

Malliavin calculus applied to mathematical finance and a new formulation of the integration-by-parts

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1 Introduction

In recent years there appear several papers in finance on jump models and on jump-diffusion models using stochastic calculus, after the success of the Black-Scholes model. Indeed, classical [1] and [16] include chapters on jump-diffusions. Recent examples are [17], [22], [10], [11], and [27]. However, fairly restricted types of jump processes have been treated, due to the technical difficulties. For example, [1] and [16] have treated the diffusion + compound Poisson model. The so-called geometric Lévy model $S_t = S_0 e^{Z_t}$, where Z_t denotes a Lévy process (with infinite jumps), has not been included in the previous typical jump models studied in many papers.

Let S_t denote a jump-diffusion given as a solution to SDE which is driven by a Lévy process. We study here as an application of Malliavin calculus of jump type the sensitivity analysis for asset prices. Basic concept is as follows.

$$price = E^Q[(pay-off)].$$

Here *price* means today's ($t = 0$) value of some contingent claim (pay-off) with respect to S_t in future ($t = T$), and Q is a risk neutral probability.

We assume the pay-off depends on some parameter λ . We consider the marginal move of the price with respect to λ by using the integration-by-parts:

$$\frac{\partial}{\partial \lambda}(price)(\lambda) = E^Q[(pay-off) \cdot (weight)(\lambda)].$$

The L.H.S. denotes the marginal move of the asset price with respect to λ , hence it serves to measure the stability of the price. Such quantities are called Greeks. Some examples of Greeks are *Delta*, *Vega*, *Gamma*, *Rho* and *Theta*. For the precise definition, see below.

The basic framework of this theory on the Wiener space has been established in [8]. We study in this paper some functionals on the Wiener-Poisson space, and develop a stochastic calculus of variations to achieve the integration-by-parts formula.

2 Jump-diffusion models in closed form

Let $N(dtdz)$ be a Poisson random measure on $[0, T] \times \mathbf{R}$ with the mean measure $dt \cdot \delta_{\{1\}}$, and W_t be a Wiener process on $\mathbf{R}$.

Let Z_t be a simple Lévy process given by

$$Z_t = \sigma_1 W_t + \sigma_2 \tilde{N}_t,$$

where $\tilde{N}_t = N_t - t$.

The price process S_t associated to this Z_t is defined by

$$\frac{dS_t}{S_{t-}} = r(t)dt + \sigma_1(t)dW_t + \sigma_2(t)d\tilde{N}_t, S_0 = x.$$

Here $r(t), \sigma_1(t), \sigma_2(t)$ are deterministic functions. Then S_t is represented explicitly in closed form

$$S_t = x \exp\left[\int_0^t \sigma_1(s)dW_s + \int_0^t (r(s) - \sigma_2(s))ds - \frac{1}{2} \int_0^t \sigma_2^2(s)ds\right] \times \prod_{k=1}^{N_t} (1 + \sigma_2(T_k))$$

where $T_1, T_2, \dots$ are jump times of N_t . cf. [1] (3.2).

More generally, assume that X_t is a jump *semimartingale*, such that it is a solution to a SDE driven by a Lévy process. The price process is defined by

$$\frac{dS_t}{S_{t-}} = r(t)dt + \sigma_1(t)dW_t + \sigma_2(t)dX_t, S_0 = x.$$

Then S_t is represented also in closed form by

$$S_t = x \exp\left[\int_0^t \sigma_1(s)dW_s + \int_0^t (r(s) - \frac{1}{2}\sigma_1^2(s))ds + \int_0^t \sigma_2(s)dX_s - \frac{1}{2} \int_0^t \sigma_2^2(s)d[X, X]_s\right] \\ \times \prod_{s=0}^t ((1 + \sigma_2(s)\Delta X_s) \exp(-\sigma_2(s)\Delta X_s + \frac{1}{2}(\sigma_2(s)\Delta X_s)^2)).$$

Note that the product is a infinite product in general.

Let, for example, $F = S_T, T > 0$. If we know explicitly the density of F via closed formulae above, then we can estimate $E[f(F)]$ directly. We may then have closed forms for Greeks for “good” f . This way is called the kernel density estimation method [13]. An example of a such density is the variance gamma distribution [18]. However this is not always the case. For example, there is no explicit formula for the price of American option.

3 Greeks

Let λ be some parameter in S_T given above, and let $F = F^\lambda$ be a functional of S_t . That is, for example, $F = S_T^{(\lambda)}, T > 0$ or $F = \int_0^T S_t^{(\lambda)} dt$. Let f be a a.e. smooth function taking values on $\mathbb{R}$. Then $f(F)$ is a random variable. An example of $f(x)$ is $f_0(x) = (x - K)_+ 1_{[K, \infty)}$, or its smooth regularization $f = f_0 * \varphi_\epsilon$, where φ_ϵ is a molifier.

