} and ∂_{Y_2/Y'_2} with the cup product to obtain a map

$$\text{comm}_B: T_1 \times T_2 \xrightarrow{\partial \times \partial} H^1(\Gamma, (Y_1/Y'_1)(1)) \times H^1(\Gamma, (Y_2/Y'_2)(1)) \xrightarrow{B} H^2(\Gamma, (\mathbb{Z}/n)(2)).$$

Lemma 11.8. For $i = 1, 2, 3, 4$, let T_i be an algebraic torus with cocharacter lattice Y_i , and $Y'_i \subset Y_i$ a Γ -submodule of the same rank as Y_i . For $i = 1, 3$, we assume the inclusions $Y'_i \subset Y_{i+1}^\#$ and $Y'_{i+1} \subset Y_i^\#$, so that a Γ -invariant bilinear form $B_{i,i+1}: Y_i \times Y_{i+1} \rightarrow \mathbb{Z}$ induces a bilinear form $Y_i/Y'_i \times Y_{i+1}/Y'_{i+1} \rightarrow \mathbb{Z}/n$. For $i = 1, 2$, let $f_i: Y_i \rightarrow Y_{i+2}$ be a Γ -equivariant homomorphism such that $f_i(Y'_i) \subset Y'_{i+2}$ and that the diagram

$$\begin{array}{ccc} Y_1/Y'_1 \times Y_2/Y'_2 & \xrightarrow{B_{1,2}} & \mathbb{Z}/n \\ f_1 \times f_2 \downarrow & & \parallel \\ Y_3/Y'_3 \times Y_4/Y'_4 & \xrightarrow{B_{3,4}} & \mathbb{Z}/n \end{array}$$

commutes. Then these maps induce the following commutative diagram:

$$\begin{array}{ccccc} T_1 \times T_2 & \xrightarrow{\partial \times \partial} & H^1(\Gamma, (Y_1/Y'_1)(1)) \times H^1(\Gamma, (Y_2/Y'_2)(1)) & \xrightarrow{B_{1,2}} & H^2(\Gamma, (\mathbb{Z}/n)(2)) \\ f_1 \times f_2 \downarrow & & \downarrow & & \parallel \\ T_3 \times T_4 & \xrightarrow{\partial \times \partial} & H^1(\Gamma, (Y_3/Y'_3)(1)) \times H^1(\Gamma, (Y_4/Y'_4)(1)) & \xrightarrow{B_{3,4}} & H^2(\Gamma, (\mathbb{Z}/n)(2)) \end{array}$$

Proof. This follows from the functoriality of cup products. $\square$

Proposition 11.9. Let $T_i, Y_i \supset Y'_i$ and $B_{i,i+1}$ be as above. For $i = 1, 2$, let $f_i: Y_i \rightarrow Y_{i+2}$ be any Γ -equivariant homomorphism making the diagram

$$\begin{array}{ccc} Y_1 \times Y_2 & \xrightarrow{B_{1,2}} & \mathbb{Z} \\ f_1 \times f_2 \downarrow & & \parallel \\ Y_3 \times Y_4 & \xrightarrow{B_{3,4}} & \mathbb{Z} \end{array}$$

commute. Then the following diagram commutes:

$$\begin{array}{ccccc} T_1 \times T_2 & \xrightarrow{\partial \times \partial} & H^1(\Gamma, (Y_1/Y'_1)(1)) \times H^1(\Gamma, (Y_2/Y'_2)(1)) & \xrightarrow{B_{1,2}} & H^2(\Gamma, (\mathbb{Z}/n)(2)) \\ f_1 \times f_2 \downarrow & & & & \parallel \\ T_3 \times T_4 & \xrightarrow{\partial \times \partial} & H^1(\Gamma, (Y_3/Y'_3)(1)) \times H^1(\Gamma, (Y_4/Y'_4)(1)) & \xrightarrow{B_{3,4}} & H^2(\Gamma, (\mathbb{Z}/n)(2)) \end{array}$$

Especially, the composite map $\text{comm}_{B_{1,2}} = B_{1,2} \circ (\partial \times \partial)$ is independent of the choice of subgroups Y'_1 and Y'_2 .

Proof. (1) First, suppose that $Y'_3 = nY_3$ and that $Y'_4 = nY_4$. Since $f_i(nY_i) \subset nY_{i+2}$ for $i = 1, 2$, we have the following commutative diagram by Lemma 11.8:

$$\begin{array}{ccccc}
T_1 \times T_2 & \longrightarrow & H^1(\Gamma, (Y_1/Y'_1)(1)) \times H^1(\Gamma, (Y_2/Y'_2)(1)) & \xrightarrow{B_{1,2}} & H^2(\Gamma, (\mathbb{Z}/n)(2)) \\
\parallel & & \downarrow & & \parallel \\
T_1 \times T_2 & \longrightarrow & H^1(\Gamma, (Y_1/Y_2^\#)(1)) \times H^1(\Gamma, (Y_2/Y_1^\#)(1)) & \xrightarrow{B_{1,2}} & H^2(\Gamma, (\mathbb{Z}/n)(2)) \\
\parallel & & \uparrow & & \parallel \\
T_1 \times T_2 & \longrightarrow & H^1(\Gamma, (Y_1/nY_1)(1)) \times H^1(\Gamma, (Y_2/nY_2)(1)) & \xrightarrow{B_{1,2}} & H^2(\Gamma, (\mathbb{Z}/n)(2)) \\
f_1 \times f_2 \downarrow & & \downarrow & & \parallel \\
T_3 \times T_4 & \longrightarrow & H^1(\Gamma, (Y_3/nY_3)(1)) \times H^1(\Gamma, (Y_4/nY_4)(1)) & \xrightarrow{B_{3,4}} & H^2(\Gamma, (\mathbb{Z}/n)(2))
\end{array}$$

(2) We may reduce the general case to (1) by the diagram

$$\begin{array}{ccccc}
T_1 \times T_2 & \longrightarrow & H^1(\Gamma, (Y_1/Y'_1)(1)) \times H^1(\Gamma, (Y_2/Y'_2)(1)) & \xrightarrow{B_{1,2}} & H^2(\Gamma, (\mathbb{Z}/n)(2)) \\
f_1 \times f_2 \downarrow & & & & \parallel \\
T_3 \times T_4 & \longrightarrow & H^1(\Gamma, (Y_3/nY_3)(1)) \times H^1(\Gamma, (Y_4/nY_4)(1)) & \xrightarrow{B_{3,4}} & H^2(\Gamma, (\mathbb{Z}/n)(2)) \\
\text{id} \times \text{id} \uparrow & & & & \parallel \\
T_3 \times T_4 & \longrightarrow & H^1(\Gamma, (Y_3/Y'_3)(1)) \times H^1(\Gamma, (Y_4/Y'_4)(1)) & \xrightarrow{B_{3,4}} & H^2(\Gamma, (\mathbb{Z}/n)(2))
\end{array}$$

□

Corollary 11.10. For $i = 1, 2$, let $Y_i \subset Y$ be a Γ -submodule such that the associated homomorphism $T_i \rightarrow T$ is injective. Then the restriction $B_{\tilde{T}}|_{Y_1 \times Y_2}: Y_1 \times Y_2 \rightarrow \mathbb{Z}$ gives the equality

$$\text{comm}_{B_{\tilde{T}}|_{Y_1 \times Y_2}} = \text{comm}_{\tilde{T}}|_{T_1 \times T_2}$$

as bilinear forms $T_1 \times T_2 \rightarrow H^2(\Gamma, (\mathbb{Z}/n)(2))$.

Note that $\text{comm}_{B_{\tilde{T}}} = \text{comm}_{\tilde{T}}$ when $B = B_{\tilde{T}}$.

Definition 11.11 (annihilators). (1) Generally, let A_1 and A_2 be finite Γ -modules, and $B: A_1 \times A_2 \rightarrow (\mathbb{Z}/n)(2)$ a Γ -equivariant bilinear form. For a subgroup $J_1 \subset H^1(\Gamma, A_1)$, we define the subgroup $J_1^\perp = J_1^{\perp B}$ of $H^1(\Gamma, A_2)$ as the annihilator for the bilinear form

$$B(\cup): H^1(\Gamma, A_1) \times H^1(\Gamma, A_2) \rightarrow H^2(\Gamma, \mathbb{Z}/n(2)).$$

That is:

$$J_1^\perp = \{h \in H^1(\Gamma, A_2) \mid \forall j_1 \in J_1, B(j_1 \cup h) = 0\}.$$

Similarly, for a subgroup $J_2 \subset H^1(\Gamma, A_2)$, we define the subgroup $J_2^\perp = J_2^{\perp_B}$ of $H^1(\Gamma, A_1)$ by

$$J_2^\perp = \{h \in H^1(\Gamma, A_1) \mid \forall j_2 \in J_2, B(h \cup j_2) = 0\}.$$

- (2) For subgroups $T_1, T_2 \subset T$, we define the subgroup $Z_{T_1}^\dagger(T_2) \subset T_1$ as the annihilator of T_2 for the commutator map $\text{comm}_{\tilde{T}}: T \times T \rightarrow \mu_n$. That is:

$$Z_{T_1}^\dagger(T_2) = \{t \in T_1 \mid \text{comm}_{\tilde{T}}(t, T_2) = \{1\} \text{ in } \mu_n\}.$$

Note that for $T_2 = T$, we have $Z_{T_1}^\dagger(T) = Z^\dagger \cap T_1$.

