

Trace formulae and principal functions of Hilbert space operators

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This paper is the results of [13], [14] and [15]. Let $\mathcal{H}$ be a complex separable Hilbert space and $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. About the trace formula, we have the following:

Theorem 1 (M. Krein, 1953). *Let A be a self-adjoint operator on $\mathcal{H}$ and K be a trace class self-adjoint operator on $\mathcal{H}$. Then there exists a unique function $\delta(t)$ such that*

$$\mathrm{Tr} \left(p(A + K) - p(A) \right) = \int p'(t) \delta(t) dt,$$

where p is a polynomial.

Let $\mathcal{C}_1$ be the trace class and $\mathcal{A}$ be the set of all Laurent polynomials; $\mathcal{P}(r, z) = \sum_{k=-N}^N p_k(r) z^k$. Let $J(\mathcal{P}, \mathcal{Q})$ be the Jacobian of $\mathcal{P}, \mathcal{Q}$.

Theorem 2 (Carey-Pincus [8], Helton-Howe [19]). *Let $T = X + iY$ be an operator on $\mathcal{H}$ with trace class self-commutator $([T^*, T] \in \mathcal{C}_1)$. Then there exists a function $g(x, y)$ such that*

$$\mathrm{Tr} \left([p(X, Y), q(X, Y)] \right) = \frac{1}{2\pi i} \int \int J(p, q)(x, y) g(x, y) dx dy,$$

where p and q are polynomials of two variables.

Functions $\delta(t)$ and $g(x, y)$ in Theorems 1 and 2 are called the phase shift of the perturbation problem $A \rightarrow A + K$, and the (Cartesian) principal function of T , respectively. Let T be hyponormal and satisfy $[T^*, T] \in \mathcal{C}_1$. For operators A and K of Theorem 1, let $A = TT^*$ and $K = T^*T - TT^* (= [T^*, T] \in \mathcal{C}_1)$. Then Theorem 1 is

$$\mathrm{Tr} \left(p(T^*T) - p(TT^*) \right) = \int p'(t) \delta(t) dt.$$

And

$$\delta(t) = \frac{1}{2\pi} \int_0^{2\pi} g(\sqrt{t} \cos \theta, \sqrt{t} \sin \theta) d\theta \quad \text{a.e. } t > 0.$$

Let $\mathcal{A}$ be the linear space of all Laurent polynomials $\mathcal{P}(r, z)$ with polynomial coefficients such that $\mathcal{P}(r, z) = \sum_{k=-N}^N p_k(r) z^k$, where N is a non-negative integer and every $p_k(r)$ is a polynomial of one variable. For $T = U|T|$ with unitary U , put $\mathcal{P}(|T|, U) = \sum_{k=-N}^N p_k(|T|) U^k$.

For the polar decomposition $T = U|T|$, we have the following:

Theorem 3 ([8],[11],[25]). *Let $T = U|T|$ be semi-hyponormal operator satisfying $[|T|, U] \in \mathcal{C}_1$ with unitary U . Then there exists a function g_T such that, for $\mathcal{P}, \mathcal{Q} \in \mathcal{A}$,*

$$\text{Tr}(\mathcal{P}(|T|, U), \mathcal{Q}(|T|, U)) = \frac{1}{2\pi} \int \int J(\mathcal{P}, \mathcal{Q})(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r) dr d\theta.$$

Definition 1. T is p -hyponormal if $(T^*T)^p \geq (TT^*)^p$. Especially, T is called hyponormal and semi-hyponormal if $p = 1$ and $p = 1/2$, respectively. It holds

$$\text{hyponormal} \implies \text{semi-hyponormal} \implies p\text{-hyponormal}.$$

(1) If $T = U|T|$ is semi-hyponormal, then $S = U|T|^{\frac{1}{2}}$ is hyponormal.

(2) If $T = U|T|$ is semi-hyponormal with $1 \notin \sigma(U)$, then $R = L^{-1}(U) + i|T|$ is hyponormal, where $L^{-1}(U) = i(U + 1)(U - 1)^{-1}$.

(3) If $T = U|T|$ is p -hyponormal, then $T_t = |T|^t U|T|^{1-t}$ is q -hyponormal (Aluthge transformation), where $q = \min\{p + t, p + 1 - t, 1\}$. Hence, if T is semi-hyponormal, then T_t is hyponormal.