So called *Greeks* associated to $f(F)$ are given as follows.

$$(1) \text{ Delta} = \exp\left\{-\int_0^T r(t)dt\right\} \frac{\partial}{\partial x} E[f(F)].$$

Delta is the derivative of the price with respect to the parameter $\lambda = x$ (the initial value of S).

$$(2) \text{ Vega} = \exp\left\{-\int_0^T r(t)dt\right\} \frac{\partial}{\partial \sigma_1} E[f(F)].$$

More precisely, for $\epsilon > 0$, let

$$\frac{dS_t^\epsilon}{S_{t-}^\epsilon} = r(t)dt + (\sigma_1(t) + \epsilon\tilde{\sigma}_1(t))dW_t + \sigma_2(t)d\tilde{N}_t, S_0^\epsilon = x.$$

We put

$$C_\epsilon \equiv \exp\left\{-\int_0^T r(t)dt\right\}E[f(S_T^\epsilon)].$$

Then $\text{Vega} = \frac{\partial C_\epsilon}{\partial \epsilon}|_{\epsilon=0}$. This is a (Fréchet) derivative of S_t with respect to $\sigma_1(\cdot)$ (coefficient of the Wiener process) in the direction $\tilde{\sigma}_1(\cdot)$.

Other Greeks are, for example,

$$(3) \text{ Gamma} = \exp\left\{-\int_0^T r(t)dt\right\}\frac{\partial^2}{\partial x^2}E[f(F)].$$

$$(4) \text{ Rho} = \frac{\partial}{\partial r}(\exp\left\{-\int_0^T r(t)dt\right\}E[f(F)]). \quad (\text{The } Rho \text{ is defined similarly as } Vega.)$$

$$(5) \text{ Theta} = \frac{\partial}{\partial T}(\exp\left\{-\int_0^T r(t)dt\right\}E[f(F)]).$$

We remark that these Greeks can be regarded as corresponding (first or second) terms in the asymptotic expansion

$$E[F^\lambda] - E[F] = c_1\lambda + \frac{1}{2}c_2\lambda^2 + \dots$$

when $\lambda > 0$ is small.

4 Weights

For the calculation of Greeks we can use Malliavin calculus for jump-diffusion processes. In this section we assume that the 1-dimensional process X_t driving the SDE above is given by $X_t = \sigma_1 W_t + \sigma_2 Z_t$, where Z_t is a Lévy process

$$Z_t = bt + \int_0^t \int_{|z| \leq 1} z \tilde{N}(dsdz) + \int_0^t \int_{|z| > 1} z N(dsdz)$$

whose Lévy measure is given by $\mu(dz)$. We do not assume $\mu(dz)$ is absolutely continuous with respect to the Lebesgue measure. It can even be a discrete measure. (If $\mu = \delta_{\{1\}}$ then Z_t is a Poisson process N_t .) In this case it is not practical to compute Greeks along the closed form expression in general.

Let $F = F^x$ be as in the previous section ($\lambda = x$). For a random variable $G^x \in L^2$ depending on x , we have

$$\frac{\partial}{\partial x}E[G^x f(F)] = E[G^x \partial f(F) \partial_x F] + E[\partial_x G^x \cdot f(F)].$$

If we choose $G^x \equiv 1$,

$$\frac{\partial}{\partial x}E[f(F)] = E[\partial f(F) \cdot \partial_x F]. \quad (0)$$

We introduce a gradient operator $D_u, u = (t, z)$, on the Poisson space on $[0, T] \times \mathbf{R}$. We assume the chain rule

$$D_u f(F) = \partial f(F) \cdot D_u F \quad (1)$$

and the local operator property

$$D_u(XY) = XD_uY + YD_uX \quad (2)$$

hold for the operator D_u . By the chain rule for the gradient D_u and by the integration by parts, we have

$$\text{R.H.S. of (0)} = E\left[\frac{D_u f(F)}{D_u F} \cdot \partial_x F\right] = E\left[D_u f(F) \cdot \frac{\partial_x F}{D_u F}\right] = E\left[f(F) \delta\left(\frac{\partial_x F}{D_u F}\right)\right]. \quad (3)$$

This leads to the calculation for *Delta*.

Here $\delta(\cdot)$ is the adjoint operator (Skorohod integral) associated to the gradient D_u , and the term $\delta(\cdot)$ is called a *weight* provided that it is square integrable. In practical computation it is important to calculate this *Weight*.

We can proceed the calculation (3) above following the formula

$$\delta(vG) = G\delta(v) - \int_0^T \int D_u G v(u) dt \mu(dz) \quad (4)$$

(cf. [6] Proposition 1).