Example 11.12. (1) For $i = 1, 2$, suppose that $T_i \subset T$ is a subtorus with cocharacter lattice $Y_i \subset Y$. Then the restriction $B_{\tilde{T}}|_{Y_1 \times Y_2}: Y_1 \times Y_2 \rightarrow \mathbb{Z}$ also describes the annihilator of a subgroup $S \subset T_2$ as

$$Z_{T_1}^\dagger(S) = \{t \in T_1 \mid \text{comm}_{B_{\tilde{T}}|_{Y_1 \times Y_2}}(t, S) = \{1\} \text{ in } \mu_n\}$$

by Corollary 11.10. Take Γ -submodules $Y_1' \subset Y_1^\#$ and $Y_2' \subset Y_2^\#$ such that $\text{rank } Y_i' = \text{rank } Y_i$ for $i = 1, 2$. Then, we may write

$$\begin{aligned} Z_{T_1}^\dagger(S) &= \{t \in T_1 \mid \forall s \in S, B(\partial_{Y_1/Y_1'}(t) \cup \partial_{Y_2/Y_2'}(s)) = 0\} \\ &= \partial_{Y_1/Y_1'}^{-1}(\partial_{Y_2/Y_2'}(S)^\perp), \end{aligned}$$

where the annihilator $\partial_{Y_2/Y_2'}(S)^\perp$ is defined for the bilinear form

$$H^1(\Gamma, (Y_1/Y_1')(1)) \times H^1(\Gamma, (Y_2/Y_2')(1)) \xrightarrow{B} H^2(\Gamma, (\mathbb{Z}/n)(2)).$$

- (2) For the connecting homomorphism $\partial = \partial_{Y/Y^\#}: T = (Y \otimes \overline{F}^\times)^\Gamma \rightarrow H^1(\Gamma, (Y/Y^\#)(1))$, we shall show in Section 12 that the annihilator of the subgroup $\partial((Y \otimes \overline{\mathcal{O}}^\times)^\Gamma) \subset H^1(\Gamma, (Y/Y^\#)(1))$ is itself under the same assumption as our main theorem:

$$\partial((Y \otimes \overline{\mathcal{O}}^\times)^\Gamma)^\perp = \partial((Y \otimes \overline{\mathcal{O}}^\times)^\Gamma).$$

By (1), we have

$$\begin{aligned} Z_T^\dagger((Y \otimes \overline{\mathcal{O}}^\times)^\Gamma) &= \partial^{-1}\partial((Y \otimes \overline{\mathcal{O}}^\times)^\Gamma) = (Y \otimes \overline{\mathcal{O}}^\times)^\Gamma + \text{Ker } \partial_{Y/Y^\#} \\ &= (Y \otimes \overline{\mathcal{O}}^\times)^\Gamma + \iota(T^\#). \end{aligned}$$

The following is another example of the annihilator $Z_{T_1}^\dagger(T_2)$ that can be described explicitly:

Proposition 11.13. For $i = 1, 2$, let $T_i \subset T$ be a subtorus with cocharacter lattice $Y_i \subset Y$. Recall that the bilinear form $B_{\tilde{T}}|_{Y_1 \times Y_2}: Y_1 \times Y_2 \rightarrow \mathbb{Z}$ defines the subgroups $Y_2^\# \subset Y_1$ and $Y_1^\# \subset Y_2$, which induce isogenies $T_2^\# \xrightarrow{\iota} T_1$ and $T_1^\# \xrightarrow{\iota} T_2$ of tori (Definition 11.6). Assume that the torus $T_1^\#$ splits over F . That is, Γ acts trivially on the submodule $Y_1^\# \subset Y_2$. Then,

$$Z_{T_1}^\dagger(T_2) = \text{Ker } \partial_{Y_1/Y_2^\#} = \iota(T_2^\#).$$

Proof. The last equality follows from the long exact sequence where the connecting homomorphism $\partial_{Y_1/Y_2^\#}$ appears. Thus, it suffices to see the first equality. Since $Z_{T_1}^\dagger(T_2) = \partial_{Y_1/Y_2^\#}^{-1}(\partial_{Y_2/Y_1^\#}(T_2)^\perp)$ by Example 11.12 (1), it suffices to see that $\partial_{Y_2/Y_1^\#}(T_2)^\perp = 0$. By assumption, the connecting homomorphism $\partial_{Y_2/Y_1^\#}: T_2 = (Y_2 \otimes \overline{F}^\times)^\Gamma \rightarrow H^1(\Gamma, (Y_2/Y_1^\#)(1))$ is surjective (Corollary 11.4 (3)). Since the bilinear form

$$H^1(\Gamma, (Y_1/Y_2^\#)(1)) \times H^1(\Gamma, (Y_2/Y_1^\#)(1)) \xrightarrow{B} H^2(\Gamma, \mathbb{Z}/n(2))$$

is non-degenerate by Proposition 11.7, we have

$$\partial_{Y_2/Y_1^\#}(T_2)^\perp = H^1(\Gamma, (Y_2/Y_1^\#)(1))^\perp = 0.$$

□

Example 11.14. The maximal split subtorus $T^\Gamma \subset T$ is expressed as $T^\Gamma = Y^\Gamma \otimes F^\times$. Note that the exact sequence $0 \rightarrow Y^\Gamma \hookrightarrow Y \rightarrow Y/Y^\Gamma \rightarrow 0$ splits, since the quotient Y/Y^Γ is a free abelian group. Hence the sequence

$$\begin{array}{ccccccc} 0 \rightarrow & (Y^\Gamma \otimes \overline{F}^\times)^\Gamma & \rightarrow & (Y \otimes \overline{F}^\times)^\Gamma & \rightarrow & ((Y/Y^\Gamma) \otimes \overline{F}^\times)^\Gamma & \rightarrow 0 \\ & \parallel & & \parallel & & & \\ & T^\Gamma & & T & & & \end{array}$$

is split exact. Since the action of Γ on Y^Γ is trivial, the pair (T, T^Γ) satisfies the assumption of Proposition 11.13. Therefore

$$Z_T^\dagger(T^\Gamma) = \iota(T^{\Gamma\#}),$$

where $T^{\Gamma\#} \xrightarrow{\iota} T$ is the isogeny corresponding to the inclusion $Y^{\Gamma\#} \hookrightarrow Y$ of cocharacter lattices.

12. MAIN THEOREM

Our main theorem aims to determine the finite abelian group $\mathcal{S}_{\tilde{T}} = Z^\dagger/\iota(T^\#)$ introduced in Definition 10.3. Recall that its dual group $\text{Hom}(\mathcal{S}_{\tilde{T}}, \mathbb{C}^\times)$ parametrizes a packet in the local Langlands correspondence for a Brylinski-Deligne covering $\mu_n \rightarrow \tilde{T} \rightarrow T$ of a torus over a non-archimedean local field F of characteristic zero. Here, $\mu_n \subset \mathbf{f}^\times$ denotes the cyclic subgroup of order n .

We prove the main theorem by reducing it to a comparison between the image $Z^\dagger \subset T$ of the center $Z(\tilde{T}) \subset \tilde{T}$ and the image of the isogeny $T^\# \xrightarrow{\iota} T$ of tori. We analyze these two subgroups of T via the exact sequence $0 \rightarrow (Y \otimes \overline{\mathcal{O}}^\times)^\Gamma \rightarrow T \xrightarrow{\text{ord}} Y^\Gamma \otimes \mathbb{Q}$ appeared in Proposition 11.3. In addition to the cocharacter lattices $Y^\# \subset Y$ of $T^\#$ and T , we employ the subgroup

$$Y^{\Gamma\#} = (Y^\Gamma)^\# = \{y \in Y \mid B_{\tilde{T}}(y, Y^\Gamma) \subset n\mathbb{Z}\}$$

of Y , where $B_{\tilde{T}}: Y \times Y \rightarrow \mathbb{Z}$ is the bilinear form attached to the Brylinski-Deligne covering $\tilde{T} \rightarrow T$. Then the inclusion $Y^{\Gamma\#} \hookrightarrow Y$ again induces an isogeny $T^{\Gamma\#} \xrightarrow{\iota} T$ of tori.

The essential part of the proof is the self-orthogonality of the subgroup

$$\partial_{Y/Y^{\#}}((Y \otimes \overline{\mathcal{O}}^{\times})^{\Gamma}) \subset H^1(\Gamma, (Y/Y^{\#})(1))$$

mentioned in Example 11.12 (2), which will be reduced to the orthogonality theorem for unramified cohomology. Let $I = I_F \subset \Gamma$ be the inertia group of the extension $\overline{F}/F$, and $\Gamma_{\mathbf{f}} = \text{Gal}(\overline{\mathbf{f}}/\mathbf{f}) = \Gamma/I$ the absolute Galois group of $\mathbf{f}$. We fix a finite Galois extension L/F of the base field where the torus $T = (Y \otimes \overline{F}^{\times})^{\Gamma}$ splits over L , and write $e \in \mathbb{Z}_{>0}$ for its ramification index.

Since $\mu_n \subset \mathbf{f}^{\times}$ by the assumption, we may use the exact sequence $0 \rightarrow (Y/Y^{\#})(1) \rightarrow Y^{\#} \otimes \overline{\mathbf{f}}^{\times} \rightarrow Y \otimes \overline{\mathbf{f}}^{\times} \rightarrow 0$ given in Example 11.2.

Lemma 12.1. Assume that the degree n of the cover is prime to the ramification index e of the fixed splitting field L/F .

(1) The inclusion $Y^{\#} \hookrightarrow Y$ induces a short exact sequence

$$0 \rightarrow (Y/Y^{\#})^I(1) \rightarrow (Y^{\#} \otimes \overline{\mathbf{f}}^{\times})^I \rightarrow (Y \otimes \overline{\mathbf{f}}^{\times})^I \rightarrow 0.$$

(2) The exact sequence in (1) induces an isomorphism

$$\iota: H^1(\Gamma_{\mathbf{f}}, (Y^{\#} \otimes \overline{\mathbf{f}}^{\times})^I) \rightarrow H^1(\Gamma_{\mathbf{f}}, (Y \otimes \overline{\mathbf{f}}^{\times})^I).$$

(3) The connecting homomorphism

$$\partial: ((Y \otimes \overline{\mathbf{f}}^{\times})^I)^{\Gamma_{\mathbf{f}}} \rightarrow H^1(\Gamma_{\mathbf{f}}, (Y/Y^{\#})^I(1))$$

induced from the exact sequence in (1) is surjective.