Spectral mapping theorem holds for these transformations; i.e., if $T = U|T|$ be semi-hyponormal, then

- (1) $\sigma(S) = \{\sqrt{r}e^{i\theta} : re^{i\theta} \in \sigma(T)\};$
- (2) $\sigma(R) = \{L^{-1}(e^{i\theta}) + ir : re^{i\theta} \in \sigma(T)\};$
- (3) $\sigma(T_t) = \sigma(T).$

Functions g and g_T of Theorems 2 and 3 are called *the principal functions* of T related to the Cartesian decomposition $T = X + iY$ and the polar decomposition $T = U|T|$, respectively. For a hyponormal operator $T = X + iY$, the principal function of T is defined by the mosaic $0 \sqsubset B(x, y) \sqsubset I$ as follows;

$$g(x, y) = \text{Tr}(B(x, y)).$$

For a semi-hyponormal operator $T = U|T|$, the principal function of T is defined by the mosaic $0 \leq B^P(e^{i\theta}, r) \leq I$ as follows;

$$g_T(e^{i\theta}, r) = \text{Tr}(B^P(e^{i\theta}, r)).$$

For an operator $T = U|T|$ let $T_t = |T|^t U |T|^{1-t}$ ($0 < t < 1$) be the Aluthge transformation of T . Let g_T and g_{T_t} be the principal functions of T and $T_t = |T|^t U |T|^{1-t}$ ($0 < t < 1$), respectively. Then we have following results ([12]):

- (1) If T is invertible p -hyponormal with $[|T|, U] \in \mathcal{C}_1$, then $g_T = g_{T_t}$.
- (2) If T is hyponormal with $[|T|, U] \in \mathcal{C}_1$, then $g(x, y) = g_T(e^{i\theta}, r)$ for $x + iy = re^{i\theta}$.

The following results are important.

Theorem 4 ([12]). *If a positive invertible operator A and an operator D satisfy $[A, D] \in \mathcal{C}_1$, then, for any real number α , we have*

$$[A^\alpha, D] \in \mathcal{C}_1.$$

Theorem 5 ([12]). *If T is an invertible operator such that $[T^*, T] \in \mathcal{C}_1$, then $[\tilde{T}^*, \tilde{T}] \in \mathcal{C}_1$, where $\tilde{T} = |T|^{1/2} U |T|^{1/2}$ (Aluthge transform of T).*

For an invertible operator $T = U|T|$, $[T^*, T] \in \mathcal{C}_1$ if and only if $[|T|, U] \in \mathcal{C}_1$, because

$$[|T|^2, U] = [T^*, T]U \quad \text{and} \quad [T^*, T] = |T|[[|T|, U]U^* + [|T|, U]|T|U^*].$$

1. Trace formula of p -hyponormal operators II

Let $\mathbf{T} = \{e^{i\theta} | 0 \leq \theta < 2\pi\}$, Σ be the set of all Borel sets in $\mathbf{T}$ and m be a measure on the measure space $(\mathbf{T}, \Sigma)$ such that $dm(\theta) = \frac{1}{2\pi} d\theta$. Then we have

Theorem 3'. *Let $T \in B(\mathcal{H})$ be semi-hyponormal and $T = U|T|$ be the polar decomposition of T . Assume that U is unitary and $[U, |T|] \in \mathcal{C}_1$. Then there exists a summable function g_T such that, for $\mathcal{P}(r, z), \mathcal{Q}(r, z) \in \mathcal{A}$, it holds*

$$\text{Tr} \left([\mathcal{P}(|T|, U), \mathcal{Q}(|T|, U)] \right) = \int \int J(\mathcal{P}, \mathcal{Q})(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r) dr dm(\theta).$$

We denote by $C^\infty(\mathbf{R})$ the set of all smooth functions on $\mathbf{R}$ and by $C_c^\infty(\mathbf{R})$ the set of all functions in $C^\infty(\mathbf{R})$ with compact support. We denote by $\mathcal{B}$ the linear space of all

Laurent polynomials $\phi(r, z)$ such that $\phi(r, z) = \sum_{k=-N}^N f_k(r) z^k$, where every $f_k \in C^\infty(\mathbf{R})$.

For $T = U|T|$ with unitary U , put $\phi(|T|, U) = \sum_{k=-N}^N f_k(|T|) U^k$ for $\phi \in \mathcal{B}$.

In [6], Carey and Pincus proved a more general version of Theorem 3. It requires complicate calculations. Using polynomial approximation, we improve Theorem 3 in the following form.