To compute *Gamma* we need to compute the second derivative

$$\frac{\partial^2}{\partial x^2} E[f(F)] = \frac{\partial}{\partial x} E[f(F) \cdot \delta(G)] = E[f(F) \frac{\partial}{\partial x} \delta(G)] + E[f(F) \delta(\delta(G)G)],$$

where $G = \frac{\partial_x F}{D_u F}$.

For the precise framework for this calculation on the Wiener-Poisson space, there seems to exist no decisive set-up up to now (e.g., gradient operator, its adjoint, norms, Sobolev spaces, ...). In the section 7 we present a new framework for this.

5 Finite difference operator and gradient operator on Poisson space

Let $Z_t = \tilde{N}_t$ for simplicity. On the Poisson space we introduce two gradients.

Let $U = [0, T] \times \mathbf{R}$. We choose u in U of the form $u = (t, 1)$. Let $F = f(T_1, \dots, T_n)$, where $f = f(x_1, \dots, x_n)$ is a smooth function and T_k denotes the k -th jump time of N_t . We introduce two gradient of F on U .

We put

$$D_u F = - \sum_{N_t < k \leq n} \partial_k f(T_1, \dots, T_n). \quad (5)$$

Here ∂_k denotes $\frac{\partial}{\partial x_k}$. This definition is due to Carlen-Pardeux [4].

We introduce a finite difference operator $\tilde{D}$ by

$$\tilde{D}_u F = f(T_1, \dots, T_{N_t}, t, T_{N_t+1}, \dots, T_{n-1}) - f(T_1, \dots, T_n) \quad (6)$$

if $N_t < n$. The above is equivalent to

$$\tilde{D}_u F = f(T_1, \dots, T_k, t, T_{k+1}, \dots, T_{n-1}) - f(T_1, \dots, T_n) \quad (7)$$

if $T_k < t \leq T_{k+1}$. This definition is due to Nualart-Vives [21] (see also Picard [23]).

The operator D_u satisfies the properties (1), (2) in Sect. 4, whereas $\tilde{D}_u$ does *not*. Instead we have by the mean value theorem when φ is differentiable :

$$\tilde{D}_u \varphi(F) = \int_0^1 \partial \varphi(F + \theta \tilde{D}_u F) d\theta \cdot \tilde{D}_u F. \quad (8)$$

And also

$$\tilde{D}_u(FG) = F \cdot \tilde{D}_u G + G \cdot \tilde{D}_u F + \tilde{D}_u F \tilde{D}_u G \quad (9)$$

(cf. [21] Lemma 6.1).

The gradient D_u is closable (in $L^2(\Omega, L^2([0, T]))$), and its adjoint is given by

$$\delta(v) = \int_0^T v(t) d\tilde{N}_t - \int_0^T D_u v(t) dt.$$

Further we have

$$E\left[\int_0^T D_u F v dt\right] = E[F\delta(v)]$$

([26] Propositions 7, 8, [19] p.104).

The formula (4) then reads

$$\delta(vG) = G\delta(v) - (v, D_u G) = G\delta(v) + \int_0^T v(t) \left(\sum_{N_t < k \leq n} \partial_k g(T_1, \dots, T_n) \right) dt$$

if $G = g(T_1, \dots, T_n)$. Hence, the formula (3) reads

$$\delta\left(\frac{\partial_x F}{D_u F}\right) = \partial_x F \delta\left(\frac{1}{D_u F}\right) - \left(\frac{1}{D_u F}, D_u(\partial_x F)\right).$$

Although, due to (9), $\tilde{D}_u$ does not satisfy the chain rule, we can show the property below between $\tilde{D}_u$ and D_u for which the chain rule holds.

Let $p_n(t) = P(N_t = n) = \frac{t^n}{n!} e^{-t}$ be the density function of T_n . We have then

$$p'_n(t) = p_{n-1}(t) - p_n(t), \quad t > 0.$$

Let $T = \infty$. In view of this formula, the formula

$$\frac{d}{du} \int_t^u g(s, u) ds = g(u, u) + \int_t^u \frac{\partial}{\partial u} g(s, u) ds,$$

and due to the fact that the jump times of a Poisson process are uniformly distributed given the number of jumps, we have the following Proposition.

Proposition Let $F = f(T_1, \dots, T_n)$ and $G = g(T_1, \dots, T_n) = \varphi(F)$. That is, $g = \varphi \circ f$. Then

$$E[D_u G / \mathcal{F}_t] = E[\tilde{D}_u G / \mathcal{F}_t].$$

The proof is due to N. Privault. It also follows from the Kabanov formula (cf. [21] Theorem. 6.2). We state the direct proof below in case $n = 5$. Proof for the general case is easy.