Proof. (1) Let $I_L \subset \text{Gal}(\overline{F}/L)$ be the inertia group for L . Note that I_L acts trivially on $Y \otimes \overline{\mathbf{f}}^{\times}$ and $Y^{\#} \otimes \overline{\mathbf{f}}^{\times}$, and that $\#(I/I_L) = e$. Then the sequence in Example 11.2 (1) gives the exact sequence

$$(Y^{\#} \otimes \overline{\mathbf{f}}^{\times})^{I/I_L} \rightarrow (Y \otimes \overline{\mathbf{f}}^{\times})^{I/I_L} \rightarrow H^1(I/I_L, (Y/Y^{\#})(1)).$$

The last term of this sequence vanishes, since $(Y/Y^{\#})(1)$ is a $\mathbb{Z}/n$ -module and the integers n and e are relatively prime by assumption [14, Proposition 6.1.10].

(2) Since the cohomological dimension of the finite field $\mathbf{f}$ is one, the two maps

$$H^1(\Gamma_{\mathbf{f}}, (Y \otimes \overline{\mathbf{f}}^{\times})^I) \xrightarrow{n} H^1(\Gamma_{\mathbf{f}}, (Y^{\#} \otimes \overline{\mathbf{f}}^{\times})^I) \xrightarrow{\iota} H^1(\Gamma_{\mathbf{f}}, (Y \otimes \overline{\mathbf{f}}^{\times})^I)$$

are surjective. Since $(Y \otimes \overline{\mathbf{f}}^{\times})^I$ is an affine algebraic group defined over $\mathbf{f}$, we know that the group $H^1(\Gamma_{\mathbf{f}}, (Y \otimes \overline{\mathbf{f}}^{\times})^I)$ has finite order [13, III.4.3]. Hence the composite $H^1(\Gamma_{\mathbf{f}}, (Y \otimes \overline{\mathbf{f}}^{\times})^I) \xrightarrow{n} H^1(\Gamma_{\mathbf{f}}, (Y \otimes \overline{\mathbf{f}}^{\times})^I)$ is an isomorphism, and so is the map ι .

(3) In the exact sequence

$$((Y \otimes \bar{\mathbf{f}}^\times)^I)^{\Gamma_{\mathbf{f}}} \xrightarrow{\partial} H^1(\Gamma_{\mathbf{f}}, (Y/Y^\#)^I(1)) \rightarrow H^1(\Gamma_{\mathbf{f}}, (Y^\# \otimes \bar{\mathbf{f}}^\times)^I) \xrightarrow{\iota} H^1(\Gamma_{\mathbf{f}}, (Y \otimes \bar{\mathbf{f}}^\times)^I),$$

the middle map is zero, since the last map ι is an isomorphism. $\square$

For a Γ -module A , let

$$\text{inf}: H^1(\Gamma_{\mathbf{f}}, A^I) \rightarrow H^1(\Gamma, A)$$

be the inflation map on the first cohomology group. This inf is injective, and fits into the exact sequence in low dimensions given by the Lyndon-Hochschild-Serre spectral sequence. Thus, we often regard $H^1(\Gamma_{\mathbf{f}}, A^I) \subset H^1(\Gamma, A)$.

Proposition 12.2. Let $A \times A' \xrightarrow{B} \mu_n$ be a non-degenerate Γ -equivariant pairing between two finite Γ -modules. Then, in the non-degenerate pairing $\beta: H^1(\Gamma, A) \times H^1(\Gamma, A') \xrightarrow{B(\cup)} H^2(\Gamma, \mu_n) = \mathbb{Z}/n$ due to the local Tate duality, we have

$$\text{inf}(H^1(\Gamma_{\mathbf{f}}, A^I))^\perp = \text{inf}(H^1(\Gamma_{\mathbf{f}}, A'^I)).$$

Proof. The restriction $H^1(\Gamma_{\mathbf{f}}, A^I) \times H^1(\Gamma_{\mathbf{f}}, A'^I) \rightarrow H^2(\Gamma, \mu_n)$ of β factors through $H^2(\Gamma_{\mathbf{f}}, \mu_n) = 0$. That is,

$$\beta(H^1(\Gamma_{\mathbf{f}}, A^I), H^1(\Gamma_{\mathbf{f}}, A'^I)) = 0.$$

It remains to show that $\# H^1(\Gamma, A) = \# H^1(\Gamma_{\mathbf{f}}, A^I) \cdot \# H^1(\Gamma_{\mathbf{f}}, A'^I)$. Indeed, we see the equalities

$$\begin{aligned} \# H^1(\Gamma, A) &= \# H^0(\Gamma, A) \cdot \# H^2(\Gamma, A) \\ &= \# H^0(\Gamma, A) \cdot \# H^0(\Gamma, A') \\ &= \# H^0(\Gamma_{\mathbf{f}}, A^I) \cdot \# H^0(\Gamma_{\mathbf{f}}, A'^I) \\ &= \# H^1(\Gamma_{\mathbf{f}}, A^I) \cdot \# H^1(\Gamma_{\mathbf{f}}, A'^I) \end{aligned}$$

as follows:

For the first equality, note that A is a finite $\mathbb{Z}/n$ -module by definition. Since n is prime to the residual characteristic p by assumption, the Euler-Poincaré characteristic is $\frac{\# H^0(\Gamma, A) \cdot \# H^2(\Gamma, A)}{\# H^1(\Gamma, A)} = 1$. The second equality follows from the local Tate duality between H^2 and H^0 . The third one holds by definition. For an explicit proof of the last equality, we may consult Harari's textbook [6, Lemma 10.14]. $\square$

Note that this lemma slightly generalizes the well-known orthogonality theorem for unramified cohomology [13, II.5.5].

Corollary 12.3. Assume that the degree n of the cover is prime to the ramification index e of the fixed splitting field L/F .

- (1) For the connecting homomorphism $\partial = \partial_{Y/Y^\#}: T = (Y \otimes \bar{F}^\times)^\Gamma \rightarrow H^1(\Gamma, (Y/Y^\#)(1))$, the annihilator of the subgroup $\partial((Y \otimes \bar{\mathcal{O}}^\times)^\Gamma) \subset H^1(\Gamma, (Y/Y^\#)(1))$ is itself:

$$\partial((Y \otimes \bar{\mathcal{O}}^\times)^\Gamma)^\perp = \partial((Y \otimes \bar{\mathcal{O}}^\times)^\Gamma).$$

- (2) As subgroups of $T = (Y \otimes \bar{F}^\times)^\Gamma$, we have

$$Z^\dagger \subset Z_T^\dagger((Y \otimes \bar{\mathcal{O}}^\times)^\Gamma) = (Y \otimes \bar{\mathcal{O}}^\times)^\Gamma + \iota(T^\#).$$

Especially, the map

$$Z^\dagger \cap (Y \otimes \bar{\mathcal{O}}^\times)^\Gamma / \iota(T^\#) \cap (Y \otimes \bar{\mathcal{O}}^\times)^\Gamma \rightarrow Z^\dagger / \iota(T^\#) = \mathcal{S}_{\tilde{T}}$$

is an isomorphism.

Proof. (1) By assumption, Lemma 12.1 (1) ensures that the horizontal sequences in the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & (Y/Y^\#)(1) & \longrightarrow & Y^\# \otimes \bar{\mathcal{O}}^\times & \longrightarrow & Y \otimes \bar{\mathcal{O}}^\times \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \text{split} \\ 0 & \longrightarrow & (Y/Y^\#)(1) & \longrightarrow & Y^\# \otimes \bar{\mathbf{f}}^\times & \longrightarrow & Y \otimes \bar{\mathbf{f}}^\times \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & (Y/Y^\#)^I(1) & \longrightarrow & (Y^\# \otimes \bar{\mathbf{f}}^\times)^I & \longrightarrow & (Y \otimes \bar{\mathbf{f}}^\times)^I \longrightarrow 0 \end{array}$$

are exact. The connecting homomorphisms ∂ on the Galois cohomology groups form the commutative diagram

$$\begin{array}{ccc} (Y \otimes \bar{\mathcal{O}}^\times)^\Gamma & \xrightarrow{\partial} & H^1(\Gamma, (Y/Y^\#)(1)) \\ \downarrow & & \parallel \\ (Y \otimes \bar{\mathbf{f}}^\times)^\Gamma & \xrightarrow{\partial} & H^1(\Gamma, (Y/Y^\#)(1)) \\ \parallel & & \uparrow \text{inf} \\ ((Y \otimes \bar{\mathbf{f}}^\times)^I)^{\Gamma_{\mathbf{f}}} & \xrightarrow{\partial} & H^1(\Gamma_{\mathbf{f}}, (Y/Y^\#)^I(1)) \end{array}$$

In the left column, the downward map $(Y \otimes \bar{\mathcal{O}}^\times)^\Gamma \rightarrow (Y \otimes \bar{\mathbf{f}}^\times)^\Gamma$ is surjective, since $Y \otimes \bar{\mathcal{O}}^\times \rightarrow Y \otimes \bar{\mathbf{f}}^\times$ is a split surjection. The bottom ∂ is also surjective by Lemma 12.1 (3). Recall that inf in the right column is injective. Then

$$\partial((Y \otimes \bar{\mathcal{O}}^\times)^\Gamma) = \partial((Y \otimes \bar{\mathbf{f}}^\times)^\Gamma) = \text{inf} \left(H^1(\Gamma_{\mathbf{f}}, (Y/Y^\#)^I(1)) \right).$$

By Proposition 12.2,

$$\partial((Y \otimes \bar{\mathcal{O}}^\times)^\Gamma) = \text{inf} \left(H^1(\Gamma_{\mathbf{f}}, (Y/Y^\#)^I(1)) \right)^\perp = \partial((Y \otimes \bar{\mathcal{O}}^\times)^\Gamma)^\perp.$$

(2) The inclusion holds by the definition of annihilators. By Example 11.12 (1), we have

$$\begin{aligned} Z_T^\dagger((Y \otimes \overline{\mathcal{O}}^\times)^\Gamma) &= \partial^{-1} \partial((Y \otimes \overline{\mathcal{O}}^\times)^\Gamma) = (Y \otimes \overline{\mathcal{O}}^\times)^\Gamma + \text{Ker } \partial_{Y/Y^\#} \\ &= (Y \otimes \overline{\mathcal{O}}^\times)^\Gamma + \iota(T^\#). \end{aligned}$$

□

Corollary 12.3 allows us to describe the intersection $Z^\dagger \cap (Y \otimes \overline{\mathcal{O}}^\times)^\Gamma$.