Theorem 1.1. *Let $T \in B(\mathcal{H})$ be semi-hyponormal and $T = U|T|$ be the polar decomposition of T . Assume that U is unitary and $[U, |T|] \in \mathcal{C}_1$. Then, for $\phi, \psi \in \mathcal{B}$, it holds*

$$\text{Tr} \left([\phi(|T|, U), \psi(|T|, U)] \right) = \int \int J(\phi, \psi)(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r) dr d\mu(\theta).$$

For the proof of Theorem 1.1, it needs Theorems 1.2 - 1.5.

Theorem 1.2. *Let $A, \{B_j\}_{j=1, \dots, n}$ be operators such that $[A, B_j] \in \mathcal{C}_1$ and $\|B_j\| \leq r$ for all j ($j = 1, 2, \dots, n$). Then*

$$\|[A, B_1 B_2 \cdots B_n]\|_1 \leq nr^{n-1} \max_j \|[A, B_j]\|_1.$$

Let $T = U|T| \in B(\mathcal{H})$ be the polar decomposition of T . Assume that U is unitary and $[U, |T|] \in \mathcal{C}_1$. Then we have

$$[U, e^{it|T|}] = \sum_{n=1}^{\infty} \frac{(it)^n}{n!} [U, |T|^n].$$

By Theorem 1.2,

$$\|[U, e^{it|T|}]\|_1 \leq \sum_{n=1}^{\infty} \frac{|t|^n n \| |T|^{n-1} \|_1}{n!} \|[U, |T|]\|_1 \leq |t| \cdot \|[U, |T|]\|_1 e^{|t| \cdot \| |T| \|_1}.$$

Definition 1.1. Under the assumption above, we define a constant c_T of an operator $T = U|T|$ satisfying $[U, |T|] \in \mathcal{C}_1$ by

$$c_T = \max_{|t| \leq 1} \|[U, e^{it|T|}]\|_1.$$

Proof of the next theorem is based on an idea of the proof of [19, Lemma 3.2].

Theorem 1.3. *Let $T = U|T| \in B(\mathcal{H})$ be the polar decomposition of T . Assume that U is unitary and $[U, |T|] \in \mathcal{C}_1$. Then, for $f \in \mathcal{S}$ and an integer n , it holds*

$$\|[U^n, f(|T|)]\|_1 \leq |n| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c_T (|t| + 1) |\hat{f}(t)| dt,$$

where c_T is the constant of Definition 1.1.

Theorem 1.4. *Let F be a compact set of $\mathbf{R}$ and $f \in C^\infty(\mathbf{R})$. Then there exist a function $f_1 \in C_c^\infty(\mathbf{R})$, a sequence $\{p_n\}$ of polynomials and a sequence $\{\gamma_n\}$ in $C_c^\infty(\mathbf{R})$ such that*

$$f(x) = f_1(x), \quad p_n(x) = \gamma_n(x) \quad \text{for } x \in F,$$

$$\begin{aligned}\sup_{y \in F} |f_1(y) - \gamma_n(y)| &\rightarrow 0 \quad (n \rightarrow \infty), \\ \sup_{y \in F} |f_1^{(1)}(y) - \gamma_n^{(1)}(y)| &\rightarrow 0 \quad (n \rightarrow \infty), \\ \sup_{t \in \mathbb{R}} |\hat{f}_1(t) - \hat{\gamma}_n(t)| &\rightarrow 0 \quad (n \rightarrow \infty),\end{aligned}$$

and

$$\sup_{t \in \mathbb{R}} |t|^3 \cdot |\hat{f}_1(t) - \hat{\gamma}_n(t)| \rightarrow 0 \quad (n \rightarrow \infty).$$

Theorem 1.5. Let $T = U|T|$ be the polar decomposition. Assume that U is unitary and $[U, |T|] \in \mathcal{C}_1$. Then, for $f, g \in \mathcal{S}$ and integers m, n , it holds

$$\begin{aligned}& ||[f(|T|)U^n, g(|T|)U^m]|| \\ & \leq |n| \cdot ||f|| \frac{1}{2\pi} \int_{-\infty}^{\infty} c_T(|t|+1) |\hat{g}(t)| dt \\ & + |m| \cdot ||g|| \frac{1}{2\pi} \int_{-\infty}^{\infty} c_T(|t|+1) |\hat{f}(t)| dt\end{aligned}$$

where c_T is the constant of Definition 1.1 and $||h|| = \sup_{x \in \sigma(|T|)} |h(x)|$.

Next we apply this result to p -hyponormal operators ($0 < p < 1/2$).