Example

Let $G = g(T_n)$. We have the equality directly as follows.

$$\begin{aligned}
E[D_u g(T_n)/\mathcal{F}_t] &= -1_{\{N_t < n\}} E[g'(T_n)/\mathcal{F}_t] \\
&= -\int_t^\infty g'(x) p_{n-1-N_t}(x-t) dx = g(t) p_{n-1-N_t}(0) + \int_t^\infty g(x) p'_{n-1-N_t}(x-t) dx \\
&= g(t) 1_{\{T_{n-1} \leq t < T_n\}} + \int_t^\infty g(x) p'_{n-1-N_t}(x-t) dx \\
&= g(t) 1_{\{T_{n-1} \leq t < T_n\}} + \int_t^\infty (p_{n-2-N_t}(x-t) - p_{n-1-N_t}(x-t)) g(x) dx \\
&= g(t) 1_{\{T_{n-1} \leq t < T_n\}} + E[1_{\{T_{n-1} > t\}} g(T_{n-1}) - 1_{\{T_n > t\}} g(T_n)/\mathcal{F}_t] \\
&= E[1_{\{T_{n-1} > t\}} g(T_{n-1}) + 1_{\{T_{n-1} \leq t < T_n\}} g(t) - 1_{\{T_n > t\}} g(T_n)/\mathcal{F}_t] \\
&= E[1_{\{N_t < n-1\}} (g(T_{n-1}) - g(T_n)) + 1_{\{N_t = n-1\}} (g(t) - g(T_n))/\mathcal{F}_t] \\
&= E[\tilde{D}_u g(T_n)/\mathcal{F}_t].
\end{aligned}$$

It is natural that they coincide with each other by the uniqueness of the Clark-Ocone formula for G , as they are the conditional expectation terms (integrands) in the 1-st stochastic integral in the Clark-Ocone expression. Cf. [28]. However we can see it directly in this case.

Proof.

Let $G = g(T_1, T_2, \dots, T_5)$.

$$\begin{aligned}
E[D_u G/\mathcal{F}_t] &= - \sum_{N_t < k \leq 5} E[\partial_k g(T_1, \dots, T_5)/\mathcal{F}_t] \\
&= - \sum_{N_t < k \leq 5} \int_0^\infty e^{-(s_5-t)} \int_t^{s_5} \dots \int_t^{s_{N_t+2}} \partial_k g(T_1, \dots, T_{N_t}, s_{N_t+1}, \dots, s_5) ds_{N_t+1} \dots ds_5 \\
&= - \sum_{k=N_t+2}^5 \int_t^\infty e^{-(s_5-t)} \int_t^{s_5} \dots \frac{\partial}{\partial s_k} \int_t^{s_k} \dots \int_t^{s_{N_t+2}} g(T_1, \dots, T_{N_t}, s_{N_t+1}, \dots, s_5) ds_{N_t+1} \dots ds_5 \\
&+ \sum_{k=N_t+2}^5 \int_t^\infty e^{-(s_5-t)} \int_t^{s_5} \dots \int_t^{s_k} \dots \int_t^{s_{N_t+2}} g(T_1, \dots, T_{N_t}, s_{N_t+1}, s_k, s_k, s_{k+1}, \dots, s_5) ds_{N_t+1} \dots ds_{k-1} \dots ds_5 \\
&- 1_{\{N_t < 5\}} \int_t^\infty e^{-(s_5-t)} \int_t^{s_5} \dots \int_t^{s_{N_t+2}} \frac{\partial}{\partial s_{N_t+1}} g(T_1, \dots, T_{N_t}, s_{N_t+1}, \dots, s_5) ds_{N_t+1} \dots ds_5 \\
&= -1_{\{N_t < 4\}} \int_t^\infty e^{-(s_5-t)} \int_t^{s_5} \dots \int_t^{s_{N_t+2}} \frac{\partial}{\partial s_5} g(T_1, \dots, T_{N_t}, s_{N_t+1}, \dots, s_5) ds_{N_t+1} \dots ds_5
\end{aligned}$$