Proposition 12.4. As in Corollary 12.3, assume that the degree n of the cover is prime to the ramification index e of the fixed splitting field L/F . Then the following equalities hold.

- (1) $T = \iota(T^\#) + T^\Gamma + (Y \otimes \overline{\mathcal{O}}^\times)^\Gamma$
- (2) $Z^\dagger \cap (Y \otimes \overline{\mathcal{O}}^\times)^\Gamma = \iota((Y^{\Gamma^\#} \otimes \overline{\mathcal{O}}^\times)^\Gamma).$

Proof. (1) As usual, we normalize the valuation map $\text{ord}: \overline{F}^\times \rightarrow \mathbb{Q}$ so that $\text{ord}(\overline{F}^\times) = \mathbb{Z}$. Then, for the induced map

$$\text{ord}: T = (Y \otimes \overline{F}^\times)^\Gamma \rightarrow (Y \otimes \mathbb{Q})^\Gamma = Y^\Gamma \otimes \mathbb{Q},$$

we have $\text{ord}(T^\Gamma) = Y^\Gamma$ and $\text{ord}(T) \subset \frac{1}{e} Y^\Gamma$. Since $nY \subset Y^\#$ by definition, we have the inclusion $n(Y \otimes \overline{F})^\Gamma \subset \iota((Y^\# \otimes \overline{F})^\Gamma)$ in $T = (Y \otimes \overline{F})^\Gamma$, and hence $n \cdot \text{ord}(T) \subset \text{ord}(\iota(T^\#))$. Since the integers n and e are relatively prime by assumption,

$$\text{ord}(T) = n \text{ord}(T) + e \text{ord}(T) \subset \text{ord}(\iota(T^\#)) + Y^\Gamma = \text{ord}(\iota(T^\#) + T^\Gamma).$$

Since the map $\text{ord}: T \rightarrow Y^\Gamma \otimes \mathbb{Q}$ has kernel $(Y \otimes \overline{\mathcal{O}}^\times)^\Gamma$, the claimed equality holds.

(2) The identity in (1) and Example 11.14 imply

$$\begin{aligned} Z^\dagger &= Z_T^\dagger(T^\Gamma) \cap Z_T^\dagger(\iota(T^\#) + (Y \otimes \overline{\mathcal{O}}^\times)^\Gamma) \\ &= \iota(T^{\Gamma^\#}) \cap Z_T^\dagger(\iota(T^\#) + (Y \otimes \overline{\mathcal{O}}^\times)^\Gamma). \end{aligned}$$

Since $Z_T^\dagger(\iota(T^\#) + (Y \otimes \overline{\mathcal{O}}^\times)^\Gamma) \supset (Y \otimes \overline{\mathcal{O}}^\times)^\Gamma$ by Corollary 12.3 (2), we see

$$Z^\dagger \cap (Y \otimes \overline{\mathcal{O}}^\times)^\Gamma = \iota(T^{\Gamma^\#}) \cap (Y \otimes \overline{\mathcal{O}}^\times)^\Gamma = \iota((Y^{\Gamma^\#} \otimes \overline{\mathcal{O}}^\times)^\Gamma).$$

Note that the last equality is the special case of Corollary 11.4 (2) for $Y' = Y^{\Gamma^\#}$. □

Now we can prove the main theorem:

Theorem 12.5. Let F be a non-archimedean local field of characteristic zero, $\mathbf{f}$ the residue field of the valuation ring of F , and $\mu_n \subset \mathbf{f}^\times$ the cyclic subgroup of order n . Suppose that an algebraic torus $T = \mathbb{T}(F)$

defined over F splits over a finite Galois extension L/F whose ramification index e is relatively prime to n . Let $\mu_n \rightarrow \tilde{T} \rightarrow T$ be a Brylinski-Deligne covering group.

Then the group $\mathcal{S} = \mathcal{S}_{\tilde{T}}$ parametrizing a packet of representations of $\tilde{T}$ is

$$\begin{aligned} \mathcal{S}_{\tilde{T}} &= \iota((Y^{\Gamma\#} \otimes_{\mathbb{Z}} \overline{\mathcal{O}}^{\times})^{\Gamma}) / \iota((Y^{\#} \otimes_{\mathbb{Z}} \overline{\mathcal{O}}^{\times})^{\Gamma}) . \\ &= \iota((Y^{\Gamma\#} \otimes_{\mathbb{Z}} \overline{\mathbf{f}}^{\times})^{\Gamma}) / \iota((Y^{\#} \otimes_{\mathbb{Z}} \overline{\mathbf{f}}^{\times})^{\Gamma}) . \end{aligned}$$

Proof. The first identity follows from Corollary 12.3 (2), Proposition 12.4 (2), and Corollary 11.4 (2) for $Y' = Y^{\#}$.

Let $\overline{\mathfrak{p}} \subset \overline{\mathcal{O}}$ be the maximal ideal of the valuation ring in the algebraic closure $\overline{F}$. Since $\mu_n \subset \mathbf{f}^{\times}$ by assumption, the inclusions $nY \subset Y^{\#} \subset Y^{\Gamma\#} \subset Y$ induces isomorphisms

$$Y^{\#} \otimes (1 + \overline{\mathfrak{p}}) \xrightarrow{\sim} Y^{\Gamma\#} \otimes (1 + \overline{\mathfrak{p}}) \xrightarrow{\sim} Y \otimes (1 + \overline{\mathfrak{p}}).$$

Then the split exact sequence $1 \rightarrow 1 + \overline{\mathfrak{p}} \rightarrow \overline{\mathcal{O}}^{\times} \rightarrow \overline{\mathbf{f}}^{\times} \rightarrow 1$ gives a commutative diagram

$$\begin{array}{ccccc} 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ (Y^{\#} \otimes (1 + \overline{\mathfrak{p}}))^{\Gamma} & \xrightarrow{\sim} & (Y^{\Gamma\#} \otimes (1 + \overline{\mathfrak{p}}))^{\Gamma} & \xrightarrow{\sim} & (Y \otimes (1 + \overline{\mathfrak{p}}))^{\Gamma} \\ \downarrow & & \downarrow & & \downarrow \\ (Y^{\#} \otimes \overline{\mathcal{O}}^{\times})^{\Gamma} & \longrightarrow & (Y^{\Gamma\#} \otimes \overline{\mathcal{O}}^{\times})^{\Gamma} & \longrightarrow & (Y \otimes \overline{\mathcal{O}}^{\times})^{\Gamma} \\ \downarrow & & \downarrow & & \downarrow \\ (Y^{\#} \otimes \overline{\mathbf{f}}^{\times})^{\Gamma} & \longrightarrow & (Y^{\Gamma\#} \otimes \overline{\mathbf{f}}^{\times})^{\Gamma} & \longrightarrow & (Y \otimes \overline{\mathbf{f}}^{\times})^{\Gamma} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

where the top horizontal maps are isomorphisms, and the vertical sequences are split exact. Thus the map

$$\mathcal{S}_{\tilde{T}} = \iota((Y^{\Gamma\#} \otimes \overline{\mathcal{O}}^{\times})^{\Gamma}) / \iota((Y^{\#} \otimes \overline{\mathcal{O}}^{\times})^{\Gamma}) \rightarrow \iota((Y^{\Gamma\#} \otimes \overline{\mathbf{f}}^{\times})^{\Gamma}) / \iota((Y^{\#} \otimes \overline{\mathbf{f}}^{\times})^{\Gamma})$$

is an isomorphism. $\square$

As a trivial example, if $\tilde{T} \rightarrow T$ is a Brylinski-Deligne cover of a split torus, then $\mathcal{S}_{\tilde{T}} = 1$ since $Y^{\Gamma} = Y$. Another elementary example is a cover $\tilde{T} \rightarrow T$ of an unramified torus, that is, a torus splitting over an unramified extension L/F of the base field. Then Theorem 12.5 regains Weissman's result [17] on covers of an unramified torus.

Thus, our main theorem is a generalization of Weissman's work on covering groups. Moreover, this theorem partially realizes his hope [17,

Remark 5.18] to parametrize a packet in the local Langlands correspondence for a cover of a ramified torus.

13. APPLICATION TO THE GLOBAL CORRESPONDENCE

Our local result has an application to the finiteness problem of a certain group describing a gap between the local and the global Langlands correspondence for a covering group of a torus. First, we review the construction of a Brylinski-Deligne covering group defined over an algebraic number field.

Let F be an algebraic number field, $\mu_n \subset F^\times$ the cyclic subgroup of order n , and $\mathbb{T} \rightarrow \operatorname{Spec} F$ an algebraic torus. Recall that $\mathbb{K}_2$ is the sheaf on the big Zariski site F_{ZAR} associated with the presheaf of Milnor's $\mathbb{K}_2$ -groups. As in the local case, a Brylinski-Deligne covering group of the torus $\mathbb{T}$ is a central extension

$$1 \rightarrow \mathbb{K}_2 \rightarrow \widetilde{\mathbb{T}} \rightarrow \mathbb{T} \rightarrow 1$$

as a sheaf of groups on the big Zariski site of $\operatorname{Spec} F$. At a place v of the number field F , the sections

$$1 \rightarrow \mathbb{K}_2(F_v) \rightarrow \widetilde{\mathbb{T}}(F_v) \rightarrow \mathbb{T}(F_v) \rightarrow 1$$

over $\operatorname{Spec} F_v$ also form a central extension, and give a covering group

$$1 \rightarrow \mu_n \rightarrow \widetilde{T}_v \rightarrow \mathbb{T}(F_v) \rightarrow 1$$

by taking the push-out via the Hilbert symbol $\mathbb{K}_2(F) \rightarrow \mu_n$. If v is a complex place, then we regard the Hilbert symbol as the trivial map.