Definition 1.2. Let $T = U|T|$ be p -hyponormal with unitary U . Put $S = U|T|^{2p}$. Then S is semi-hyponormal with unitary U . Hence there exists the Pincus principal function g_S of S and we define the principal function g_T of T by

$$g_T(e^{i\theta}, r) = g_S(e^{i\theta}, r^{\frac{1}{2p}})$$

(see [11, Definition 3]).

The following theorem is a generalization of Theorem 10 of [11].

Theorem 1.6. Let $T = U|T|$ be an invertible p -hyponormal operator. If $|T|^{2p} - U|T|^{2p}U^* \in \mathcal{C}_1$, then for $\mathcal{P}(r, z), \mathcal{Q}(r, z) \in \mathcal{A}$ it holds

$$\text{Tr} \left([\mathcal{P}(|T|, U), \mathcal{Q}(|T|, U)] \right) = \int \int J(\mathcal{P}, \mathcal{Q})(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r) dr dm(\theta).$$

2. Trace formulae associated with the polar decomposition of operators

Let $\mathcal{S}(\mathbb{R}^2)$ be the Schwartz space of rapidly decreasing functions at infinity. For $T = X + iY$, let $\mathcal{E}$ and $\mathcal{F}$ be the spectral measures of self-adjoint operators X and Y , respectively. We define τ on $\mathcal{S}(\mathbb{R}^2)$ by

$$(*) \quad \tau(\phi) = \int \int \phi(x, y) d\mathcal{E}(x) d\mathcal{F}(y) \quad (\phi \in \mathcal{S}(\mathbb{R}^2)).$$

By a standard argument, we have

$$\int \int e^{itX} e^{isY} \hat{\phi}(t, s) dt ds = \int \int \phi(x, y) d\mathcal{E}(x) d\mathcal{F}(y),$$

where

$$\hat{\phi}(t, s) = \frac{1}{2\pi} \int \int e^{-i(tx+sy)} \phi(x, y) dx dy$$

is the Fourier transform of the function ϕ (see, for example, [22, p.237]).

Put $\nu(E) = \int \int_E \hat{\phi}(t, s) dt ds$ for a measurable set $E \subset \mathbf{R}^2$. Since $\hat{\phi}(t, s) \in \mathcal{S}(\mathbf{R}^2)$, we have

$$\int \int (1 + |t|)(1 + |s|) |\hat{\phi}(t, s)| dt ds < \infty.$$

Following Carey-Pincus [8], put $G(x, y) = \int \int e^{itx+isy} d\nu(t, s)$ and define

$$G(X, Y) = \int \int G(x, y) d\mathcal{E}(x) d\mathcal{F}(y).$$

Then

$$\tau(\phi) = \int \int e^{itX} e^{isY} \nu(t, s) dt ds = G(X, Y).$$

Note here that we have $\tau(\psi) = \tau(\phi)$ for any smooth function $\psi(x, y)$ which coincides with $\phi(x, y)$ on $\text{supp}(\tau)$.

The map $\tau : \mathcal{S}(\mathbf{R}^2) \rightarrow B(\mathcal{H})$ has the following properties [22, Chapter X, §2];

- (1) τ is linear, continuous and $\text{supp}(\tau) \subseteq \sigma(X) \times \sigma(Y)$,
- (2) $\tau(1) = I, \tau(p + q) = p(X) + q(Y)$ for polynomials p and q of one variable of x and y , respectively.
- (3) $\tau(\phi)\tau(\psi) - \tau(\phi\psi) \in \mathcal{C}_1$ for $\phi, \psi \in \mathcal{S}(\mathbf{R}^2)$,
- (4) $\tau(\phi)^* - \tau(\bar{\phi}) \in \mathcal{C}_1$.

By (3) we have an important property $[\tau(\phi), \tau(\psi)] \in \mathcal{C}_1$ for $\phi, \psi \in \mathcal{S}(\mathbf{R}^2)$.

The following theorem is a basis.

Theorem 6 (Carey-Pincus, [8, Theorem 5.1]). *Let $T = X + iY$ be an operator with $[T^*, T] \in \mathcal{C}_1$. Let $\mathcal{E}, \mathcal{F}$ be the spectral measures of X and Y , respectively and τ be given by (*). Then there exists a summable function g such that, for $\phi, \psi \in \mathcal{S}(\mathbf{R}^2)$,*

$$\text{Tr}([\tau(\phi), \tau(\psi)]) = \frac{1}{2\pi i} \int \int J(\phi, \psi)(x, y) g(x, y) dx dy.$$

Moreover, if T is hyponormal, then $g \geq 0$ and $g(x, y) = 0$ for $x + iy \notin \sigma(T)$.