$$\begin{aligned}
& - \sum_{k=N_t+2}^4 \int_t^\infty e^{-(s_5-t)} \int_t^{s_5} \dots \int_t^{s_{k+1}} \frac{\partial}{\partial s_k} \int_t^{s_k} \dots \int_t^{s_{N_t+2}} g(T_1, \dots, T_{N_t}, s_{N_t+1}, \dots, s_5) ds_{N_t+1} \dots ds_5 \\
& + \sum_{k=N_t+2}^5 \int_t^\infty e^{-(s_5-t)} \int_t^{s_5} \dots \int_t^{s_k} \dots \int_t^{s_{N_t+2}} g(T_1, \dots, T_{N_t}, s_{N_t+1}, s_k, s_k, s_{k+1}, \dots, s_5) ds_{N_t+1} \dots d\hat{s}_{k-1} \dots ds_5 \\
& \quad - 1_{\{N_t < 5\}} \int_t^\infty e^{-(s_5-t)} \int_t^{s_5} \dots \int_t^{s_{N_t+2}} \frac{\partial}{\partial s_{N_t+1}} g(T_1, \dots, T_{N_t}, s_{N_t+1}, \dots, s_5) ds_{N_t+1} \dots ds_5 \\
& \quad = -1_{\{N_t < 4\}} \int_t^\infty e^{-(s_5-t)} \frac{\partial}{\partial s_5} \int_t^{s_5} \int_t^{s_{N_t+2}} g(T_1, \dots, T_{N_t}, s_{N_t+1}, \dots, s_5) ds_{N_t+1} \dots ds_5 \\
& - \sum_{k=N_t+2}^4 \int_t^\infty e^{-(s_5-t)} \int_t^{s_5} \int_t^{s_{N_t+2}} g(T_1, \dots, T_{N_t}, s_{N_t+1}, \dots, s_{k-1}, s_{k+1}, s_{k+1}, \dots, s_5) ds_{N_t+1} \dots d\hat{s}_k \dots ds_5 \\
& + \sum_{k=N_t+2}^5 \int_t^\infty e^{-(s_5-t)} \int_t^{s_5} \dots \int_t^{s_k} \dots \int_t^{s_{N_t+2}} g(T_1, \dots, T_{N_t}, s_{N_t+1}, s_k, s_k, s_{k+1}, \dots, s_5) ds_{N_t+1} \dots d\hat{s}_{k-1} \dots ds_5 \\
& \quad - 1_{\{N_t < 5\}} \int_t^\infty e^{-(s_5-t)} \int_t^{s_5} \dots \int_t^{s_{N_t+2}} \frac{\partial}{\partial s_{N_t+1}} g(T_1, \dots, T_{N_t}, s_{N_t+1}, \dots, s_5) ds_{N_t+1} \dots ds_5 \\
& \quad = -1_{\{N_t < 4\}} \int_t^\infty e^{-(s_5-t)} \int_t^{s_5} \dots \int_t^{s_{N_t+2}} g(T_1, \dots, T_{N_t}, s_{N_t+1}, \dots, s_5) ds_{N_t+1} \dots ds_5 \\
& \quad + 1_{\{N_t < 4\}} \int_t^\infty e^{-(s_5-t)} \int_t^{s_5} \dots \int_t^{s_{N_t+2}} g(T_1, \dots, T_{N_t}, s_{N_t+2}, s_{N_t+2}, \dots, s_5) ds_{N_t+1} \dots ds_5 \\
& \quad - 1_{\{N_t < 5\}} \int_t^\infty e^{-(s_5-t)} \int_t^{s_5} \dots \int_t^{s_{N_t+2}} \frac{\partial}{\partial s_{N_t+1}} g(T_1, \dots, T_{N_t}, s_{N_t+1}, \dots, s_5) ds_{N_t+1} \dots ds_5 \\
& \quad = -1_{\{N_t < 4\}} \int_t^\infty e^{-(s_5-t)} \int_t^{s_5} \dots \int_t^{s_{N_t+2}} g(T_1, \dots, T_{N_t}, s_{N_t+1}, \dots, s_5) ds_{N_t+1} \dots ds_5 \\
& \quad + 1_{\{N_t < 4\}} \int_t^\infty e^{-(s_5-t)} \int_t^{s_5} \dots \int_t^{s_{N_t+2}} g(T_1, \dots, T_{N_t}, s_{N_t+2}, s_{N_t+2}, \dots, s_5) ds_{N_t+1} \dots ds_5 \\
& \quad - 1_{\{N_t < 4\}} \int_t^\infty e^{-(s_5-t)} \int_t^{s_5} \dots \int_t^{s_{N_t+2}} g(T_1, \dots, T_{N_t}, s_{N_t+2}, s_{N_t+2}, \dots, s_5) ds_{N_t+1} \dots ds_5 \\
& \quad + 1_{\{N_t < 4\}} \int_t^\infty e^{-(s_5-t)} \int_t^{s_5} \dots \int_t^{s_{N_t+2}} g(T_1, \dots, T_{N_t}, t, s_{N_t+2}, \dots, s_5) ds_{N_t+2} \dots ds_5 \\
& \quad \quad - 1_{\{5=N_t+1\}} \int_t^\infty e^{-(s_5-t)} \int_t^{s_5} g(T_1, \dots, T_{n-1}, s_5) ds_5 \\
& \quad \quad + 1_{\{5=N_t+1\}} g(T_1, \dots, T_{n-1}, t) \\
& = -1_{\{N_t < 5\}} \int_t^\infty e^{-(s_5-t)} \int_t^{s_5} \dots \int_t^{s_{N_t+2}} g(T_1, \dots, T_{N_t}, s_{N_t+1}, \dots, s_5) ds_{N_t+1} \dots ds_5 \\
& \quad + 1_{\{N_t < 5\}} \int_t^\infty e^{-(s_5-t)} \int_t^{s_5} \dots \int_t^{s_{N_t+2}} g(T_1, \dots, T_{N_t}, t, s_{N_t+2}, \dots, s_5) ds_{N_t+2} \dots ds_5 \\
& = E[\tilde{D}_u G / \mathcal{F}_t].
\end{aligned}$$