At almost every place v of the number field F , Brylinski and Deligne found a split monomorphism $\mathbb{T}(\mathcal{O}_v) \rightarrow \widetilde{T}_v$. Thus, we may construct the restricted direct product $\prod'_v \widetilde{T}_v$ with respect to the family $\{\mathbb{T}(\mathcal{O}_v)\}_v$ of compact groups. This restricted direct product is a central extension $1 \rightarrow \bigoplus_v \mu_n \rightarrow \prod'_v \widetilde{T}_v \rightarrow \mathbb{T}(\mathbb{A}_F) \rightarrow 1$, and gives a covering group

$$1 \rightarrow \mu_n \rightarrow \widetilde{\mathbb{T}(\mathbb{A}_F)} \rightarrow \mathbb{T}(\mathbb{A}_F) \rightarrow 1$$

by taking the push-out via the product map $\bigoplus_v \mu_n \rightarrow \mu_n$.

Definition 13.1. Let F be a number field, $\mathbb{A} = \mathbb{A}_F$ the adèle ring, and $\mathbb{T} \rightarrow \operatorname{Spec} F$ an algebraic torus. Then a *Brylinski-Deligne covering group of the adèle group* $\mathbb{T}(\mathbb{A})$ is the cover

$$\mu_n \rightarrow \widetilde{\mathbb{T}(\mathbb{A})} \rightarrow \mathbb{T}(\mathbb{A})$$

constructed from a Brylinski-Deligne cover $1 \rightarrow \mathbb{K}_2 \rightarrow \widetilde{\mathbb{T}} \rightarrow \mathbb{T} \rightarrow 1$ of the torus $\mathbb{T}$.

Metaplectic groups and other well-known covering groups are examples of Brylinski-Deligne covers of adèle groups.

We study the automorphic representations of a Brylinski-Deligne covering group $\widetilde{\mathbb{T}(\mathbb{A})}$ in terms of global Langlands correspondence. As in

the local case, we require the representations to be genuine. To simplify the formulation, we further focus on unitary representations.

Proposition 13.2. Let $\mu_n \rightarrow \widetilde{\mathbb{T}(\mathbb{A})} \rightarrow \mathbb{T}(\mathbb{A})$ be a Brylinski-Deligne covering of an adelic torus. Then the genuine unitary automorphic representations of the group $\widetilde{\mathbb{T}(\mathbb{A})}$ has genuine unitary central characters. Moreover, this correspondence is bijective onto the set of genuine unitary characters of the center $Z(\mathbb{T}(\mathbb{A}))$ [17, Theorem 4.5].

Thus, to establish the global Langlands correspondence for a Brylinski-Deligne covering group $\widetilde{\mathbb{T}(\mathbb{A})} \rightarrow \mathbb{T}(\mathbb{A})$, it suffices to parametrize the genuine characters of the center $Z(\widetilde{\mathbb{T}(\mathbb{A})})$. To estimate $Z(\widetilde{\mathbb{T}(\mathbb{A})}) \subset \mathbb{T}(\mathbb{A})$, we again choose the isogeny $\mathbb{T}^\# \xrightarrow{\iota} \mathbb{T}$ of tori, which approximates the image $Z^\dagger(\mathbb{A}) \subset \mathbb{T}$ of $Z(\mathbb{T}(\mathbb{A}))$.

Definition 13.3. For a Brylinski-Deligne covering group $\widetilde{\mathbb{T}(\mathbb{A})} \rightarrow \mathbb{T}(\mathbb{A})$, we define the abelian group $\mathcal{S}_{\widetilde{\mathbb{T}}, F} = \mathcal{S}_{\widetilde{\mathbb{T}}, F}$ as the quotient

$$\mathcal{S}_{\widetilde{\mathbb{T}}, F} = (Z^\dagger(\mathbb{A}) \cap \mathbb{T}(F)) / (\iota(\mathbb{T}^\#(\mathbb{A})) \cap \mathbb{T}(F)) .$$

This may be called the *global packet group*.

In the rest of this section, we explain the role and the finiteness of this group $\mathcal{S}_{\widetilde{\mathbb{T}}, F}$. At a place v of the number field F , let T_v denote the local torus $\mathbb{T}(F_v)$, and $Z_v^\dagger \subset T_v$ the image of the center $Z(\widetilde{T}_v)$ of the cover. By the construction of the Brylinski-Deligne cover $\widetilde{\mathbb{T}(\mathbb{A})}$, the image $Z^\dagger(\mathbb{A})$ of the center is determined place-wise. Thus the group $Z^\dagger(\mathbb{A})$ decomposes into the restricted direct product $\prod'_v Z_v^\dagger$ with respect to the family $\{Z_v^\dagger \cap \mathbb{T}(\mathcal{O}_v)\}_v$ of intersections. Hence, we have the inclusion

$$\iota(\mathbb{T}(\mathbb{A})) \subset Z^\dagger(\mathbb{A}),$$

whose quotient group $Z^\dagger(\mathbb{A})/\iota(\mathbb{T}(\mathbb{A}))$ is the direct product $\prod_v \mathcal{S}_{\widetilde{T}_v}$ of local packet groups. Note that the sequences in the commutative diagram

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & \iota(\mathbb{T}^\#(\mathbb{A})) \cap \mathbb{T}(F) & \longrightarrow & Z^\dagger(\mathbb{A}) \cap \mathbb{T}(F) & \longrightarrow & \mathcal{S}_{\widetilde{\mathbb{T}}, F} \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & \iota(\mathbb{T}^\#(\mathbb{A})) & \longrightarrow & Z^\dagger(\mathbb{A}) & \longrightarrow & \prod_v \mathcal{S}_{\widetilde{T}_v} \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & \iota(\mathbb{T}^\#(\mathbb{A})/\mathbb{T}^\#(F)) & \longrightarrow & Z^\dagger(\mathbb{A})/(Z^\dagger(\mathbb{A}) \cap \mathbb{T}(F)) & \longrightarrow & (\prod_v \mathcal{S}_{\widetilde{T}_v})/\mathcal{S}_{\widetilde{\mathbb{T}}, F} \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 1 & & 1 & & 1
 \end{array}$$

are all exact. The bottom sequence fits into another diagram

$$\begin{array}{ccccccc}
 & & 1 & & & & \\
 & & \uparrow & & & & \\
 & & (\prod_v \mathcal{S}_{\widetilde{T}_v})/\mathcal{S}_{\widetilde{\mathbb{T}}, F} & & & & \\
 & & \uparrow & & & & \\
 1 & \longleftarrow & Z^\dagger(\mathbb{A})/(Z^\dagger(\mathbb{A}) \cap \mathbb{T}(F)) & \longleftarrow & Z(\widetilde{\mathbb{T}(\mathbb{A})})/(Z(\widetilde{\mathbb{T}(\mathbb{A})}) \cap \mathbb{T}(F)) & \longleftarrow & \mu_n \longleftarrow 1 \\
 & & \uparrow & & & & \\
 1 & \longleftarrow & \iota(\mathbb{T}^\#(\mathbb{A})/\mathbb{T}^\#(F)) & \xleftarrow{\iota} & \mathbb{T}^\#(\mathbb{A})/\mathbb{T}^\#(F) & & \\
 & & \uparrow & & & & \\
 & & 1 & & & &
 \end{array}$$

of exact sequences. We take the Pontryagin dual $(\)^\vee = \text{Hom}(\ , \mathbb{C}^1)$ of these groups, noting the following fact.

Proposition 13.4 (Global Langlands correspondence for a torus [7, Théorème 6.4]). Let W_F be the Weil group of the algebraic number field F . Then we have a natural surjection

$$\text{GLC}_{\mathbb{T}^\#} : H^1(W_F, (Y^\#)^\vee) \rightarrow \text{Hom}(\mathbb{T}^\#(\mathbb{A})/\mathbb{T}^\#(F), \mathbb{C}^1)$$

with finite kernel. Here, the Pontryagin dual $(Y^\#)^\vee = \text{Hom}(Y^\#, \mathbb{C}^1)$ is the maximal compact subgroup of the Langlands dual $\widehat{\mathbb{T}^\#}$ of the torus $\mathbb{T}^\#$.

Thus, we obtain the following frame of the global Langlands correspondence for the covering group $\widehat{\mathbb{T}(\mathbb{A})} \rightarrow \mathbb{T}(\mathbb{A})$.

Proposition 13.5. Recall that $j: \mu_n \hookrightarrow \mathbb{C}^1$ is the fixed embedding to define genuine representations. Then the objects introduced above fit in the diagram

$$\begin{array}{ccccc}
1 & & \left\{ \begin{array}{c} \text{genuine unitary automorphic} \\ \text{representations of } \widetilde{T} \end{array} \right\} & & \\
\downarrow & & \downarrow \scriptstyle (\cdot)|_{Z(\widetilde{\mathbb{T}(\mathbb{A})})} \text{ 1:1} & & \\
\left(\left(\prod_v \mathcal{S}_{\widetilde{T}_v} \right) / \mathcal{S}_{\widetilde{\mathbb{T}}, F} \right)^\vee & & \{\text{genuine characters}\} & \rightarrow & \{j\} \\
\downarrow & & \cap & & \cap \\
1 \rightarrow (Z^\dagger(\mathbb{A}) / (Z^\dagger(\mathbb{A}) \cap \mathbb{T}(F)))^\vee & \rightarrow & \left(Z(\widetilde{\mathbb{T}(\mathbb{A})}) / \left(Z(\widetilde{\mathbb{T}(\mathbb{A})}) \cap \mathbb{T}(F) \right) \right)^\vee & \rightarrow & \mu_n^\vee \rightarrow 1 \\
\downarrow \scriptstyle (\cdot)|_{\iota(\mathbb{T}^\#(\mathbb{A})/\mathbb{T}^\#(F))} & & & & \\
(\iota(\mathbb{T}^\#(\mathbb{A})/\mathbb{T}^\#(F)))^\vee & \xrightarrow{\iota^*} & (\mathbb{T}^\#(\mathbb{A})/\mathbb{T}^\#(F))^\vee & & \\
\downarrow & & \parallel & & \\
1 & & H^1(W_F, (Y^\#)^\vee) / \text{Ker GLC}_{\mathbb{T}^\#} & &
\end{array}$$

where the middle horizontal sequence and the left vertical one are exact.