In this section, the main theorem is Theorem 2.7. We prepare some results.

Lemma 2.1. Let A be a positive invertible operator and operators D, E, F satisfying $[A, D], [E, D], [F, D] \in \mathcal{C}_1$. Then for any real number α , we have

$$[EA^\alpha F, D] \in \mathcal{C}_1.$$

Lemma 2.2 (cf. p.158 of [8]). Let $T = X + iY$ be an invertible operator such that $[T^*, T] \in \mathcal{C}_1$. Let $\psi \in \mathcal{S}(\mathbb{R}^2)$, $D = \tau(\psi)$ and operators E, F satisfy $[E, D], [F, D] \in \mathcal{C}_1$. Then, for $\phi(x, y) = (x^2 + y^2)^\alpha$ with a real number α ,

$$\text{Tr}([E\tau(\phi)F, D]) = \text{Tr}([E|T|^{2\alpha}F, D]).$$

Theorem 2.3. Let $T = U|T|$ be an invertible operator with $[T^*, T] \in \mathcal{C}_1$ and let g be the principal function associated with the Cartesian decomposition of $T = X + iY$. Then there exists a summable function g_T such that, for $\mathcal{P}, \mathcal{Q} \in \mathcal{A}$,

$$\text{Tr}([\mathcal{P}(|T|, U), \mathcal{Q}(|T|, U)]) = \frac{1}{2\pi} \int \int J(\mathcal{P}, \mathcal{Q})(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r) dr d\theta,$$

and $g_T(e^{i\theta}, r) = g(x, y)$ almost everywhere $x + iy = re^{i\theta}$ on $\mathbb{C}$.

An invertible operator T is said to be log-hyponormal if $\log T^*T \geq \log TT^*$ [17].

For the proof of the next result, we need the following two lemmas. For an operator T , let $\sigma_{ap}(T)$ and $\sigma_p(T)$ be the approximate point spectrum and the point spectrum of T , respectively. The following lemmas are important for the main theorem.

Lemma 2.4. Let $T = U|T|$ be an invertible semi-hyponormal operator with $[|T|, U] \in \mathcal{C}_1$. Then the principal function g^P associated with the polar decomposition $T = U|T|$ of T satisfies $g_T(e^{i\theta}, r) = 0$ for $re^{i\theta} \notin \sigma(T)$.

Lemma 2.5. Let $T = U|T|$ be an operator with unitary U and put $S = U(|T| + I)$. If $z \in \partial\sigma(S)$, then $|z| \geq 1$. Therefore, if $z \in \sigma(S)$, then $|z| \geq 1$.

By the above lemmas, for the next theorem we can give another proof of [11, Theorem 9].

Theorem 2.6. Let $T = U|T|$ be a semi-hyponormal operator with unitary U and $[|T|, U] \in \mathcal{C}_1$. Then there exists a summable function g_T such that, for $\mathcal{P}, \mathcal{Q} \in \mathcal{A}$,

$$\text{Tr}([\mathcal{P}(|T|, U), \mathcal{Q}(|T|, U)]) = \frac{1}{2\pi} \int \int J(\mathcal{P}, \mathcal{Q})(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r) dr d\theta.$$

Theorem 2.7. Let $T = X + iY = U|T|$ be a semi-hyponormal operator with unitary U and $[|T|, U] \in \mathcal{C}_1$. If g and g_T are the principal function associated with the Cartesian decomposition of T and the summable function in Theorem 2.6, respectively, then

$$g(x, y) = g_T(e^{i\theta}, r)$$

almost everywhere $x + iy = re^{i\theta}$ on $\mathbb{C}$.

3. Principal functions for high powers of operators

In this section, we denote g_T and g_T^P be the principal functions of the Cartesian decomposition and the polar decomposition of T , respectively. C.A. Berger gave the principal functions g_{T^n} of powers T^n of T in terms of g_T and proved that for a sufficiently high n , T^n has a non-trivial invariant subspace for a hyponormal operator T ([5]). For a hyponormal operator T with $[T^*, T] \in \mathcal{C}_1$, it holds that

$$g_{T^n}(z) = \sum_{k=1}^n g_T(\zeta_k), \quad \text{where } \zeta_k^n = z \ (k = 1, \dots, n).$$