The operators D_u and δ can be extended to the case

$$Z_t = \sum_{k=1}^m \tilde{N}_k(t),$$

where N_k 's are independent Poisson processes, by composing a direct sum of independent Poisson spaces (cf. [19] p.103). On the other hand, for the adjoint of $\tilde{D}_u$, see the section 7.

6 Integration-by-parts setting by Bismut

In this section we state the integration-by-parts formula by using Bismut perturbation. We sketch the idea below in case $d = 1$. We assume in this section that $\mu(dz) = g(z)dz$, where $g(z)$ is a smooth function on $\mathbf{R}$ having compact support.

Let v be a bounded predictable process on $[0, +\infty)$ to $\mathbf{R}$. We consider the perturbation

$$\theta^\lambda : z \mapsto z + \lambda \nu(z)v, \quad \lambda \in \mathbf{R}.$$

Here $\nu(z)$ is a smooth function which is $O(z^2)$ near $z = 0$. Let $N^\lambda(ds dz)$ be the Poisson random measure defined by

$$\int_0^t \int \phi(z) N^\lambda(ds dz) = \int_0^t \int \phi(\theta^\lambda(z)) N(ds dz), \quad \phi \in C_0^\infty(\mathbf{R}).$$

We put $Z_s^\lambda = \int_0^t \int z N^\lambda(du dz)$, and denote by P^λ its law. Set $\Lambda^\lambda(z) = \{1 + \lambda \nu'(z)v\} \frac{g(\theta^\lambda(z))}{g(z)}$, and

$$U_t^\lambda = \exp\left[\int_0^t \int \log \Lambda^\lambda(z) N(ds dz) - \int_0^t \int (\Lambda^\lambda(z) - 1)g(z)dz\right].$$

Then Z_t^λ is a martingale, and P^λ has the derivative

$$\frac{dP^\lambda}{dP} = U_t^\lambda \quad \text{on } \mathcal{F}_t.$$

where $\mathcal{F}_t$ denotes the σ -field generated by Z_t (cf. [2] Theorem 6-16, Bismut [3], (2.34)).

Consider the perturbed process F_s^λ which is defined by a SDE driven by Z^λ in place of Z . Then $E^P[f(F_t)] = E^{P^\lambda}[f(F_t^\lambda)] = E^P[f(F_t^\lambda)U_t^\lambda]$, and we have $0 = \frac{\partial}{\partial \lambda} E^P[f(F_t^\lambda)U_t^\lambda]$, $f \in C_0^\infty(\mathbf{R})$. By the chain rule, for $|\lambda|$ small, we have

$$\frac{\partial f}{\partial \lambda}(F_t^\lambda) = D_x f(F_t^\lambda) \cdot \frac{\partial F_t^\lambda}{\partial \lambda}, \quad f \in C_0^\infty(\mathbf{R}).$$

We have for $\lambda = 0$

$$E^P[D_x f(F_t) \cdot \frac{\partial F_t^\lambda}{\partial \lambda}|_{\lambda=0}] = -E^P[f(F_t) \frac{\partial}{\partial \lambda} U_t^\lambda|_{\lambda=0}].$$

By Corollary 6-17 of [2], we may differentiate U_t^λ with respect to λ , to obtain

$$R_t \equiv \frac{\partial}{\partial \lambda} U_t^\lambda|_{\lambda=0} = \int_0^t \int \frac{\text{div} \{g(\cdot)v\nu(\cdot)\}(z)}{g(z)} \{N(ds dz) - ds g(z)dz\}.$$