By definition, the group $\left(\left(\prod_v \mathcal{S}_{\widetilde{T}_v} \right) / \mathcal{S}_{\widetilde{\mathbb{T}}, F} \right)^\vee$ acts simply transitive on the set of genuine unitary characters of $Z(\widetilde{\mathbb{T}(\mathbb{A})}) / \left(Z(\widetilde{\mathbb{T}(\mathbb{A})}) \cap \mathbb{T}(F) \right)$. Thus, fixing such a character χ as a base point defines a bijection

$$\begin{array}{c}
\left\{ \text{genuine unitary characters of } Z(\widetilde{\mathbb{T}(\mathbb{A})}) / \left(Z(\widetilde{\mathbb{T}(\mathbb{A})}) \cap \mathbb{T}(F) \right) \right\} \\
\downarrow \scriptstyle b_\chi \\
(Z^\dagger(\mathbb{A}) / (Z^\dagger(\mathbb{A}) \cap \mathbb{T}(F)))^\vee
\end{array}$$

By Proposition 10.5, the composed map $\text{GLC}_{\mathbb{T}^\#}^{-1} \circ \iota^* \circ (\cdot)|_{\iota(\mathbb{T}^\#(\mathbb{A})/\mathbb{T}^\#(F))} :$

$$(Z^\dagger(\mathbb{A}) / (Z^\dagger(\mathbb{A}) \cap \mathbb{T}(F)))^\vee \rightarrow H^1(W_F, (Y^\#)^\vee) / \text{Ker GLC}_{\mathbb{T}^\#}$$

has kernel $\left(\left(\prod_v \mathcal{S}_{\widetilde{T}_v} \right) / \mathcal{S}_{\widetilde{\mathbb{T}}, F} \right)^\vee$.

Definition 13.6. Fix a genuine unitary character χ of the quotient group $Z(\widetilde{\mathbb{T}(\mathbb{A})}) / \left(Z(\widetilde{\mathbb{T}(\mathbb{A})}) \cap \mathbb{T}(F) \right)$. Then the *global Langlands correspondence* for the Brylinski-Deligne cover $\widetilde{\mathbb{T}(\mathbb{A})} \rightarrow \mathbb{T}(\mathbb{A})$ is the composition

$$\begin{array}{c}
\left\{ \text{genuine unitary automorphic representations of } \widetilde{\mathbb{T}(\mathbb{A})} \right\} \\
\downarrow \scriptstyle b_\chi \circ (\cdot)|_{Z(\widetilde{\mathbb{T}(\mathbb{A})})} \text{ 1:1} \\
(Z^\dagger(\mathbb{A}) / (Z^\dagger(\mathbb{A}) \cap \mathbb{T}(F)))^\vee \\
\downarrow \scriptstyle \text{GLC}_{\mathbb{T}^\#}^{-1} \circ \iota^* \circ (\cdot)|_{\iota(\mathbb{T}^\#(\mathbb{A})/\mathbb{T}^\#(F))} \\
H^1(W_F, (Y^\#)^\vee) / \text{Ker GLC}_{\mathbb{T}^\#}
\end{array}$$

Note that some packets, or fibers, may be empty. By definition, the non-empty packets are described by the group $(\prod_v \mathcal{S}_{\widetilde{T}_v}) / \mathcal{S}_{\widetilde{\mathbb{T}}, F}$, as follows.

Proposition 13.7 (role of $\mathcal{S}_{\widetilde{\mathbb{T}}, F}$). In the global Langlands correspondence defined above, each non-empty packet Π is just an orbit under the Pontryagin dual group $\left((\prod_v \mathcal{S}_{\widetilde{T}_v}) / \mathcal{S}_{\widetilde{\mathbb{T}}, F} \right)^\vee$. Thus, Choosing a base point in the packet Π defines a one-to-one correspondence between Π and $\left((\prod_v \mathcal{S}_{\widetilde{T}_v}) / \mathcal{S}_{\widetilde{\mathbb{T}}, F} \right)^\vee$.

Therefore, a unitary character of $\prod_v \mathcal{S}_{\widetilde{T}_v}$ corresponds to a genuine unitary automorphic representation if and only if it is trivial on the global packet group $\mathcal{S}_{\widetilde{\mathbb{T}}, F}$. Our local result implies that $\mathcal{S}_{\widetilde{\mathbb{T}}, F}$ is a finite group:

Proposition 13.8. Let F be an algebraic number field, and $\mu_n \subset F^\times$ the cyclic subgroup of order n . Let $\mathbb{T}$ be an algebraic torus defined over F , and $\mu_n \rightarrow \widetilde{\mathbb{T}(\mathbb{A})} \rightarrow \mathbb{T}(\mathbb{A})$ a Brylinski-Deligne covering group. Then the global packet group

$$\mathcal{S}_{\widetilde{\mathbb{T}}, F} = (Z^\dagger(\mathbb{A}) \cap \mathbb{T}(F)) / (\iota(\mathbb{T}^\#(\mathbb{A})) \cap \mathbb{T}(F)).$$

has finite order.

Proof. The torus $\mathbb{T}$ splits over a finite Galois extension L/F . Let S_0 be the finite set consisting of the following three kinds of places of F : archimedean ones, ones dividing the integer n , and ones at which the ramification indices of L/F are prime to n . We may combine the local result proved in Corollary 12.3 (2) to obtain

$$Z^\dagger(\mathbb{A}_{S_0}) \subset \iota(\mathbb{T}^\#(\mathbb{A}_{S_0})) \cdot \prod_{v \notin S_0} \mathbb{T}(\mathcal{O}_v).$$

Since the class number of the torus $\mathbb{T}^\#$ is finite [11, Theorem 5.8], we find a finite set $S \supset S_0$ of places such that the equality $\mathbb{T}^\#(\mathbb{A}_S) = \mathbb{T}^\#(F) \cdot \prod_{v \notin S} \mathbb{T}^\#(\mathcal{O}_v)$ holds. Thus, $Z^\dagger(\mathbb{A}_S) \subset \iota(\mathbb{T}^\#(F)) \cdot \prod_{v \notin S} \mathbb{T}(\mathcal{O}_v)$, and therefore

$$\begin{aligned} \mathbb{T}(F) \cap Z^\dagger(\mathbb{A}) &\subset \mathbb{T}(F) \cap Z^\dagger(\mathbb{A}_S) \subset \iota(\mathbb{T}^\#(F)) \cdot \mathbb{T}(\mathcal{O}_S) \\ &\subset (\iota(\mathbb{T}^\#(\mathbb{A})) \cap \mathbb{T}(F)) \cdot \mathbb{T}(\mathcal{O}_S). \end{aligned}$$

In the quotients

$$(\mathbb{T}(F) \cap Z^\dagger(\mathbb{A})) / \iota(\mathbb{T}^\#(\mathcal{O}_S)) \subset (\iota(\mathbb{T}^\#(\mathbb{A})) \cap \mathbb{T}(F)) / \iota(\mathbb{T}^\#(\mathcal{O}_S)) \cdot \mathbb{T}(\mathcal{O}_S) / \iota(\mathbb{T}^\#(\mathcal{O}_S)),$$

the last factor $\mathbb{T}(\mathcal{O}_S) / \iota(\mathbb{T}^\#(\mathcal{O}_S))$ is finite, since the group $\mathbb{T}(\mathcal{O}_S)$ of S -integer points is finitely generated. Therefore, the global packet group

$$\mathcal{S}_{\widetilde{\mathbb{T}}, F} = (Z^\dagger(\mathbb{A}) \cap \mathbb{T}(F)) / (\iota(\mathbb{T}^\#(\mathbb{A})) \cap \mathbb{T}(F))$$

is also finite. $\square$

APPENDIX A. 2-COCYCLES ON THE CHEVALLEY GROUP SL_2

The structure of a Chevalley group allows us to construct a 2-cocycle. Let k be an infinite field, and $1 \rightarrow K_2(k) \rightarrow \widetilde{\mathrm{SL}_2(k)} \rightarrow \mathrm{SL}_2(k) \rightarrow 1$ the Matsumoto extension. For $t \in k$, we define two kinds of elements

$$\begin{aligned}\bar{x}(t) &= \overline{x_\alpha}(t) = \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}, \\ \overline{x_{-\alpha}}(t) &= \begin{pmatrix} 1 & \\ t & 1 \end{pmatrix}\end{aligned}$$

in SL_2 . Let $x(t) = x_\alpha(t)$ and $x_{-\alpha}(t)$ be the corresponding generators of $\widetilde{\mathrm{SL}_2(k)}$. For $\gamma \in \{\pm\alpha\}$, let

$$\begin{aligned}w_\gamma(t) &= x_\gamma(t)x_{-\gamma}(-t^{-1})x_\gamma(t) \\ h_\gamma(t) &= w_\gamma(t)w_\gamma(-1).\end{aligned}$$

For brevity, we write $w(t) = w_\alpha(t)$ and $h(t) = h_\alpha(t)$. Let $\langle, \rangle$ be the K_2 -symbol on the field k .

Proposition A.1 (Relations among $h(t)$ and $w(t')$). Any two elements $t, t' \in k^\times$ satisfies the following identities:

- (1) $h(t)h(t') = h(tt') \cdot \langle t, t' \rangle.$
- (2) $h(t)h(t')^{-1} = \langle -t, t' \rangle^{-1} \cdot h(tt'^{-1}).$
- (3) $h(t)w(t') = \langle t, t' \rangle \cdot w(tt').$
- (4) $w(t)w(t') = \langle -t, -t' \rangle^{-1} \cdot h(-tt'^{-1}).$

Proof. (1) is regarded as the definition of the K_2 -symbol $\langle, \rangle$.