More generally, in [4], for a polynomial p , it holds

$$g_{p(T)}(z) = \sum_{\zeta} \{ g_T(\zeta) : p(\zeta) = z \}.$$

Theorem 3.1. *Let $T = X + iY = U|T|$ be an operator satisfying the following trace formula:*

$$\text{Tr}([\phi(|T|, U), \psi(|T|, U)]) = \frac{1}{2\pi} \int \int J(\phi, \psi)(r, e^{i\theta}) e^{i\theta} g_T^P(e^{i\theta}, r) dr d\theta$$

for any Laurent polynomials ϕ and ψ . Then the principal function $g_T(x, y)$ related to the Cartesian decomposition $T = X + iY$ of T exists and it is given by $g_T(x, y) = g_T^P(e^{i\theta}, r)$, where $x + iy = re^{i\theta}$.

If an operator $T = U|T|$ is invertible and $[|T|, U] \in \mathcal{C}_1$, then $[T^*, T] \in \mathcal{C}_1$. So we have the following

Corollary 3.2. *If an invertible operator $T = X + iY = U|T|$ satisfies $[|T|, U] \in \mathcal{C}_1$, then $g_T(x, y) = g_T^P(e^{i\theta}, r)$, where $x + iy = re^{i\theta}$.*

For a relation between g_T^P and $g_{T^n}^P$, we need the following Berger's result:

Theorem 3.3 (Berger, [5, Theorem 4]). *For an operator T , if $[T^*, T] \in \mathcal{C}_1$, then for a positive integer n ,*

$$g_{T^n}(x, y) = \sum_{(u+iv)^n = x+iy} g_T(u, v).$$

Theorem 3.4. *For an operator T with $[T^*, T] \in \mathcal{C}_1$, if $\int \int g(x, y) dx dy \neq 0$, then*

$$\lim_{n \rightarrow \infty} \text{ess sup } |g_{T^n}| = \infty.$$

Applying Corollary 3.2 and Theorem 3.4 to T^n , we have the following.

Corollary 3.5. *For an operator $T = U|T|$, let $T^n = U_n|T^n|$ be the polar decomposition of T^n ($n = 1, 2, \dots$). If $[|T^n|, U_n] \in \mathcal{C}_1$ for every non-negative integer n and*

$\int \int g_T^P(e^{i\theta}, r) r d\theta dr \neq 0$, then

$$\lim_{n \rightarrow \infty} \operatorname{ess\,sup} |g_{T^n}^P| = \infty.$$

Let $T = U|T|$ and $T^n = U_n|T^n|$ be the polar decompositions of T and T^n , respectively. Then it holds that if $[T^*, T] \in \mathcal{C}_1$, then $[T^{*n}, T^n] \in \mathcal{C}_1$ for any positive integer n . On the other hand, in the polar decomposition case, it is not clear whether $[|T|, U] \in \mathcal{C}_1$ implies $[|T^n|, U_n] \in \mathcal{C}_1$ even if $n = 2$. If T is invertible and $[|T|, U] \in \mathcal{C}_1$, then, for every n , it holds $[|T^n|, U_n] \in \mathcal{C}_1$ by [11, Theorem 3].

Next we consider operators with cyclic vectors. First we need the following result.

Theorem 3.6 (Martin and Putinar, Th.X.4.3 [22]). *Let g_T and g_V be the principal functions of operators T and V such that $[T^*, T], [V^*, V] \in \mathcal{C}_1$, respectively. If there exists an operator $A \in \mathcal{C}_1$ such that $AV = TA$ and $\ker(A) = \ker(A^*) = \{0\}$, then $g_T \sqsubset g_V$.*

Proof of the following lemma is based on it of [22, Corollary X.4.4].

Lemma 3.7. *Let T be an operator such that $[T^*, T] \in \mathcal{C}_1$ and $\sigma(T)$ is an infinite set. If T has a cyclic vector, then $g_T \sqsubset 1$.*

Let S be an operator having the principal function g_S related to the Cartesian decomposition $S = X + iY$. Then $g_{S^*}(x, y) = -g_S(x, -y)$. Hence, as a corollary of Lemma 3.7, we have the following.

Lemma 3.8. *Let T be an operator such that $[T^*, T] \in \mathcal{C}_1$ and $\sigma(T)$ is an infinite set. If T^* has a cyclic vector, then $-1 \sqsubset g_T$.*

Theorem 3.9. *Let T be an operator such that $[T^*, T] \in \mathcal{C}_1$ and $\sigma(T)$ is an infinite set. If $\int \int g_T(x, y) dx dy \neq 0$, then, for a sufficiently high n , T^n has a non-trivial invariant subspace.*

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