Next we compute $H_t^\lambda \equiv \frac{\partial F_t^\lambda}{\partial \lambda}$. F_t^λ is differentiable a.s. for $|\lambda|$ small, and its derivative at $\lambda = 0$, $H_t = H_t^0$ is obtained explicitly as the solution of a SDE (cf. [2] Theorem 6-24). We put $DH_t = \frac{\partial}{\partial \lambda} H_t^\lambda|_{\lambda=0}$, where $\frac{\partial}{\partial \lambda} H_t^\lambda$ is the second Fréchet derivative of F_t^λ defined as in [2] Theorem 6-44. Then $\frac{\partial}{\partial \lambda} H_t^{\lambda,-1}|_{\lambda=0} = -H_t^{-1}DH_tH_t^{-1}$. Here $\frac{\partial}{\partial \lambda} H_t^{\lambda,-1}$ is defined by $\langle \frac{\partial}{\partial \lambda} H_t^{\lambda,-1}, e \rangle = \text{trace } [e' \mapsto \langle -H_t^{\lambda,-1}(\frac{\partial}{\partial \lambda} H_t^\lambda \cdot e')H_t^{\lambda,-1}, e \rangle]$, $e \in \mathbf{R}$.

We carry out the integration-by-parts procedure for $G_t^\lambda = f(F_t^\lambda)H_t^{\lambda,-1}$. Recall we have $E[G_t^0] = E[G_t^\lambda \cdot U_t^\lambda]$. Taking the Fréchet derivation $\frac{\partial}{\partial \lambda}|_{\lambda=0}$ for both sides yields

$$0 = E[D_x f(F_t)H_t^{-1}H_t] + E[f(F_t)\frac{\partial}{\partial \lambda} H_t^{\lambda,-1}|_{\lambda=0}] + E[f(F_t)H_t^{-1} \cdot R_t].$$

This yields

$$E[D_x f(F_t)] = E[f(F_t)\mathcal{A}_t^{(1)}]$$

where

$$\mathcal{A}_t^{(1)} = \{H_t^{-1}DH_tH_t^{-1} - H_t^{-1}R_t\}.$$

This is the integration-by-parts formula in Bismut setting. We can calculate $H_t^{-1}DH_tH_t^{-1}$ explicitly.

7 New integration-by-parts setting for jump diffusion

This is a joint work with Prof. H. Kunita. [12]

From the gradient-adjoint formula to the integration-by-parts formula for $f(F)$, there are several attempts. Here we recall one which is based on Picard's method.

In this section, let $N(dtdz)$ be a Poisson random measure on $[0, T] \times \mathbf{R}^m$ and W_t be a Wiener process on $\mathbf{R}^m$, $m \geq 1$.

Let T_0 be a positive number and let $T = [0, T_0]$. Let Ω_1 be the set of all continuous maps $\omega_1 : T \rightarrow \mathbf{R}^m$ such that $\omega_1(0) = 0$ and let $\mathcal{F}_1$ be the smallest σ -field of Ω_1 with respect to which $\{\omega_1(t), t \in [0, T]\}$ are measurable. Let P_1 be a probability measure on $(\Omega_1, \mathcal{F}_1)$ such that $W(t) := \omega_1(t)$ is a standard 1-dimensional Brownian motion.

Set

$$\varphi(\rho) = \int_{|z| \leq \rho} |z|^2 \mu(dz). \quad (10)$$

We say that the measure μ satisfies an *order condition* if there exists $0 < \alpha < 2$ such that

$$\liminf_{\rho \rightarrow 0} \frac{\varphi(\rho)}{\rho^\alpha} > 0. \quad (11)$$

Note that Lévy measures with finite mass do not satisfy the order condition, because $\liminf_{\rho \rightarrow 0} \frac{\varphi(\rho)}{\rho^\alpha} = 0$ holds for any $\alpha \in (0, 2)$ then. On the other hand, Lévy measures of stable laws with exponent β satisfies the order condition with $\alpha = 2 - \beta$.

Let T_0 be a positive number and let $T = [0, T_0]$. Let Ω_1 be the set of all continuous maps $\omega_1 : T \rightarrow \mathbf{R}^m$ such that $\omega_1(0) = 0$ and let $\mathcal{F}_1$ be the smallest σ -field of Ω_1 with

respect to which $\{w_1(t), t \in [0, T]\}$ are measurable. Let P_1 be a probability measure on $(\Omega_1, \mathcal{F}_1)$ such that $W(t) := \omega_1(t)$ is a standard 1-dimensional Brownian motion.

Let Ω_2 be the set of all integer valued measures on $U = T \times \mathbf{R}^m$ such that $\omega_2(T \times \{0\}) = 0$ and let $\mathcal{F}_2$ be the smallest σ -field of Ω_2 with respect to which $\{\omega_2(E); E \text{ are Borel sets in } U\}$ are measurable. Let P_2 be a probability measure on $(\Omega_2, \mathcal{F}_2)$ such that $N(dtdz) := \omega_2(dtdz)$ is a Poisson random measure with intensity measure $\hat{N}(dtdz) := dt\mu(dz)$, where μ is a Lévy measure.