(2) is verified as follows:

$$\begin{aligned}h(t)h(t')^{-1} &= h(t) \cdot h(t')^{-1}h(tt'^{-1})^{-1} \cdot h(tt'^{-1}) \\ &= h(t) \cdot (h(tt'^{-1})h(t'))^{-1} \cdot h(tt'^{-1}) \\ &= h(t)h(t)^{-1} \cdot \langle tt'^{-1}, t' \rangle^{-1} \cdot h(tt'^{-1}) \\ &= \langle -t, t' \rangle^{-1} \cdot h(tt'^{-1}).\end{aligned}$$

(3) is deduced by the equalities

$$\begin{aligned}h(t)w(t') &= h(t)h(t')w(1) = \langle t, t' \rangle h(tt')w(1) \\ &= \langle t, t' \rangle \cdot w(tt').\end{aligned}$$

(4) is derived as follows.

$$\begin{aligned}
w(t)w(t') &= w(t)h(t')w(1) \\
&= h(t') \cdot h(t')^{-1}w(t)h(t') \cdot w(1) \\
&= h(t')w(1)w(-1) \cdot w(t'^{-2}t)w(1) \\
&= h(t')w(-1)^{-1} \cdot w_{-\alpha}(-t'^{-2}t) \\
&= h(t')w(-1)^{-1} \cdot w_{\alpha}(t'^2t^{-1}) \\
&= h(t')w(-1)^{-1}w_{\alpha}(-t'^2t^{-1})^{-1} \\
&= h(t')h(-t'^2t^{-1})^{-1} \\
&= \langle -t', -t'^2t^{-1} \rangle^{-1} \cdot h(-t'^{-1}t) \\
&= \langle -t', -t^{-1} \rangle^{-1} \cdot h(-tt'^{-1}) \\
&= \langle -t, -t' \rangle^{-1} \cdot h(-tt'^{-1}).
\end{aligned}$$

□

First, we fill in the routine details that Moore omitted [10, Section 9], and correct his description of a 2-cocycle. We compute it according to five cases, as usual. We define a naive section $s: \mathrm{SL}_2(k) \rightarrow \widetilde{\mathrm{SL}_2(k)}$ by

$$\begin{aligned}
&\begin{pmatrix} t & ut^{-1} \\ & t^{-1} \end{pmatrix} \mapsto x(u)h(t) \\
\begin{pmatrix} -ut^{-1} & * \\ -t^{-1} & -vt^{-1} \end{pmatrix} &= \left(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ -t^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & v \\ & 1 \end{pmatrix} \right) \mapsto x(u)w(t)x(v),
\end{aligned}$$

due to Moore.

On the minimal Bruhat cell, we obtain the following.

$$\begin{aligned}
s \begin{pmatrix} t & ut^{-1} \\ & t^{-1} \end{pmatrix} s \begin{pmatrix} t' & u't'^{-1} \\ & t'^{-1} \end{pmatrix} &= x(u)h(t) \cdot x(u')h(t') \\
&= x(u) \cdot h(t)x(u')h(t)^{-1} \cdot h(t)h(t') \\
&= x(u) \cdot x(t^2u') \cdot h(tt')\langle t, t' \rangle \\
&= x(u + t^2u') \cdot h(tt')\langle t, t' \rangle \\
&= (s \circ \mathrm{pr}) \left(s \begin{pmatrix} t & ut^{-1} \\ & t^{-1} \end{pmatrix} s \begin{pmatrix} t' & u't'^{-1} \\ & t'^{-1} \end{pmatrix} \right) \cdot \langle t, t' \rangle \\
&= s \left(\begin{pmatrix} t & ut^{-1} \\ & t^{-1} \end{pmatrix} \begin{pmatrix} t' & u't'^{-1} \\ & t'^{-1} \end{pmatrix} \right) \cdot \langle t, t' \rangle.
\end{aligned}$$

Between the minimal and the maximal Bruhat cells, we have the following.

$$\begin{aligned}
& s \begin{pmatrix} t & ut^{-1} \\ & t^{-1} \end{pmatrix} s \begin{pmatrix} -u't'^{-1} & * \\ -t'^{-1} & -v't'^{-1} \end{pmatrix} \\
&= s \left(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \right) \cdot s \left(\begin{pmatrix} 1 & u' \\ & 1 \end{pmatrix} \begin{pmatrix} -t'^{-1} & t' \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & v' \\ & 1 \end{pmatrix} \right) \\
&= x(u)h(t) \cdot x(u')w(t')x(v') \\
&= x(u)x(t^2u') \cdot h(t)w(t') \cdot x(v') \\
&= x(u + t^2u') \cdot w(tt')\langle t, t' \rangle \cdot x(v') \\
&= s \left(\begin{pmatrix} t & ut^{-1} \\ & t^{-1} \end{pmatrix} \begin{pmatrix} -u't'^{-1} & * \\ -t'^{-1} & -v't'^{-1} \end{pmatrix} \right) \cdot \langle t, t' \rangle.
\end{aligned}$$

The reversed case is described as follows.

$$\begin{aligned}
& s \begin{pmatrix} -ut^{-1} & * \\ -t^{-1} & -vt^{-1} \end{pmatrix} s \begin{pmatrix} t' & u't'^{-1} \\ & t'^{-1} \end{pmatrix} \\
&= x(u)w(t)x(v) \cdot x(u')h(t') \\
&= x(u) \cdot w(t)x(v + u')h(t') \\
&= x(u)h(t') \cdot h(t')^{-1}w(t)h(t') \cdot h(t')^{-1}x(v + u')h(t') \\
&= x(u) \cdot h(t')w(t'^{-2}t) \cdot x(t'^{-2}(v + u')) \\
&= x(u) \cdot w(t'^{-1}t)\langle t', t'^{-2}t \rangle \cdot x(t'^{-2}(v + u')) \\
&= x(u) \cdot w(t'^{-1}t)x(t'^{-2}(v + u')) \cdot \langle t', -t'^{-1} \rangle^2 \langle t', t \rangle \\
&= s \left(\begin{pmatrix} -ut^{-1} & * \\ -t^{-1} & -vt^{-1} \end{pmatrix} \begin{pmatrix} t' & u't'^{-1} \\ & t'^{-1} \end{pmatrix} \right) \langle t, t' \rangle^{-1}.
\end{aligned}$$

Assume the equality $v + u' = 0$. Then we see the following.

$$\begin{aligned}
& s \begin{pmatrix} -ut^{-1} & * \\ -t^{-1} & -vt^{-1} \end{pmatrix} s \begin{pmatrix} -u't'^{-1} & * \\ -t'^{-1} & -v't'^{-1} \end{pmatrix} \\
&= x(u)w(t)x(v) \cdot x(u')w(t')x(v') \\
&= x(u)w(t)w(t')x(v') \\
&= x(u) \cdot w(t)w(t')x(v')w(-t')w(t) \cdot w(t)w(t') \\
&= x(u) \cdot w_\alpha(t)x_{-\alpha}(-t'^{-2}v')w_\alpha(t) \cdot h(-tt'^{-1})\langle -t, -t' \rangle^{-1} \\
&= x(u) \cdot x_\alpha(t^2t'^{-2}v') \cdot h(-tt'^{-1})\langle -t, -t' \rangle^{-1} \\
&= x(u + t^2t'^{-2}v')h(-tt'^{-1})\langle -t, -t' \rangle^{-1} \\
&= s \left(\begin{pmatrix} -ut^{-1} & * \\ -t^{-1} & -vt^{-1} \end{pmatrix} \begin{pmatrix} -u't'^{-1} & * \\ -t'^{-1} & -v't'^{-1} \end{pmatrix} \right) \cdot \langle -t, -t' \rangle^{-1}.
\end{aligned}$$

Assume that $v + u' \neq 0$. Then we have the following.

$$\begin{aligned}
& s \begin{pmatrix} -ut^{-1} & * \\ -t^{-1} & -vt^{-1} \end{pmatrix} s \begin{pmatrix} -u't'^{-1} & * \\ -t'^{-1} & -v't'^{-1} \end{pmatrix} \\
&= x(u)w(t)x(v) \cdot x(u')w(t')x(v') \\
&= x(u) \cdot w(t)x(v+u')w(-t) \cdot w(t)w(t') \cdot x(v') \\
&= x(u) \cdot x_{-\alpha}(-t^{-2}(v+u')) \cdot w(t)w(t') \cdot x(v') \\
&= x(u)x_{\alpha}(-t^2(v+u')^{-1}) \\
&\quad \cdot x_{\alpha}(t^2(v+u')^{-1})x_{-\alpha}(-t^{-2}(v+u'))x_{\alpha}(t^2(v+u')^{-1}) \\
&\quad \cdot x_{\alpha}(-t^2(v+u')^{-1}) \cdot w(t)w(t')x(v') \\
&= x(u-t^2(v+u')^{-1}) \cdot w(t^2(v+u')^{-1}) \cdot w(t)w(t') \\
&\quad \cdot w(-t')w(-t)x_{\alpha}(-t^2(v+u')^{-1})w(t)w(t') \cdot x(v') \\
&= x(u-t^2(v+u')^{-1}) \cdot \langle -t^2(v+u')^{-1}, -t \rangle^{-1} h(-t(v+u')^{-1})w(t') \\
&\quad \cdot w_{\alpha}(-t')x_{-\alpha}((v+u')^{-1})w_{\alpha}(t') \cdot x(v') \\
&= x(u-t^2(v+u')^{-1}) \cdot \langle -t(v+u')^{-1}, -t \rangle^{-1} \langle -t(v+u')^{-1}, t' \rangle w(-tt'(v+u')^{-1}) \\
&\quad \cdot x_{\alpha}(-t'^2(v+u')^{-1})x(v') \\
&= x(u-t^2(v+u')^{-1}) \cdot \langle -t(v+u')^{-1}, -t^{-1}t' \rangle w(-tt'(v+u')^{-1}) \\
&\quad \cdot x_{\alpha}(-t'^2(v+u')^{-1}+v') \\
&= x(u-t^2(v+u')^{-1})w(-tt'(v+u')^{-1})x(v'-t'^2(v+u')^{-1}) \\
&\quad \cdot \langle -t(v+u')^{-1}, -t^{-1}t' \cdot t(v+u')^{-1} \rangle \\
&= s \left(\begin{pmatrix} -ut^{-1} & * \\ -t^{-1} & -vt^{-1} \end{pmatrix} \begin{pmatrix} -u't'^{-1} & * \\ -t'^{-1} & -v't'^{-1} \end{pmatrix} \right) \langle -t(v+u')^{-1}, -t'(v+u')^{-1} \rangle.
\end{aligned}$$

Therefore, the equation (1) in p.198 of Moore's paper [10] should be replaced with

$$b(g_2(u, t, v), g_2(u', t', v')) = b(tw^{-1}, w)^{-1}b(tw^{-1}, t'),$$

by removing the inverse on the second w .

To obtain Kubota's 2-cocycle, Matsumoto [9, n° 5] defined a modified section $s': \mathrm{SL}_2(k) \rightarrow \widehat{\mathrm{SL}_2(k)}$ by

$$\begin{aligned}
& \begin{pmatrix} t & ut^{-1} \\ & t^{-1} \end{pmatrix} \mapsto x(u)h(t) \cdot \langle t, -1 \rangle \\
& \begin{pmatrix} -ut^{-1} & * \\ -t^{-1} & -vt^{-1} \end{pmatrix} = \left(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} & t \\ -t^{-1} & \end{pmatrix} \begin{pmatrix} 1 & v \\ & 1 \end{pmatrix} \right) \mapsto x(u)w(t)x(v).
\end{aligned}$$

We verify that this section s' indeed yields Kubota's 2-cocycle, again, according to five cases.

On the minimal Bruhat cell, we see the following.

$$\begin{aligned}
s' \begin{pmatrix} t & ut^{-1} \\ & t^{-1} \end{pmatrix} s' \begin{pmatrix} t' & u't'^{-1} \\ & t'^{-1} \end{pmatrix} &= x(u)h(t)\langle t, -1 \rangle \cdot x(u')h(t')\langle t', -1 \rangle \\
&= x(u + t^2u') \cdot h(tt')\langle tt', -1 \rangle \langle t, t' \rangle \\
&= s' \left(\begin{pmatrix} t & ut^{-1} \\ & t^{-1} \end{pmatrix} \begin{pmatrix} t' & u't'^{-1} \\ & t'^{-1} \end{pmatrix} \right) \cdot \langle t, t' \rangle.
\end{aligned}$$

Between the minimal and the maximal Bruhat cells, we have the following.

$$\begin{aligned}
s' \begin{pmatrix} t & ut^{-1} \\ & t^{-1} \end{pmatrix} s' \begin{pmatrix} -u't'^{-1} & * \\ -t'^{-1} & -v't'^{-1} \end{pmatrix} \\
&= x(u)h(t)\langle t, -1 \rangle \cdot x(u')w(t')x(v') \\
&= x(u + t^2u') \cdot w(tt')\langle t, -t' \rangle \cdot x(v') \\
&= s' \left(\begin{pmatrix} t & ut^{-1} \\ & t^{-1} \end{pmatrix} \begin{pmatrix} -u't'^{-1} & * \\ -t'^{-1} & -v't'^{-1} \end{pmatrix} \right) \cdot \langle t, -t' \rangle.
\end{aligned}$$

The reversed case is described as follows.

$$\begin{aligned}
s' \begin{pmatrix} -ut^{-1} & * \\ -t^{-1} & -vt^{-1} \end{pmatrix} s' \begin{pmatrix} t' & u't'^{-1} \\ & t'^{-1} \end{pmatrix} \\
&= x(u)w(t)x(v) \cdot x(u')h(t')\langle t', -1 \rangle \\
&= x(u) \cdot w(t'^{-1}t)x(t'^{-2}(v + u')) \cdot \langle t', -t \rangle \\
&= s' \left(\begin{pmatrix} -ut^{-1} & * \\ -t^{-1} & -vt^{-1} \end{pmatrix} \begin{pmatrix} t' & u't'^{-1} \\ & t'^{-1} \end{pmatrix} \right) \langle -t, t' \rangle^{-1}.
\end{aligned}$$

Assume the equality $v + u' = 0$. Then we obtain the following.

$$\begin{aligned}
s' \begin{pmatrix} -ut^{-1} & * \\ -t^{-1} & -vt^{-1} \end{pmatrix} s' \begin{pmatrix} -u't'^{-1} & * \\ -t'^{-1} & -v't'^{-1} \end{pmatrix} \\
&= x(u)w(t)x(v) \cdot x(u')w(t')x(v') \\
&= x(u + t^2t'^{-2}v')h(-tt'^{-1})\langle -tt'^{-1}, -1 \rangle \langle t, t' \rangle^{-1} \\
&= s' \left(\begin{pmatrix} -ut^{-1} & * \\ -t^{-1} & -vt^{-1} \end{pmatrix} \begin{pmatrix} -u't'^{-1} & * \\ -t'^{-1} & -v't'^{-1} \end{pmatrix} \right) \cdot \langle t, t' \rangle^{-1}.
\end{aligned}$$

Assume that $v + u' \neq 0$. Then we have the following.

$$\begin{aligned}
& s' \begin{pmatrix} -ut^{-1} & * \\ -t^{-1} & -vt^{-1} \end{pmatrix} s' \begin{pmatrix} -u't'^{-1} & * \\ -t'^{-1} & -v't'^{-1} \end{pmatrix} \\
&= x(u)w(t)x(v) \cdot x(u')w(t')x(v') \\
&= x(u - t^2(v + u')^{-1})w(-tt'(v + u')^{-1})x(v' - t'^2(v + u')^{-1}) \\
&\quad \cdot \langle -t(v + u')^{-1}, -t^{-1}t' \cdot t(v + u')^{-1} \rangle \\
&= s' \left(\begin{pmatrix} -ut^{-1} & * \\ -t^{-1} & -vt^{-1} \end{pmatrix} \begin{pmatrix} -u't'^{-1} & * \\ -t'^{-1} & -v't'^{-1} \end{pmatrix} \right) \langle -t(v + u')^{-1}, -t'(v + u')^{-1} \rangle.
\end{aligned}$$

ACKNOWLEDGEMENTS

The author would like to thank his advisor Professor Tamotsu Ikeda for fruitful discussions, and thank Miyu Suzuki for providing me with valuable information. This work was supported by JSPS KAKENHI Grant Number JP23KJ1298.

REFERENCES

- [1] *Théorie des topos et cohomologie étale des schémas. Tome 3.* Lecture Notes in Mathematics, Vol. 305. Springer-Verlag, Berlin-New York, 1973. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat.
- [2] J.-L. Brylinski and P. Deligne. Central extensions of reductive groups by $\mathbf{K}_2$. *Publ. Math. Inst. Hautes Études Sci.*, (94):5–85, 2001.
- [3] D. Bump and D. Ginzburg. Symmetric square L -functions on $\mathrm{GL}(r)$. *Ann. of Math. (2)*, 136(1):137–205, 1992.
- [4] J. Dixmier. Quelques propriétés des groupes abéliens localement compacts. *Bull. Sci. Math. (2)*, 81:38–48, 1957.
- [5] W. T. Gan and F. Gao. The Langlands-Weissman program for Brylinski-Deligne extensions. In *L-groups and the Langlands program for covering groups*, number 398 in Astérisque, pages 187–275. 2018.
- [6] D. Harari. *Galois cohomology and class field theory*. Universitext. Springer, Cham, 2020. Translated from the 2017 French original by Andrei Yafaev.
- [7] J.-P. Labesse. Cohomologie, L -groupes et fonctorialité. *Compositio Math.*, 55(2):163–184, 1985.
- [8] S. Mac Lane and I. Moerdijk. *Sheaves in geometry and logic*. Universitext. Springer-Verlag, New York, 1994. A first introduction to topos theory, Corrected reprint of the 1992 edition.
- [9] H. Matsumoto. Sur les sous-groupes arithmétiques des groupes semi-simples déployés. *Ann. Sci. École Norm. Sup. (4)*, 2:1–62, 1969.
- [10] C. C. Moore. Group extensions of p -adic and adelic linear groups. *Inst. Hautes Études Sci. Publ. Math.*, (35):157–222, 1968.
- [11] V. Platonov, A. Rapinchuk, and I. Rapinchuk. *Algebraic groups and number theory. Vol. I*, volume 205 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2023.

- [12] D. Quillen. Higher algebraic K -theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, volume Vol. 341 of *Lecture Notes in Math.*, pages 85–147. Springer, Berlin-New York, 1973.
- [13] J.-P. Serre. *Galois cohomology*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, english edition, 2002. Translated from the French by Patrick Ion and revised by the author.
- [14] C. A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.
- [15] C. A. Weibel. *The K-book*, volume 145 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2013. An introduction to algebraic K -theory.
- [16] A. Weil. Sur la formule de Siegel dans la théorie des groupes classiques. *Acta Math.*, 113:1–87, 1965.
- [17] M. H. Weissman. Metaplectic tori over local fields. *Pacific J. Math.*, 241(1):169–200, 2009.
- [18] M. H. Weissman. Covers of tori over local and global fields. *Amer. J. Math.*, 138(6):1533–1573, 2016.
- [19] M. H. Weissman. L-groups and parameters for covering groups. Number 398, pages 33–186. 2018. L-groups and the Langlands program for covering groups.

DEPARTMENT OF MATHEMATICS, KYOTO UNIVERSITY, KYOTO 606-8502,
JAPAN