Let $H = L^2(T; \mathbf{R}^m)$. For $h_l \in H$, we set

$$W(h_l) = \int_T h_l(s) dW_s.$$

We denote by S_1 the collection of random variables X written as

$$X = f(W(h_1), \dots, W(h_{n_1})),$$

where $f(x_1, \dots, x_{n_1})$ is bounded $\mathcal{B}(\mathbf{R}^{n_1})$ measurable, smooth in $(x_1, \dots, x_{n_1})$, $n_1 \in \mathbf{N}$. The Malliavin-Shigekawa's derivative of X (with respect to the first variable ω_1) is an 1-dimensional row vector stochastic process given by

$$D_t X = \sum_i \frac{\partial f}{\partial x_i}(W(h_1), \dots, W(h_{n_1})) h_i(t). \quad (12)$$

Next, we shall introduce difference operators $\tilde{D}_u, u \in U$, acting on the Poisson space. For each $u = (t, z) = (t, z_1) \in U$, we define a map $\varepsilon_u^- : \Omega_2 \rightarrow \Omega_2$ by $\varepsilon_u^- \omega_2(A) = \omega_2(A \cap \{u\}^c)$, and $\varepsilon_u^+ : \Omega_2 \rightarrow \Omega_2$ by $\varepsilon_u^+ \omega_2(A) = \omega_2(A \cap \{u\}^c) + 1_A(u)$. (These are extended to Ω by setting $\varepsilon_u^\pm(\omega_1, \omega_2) = (\omega_1, \varepsilon_u^\pm \omega_2)$) It holds $\varepsilon_u^- \omega = \omega$ a.s. P for any u since $\omega_2(\{u\}) = 0$ holds for almost all ω_2 for any u . The difference operators $\tilde{D}_u$ for a $\mathcal{F}_2$ -measurable random variable X is defined after Picard [23] by

$$\tilde{D}_u X = X \circ \varepsilon_u^+ - X. \quad (13)$$

Let $\mathbf{u} = (u^1, \dots, u^k) = ((t_1, z^1), \dots, (t_k, z^k)) = (\mathbf{t}, \mathbf{z})$. We set $|\mathbf{u}| = |\mathbf{z}| = \max_{1 \leq i \leq k} |z^i|$ and $\gamma(\mathbf{u}) = |z^1| \cdots |z^k|$. We define $\varepsilon_{\mathbf{u}}^+ = \varepsilon_{u_1}^+ \circ \dots \circ \varepsilon_{u_k}^+$ and $\tilde{D}_{\mathbf{u}} = \tilde{D}_{\mathbf{u}}^k = \tilde{D}_{u_1} \cdots \tilde{D}_{u_k}$. Further for $\mathbf{z} = (z^1, \dots, z^k)$ where $z^i \in \mathbf{R}^m$, we set $\partial_{\mathbf{z}} g = \partial_{z^1} \cdots \partial_{z^k} g$. It is an k -vector function.

Let S_2 be the collection of random variables X written as

$$X = f(N(\varphi_1), \dots, N(\varphi_{n_2})),$$

where $f(x_1, \dots, x_{n_2})$ is bounded $\mathcal{B}(\mathbf{R}^{n_2})$ measurable, smooth in $(x_1, \dots, x_{n_2})$, $n_2 \in \mathbf{N}$.

Let $\mathcal{S} = S_1 \otimes S_2$. Spaces S_1, S_2 are identified with $S_1 \otimes 1, 1 \otimes S_2$ respectively. The space $\mathcal{S}$ is the linear span of functionals X such that

$$X = \sum_{i+j=k} X_1^{(i)} X_2^{(j)}, k \in \mathbf{N},$$

where $X_1^{(i)} = f_1^{(i)}(W(h_1), \dots, W(h_i))$ and $X_2^{(j)} = f_2^{(j)}(N(\varphi_1), \dots, N(\varphi_j))$. Here $f_1^{(i)}$ and $f_2^{(j)}$ are bounded $\mathcal{B}(\mathbf{R}^i)$ ($\mathcal{B}(\mathbf{R}^j)$) measurable, smooth functions of i (j) variables, respectively.

The adjoint $\tilde{\delta}$ of the operators $\tilde{D} = (\tilde{D}_u)_{u \in U}$ is defined as follows. Let $Z_u = Z_{t,z}$ be an $\mathcal{F}$ -measurable random field, integrable with respect to $\tilde{N} = N - \hat{N}$, i.e.,

$$E\left[\int_U |Z_u \circ \varepsilon_u^-| (N + \hat{N})(du)\right] < \infty.$$

We set

$$\tilde{\delta}(Z) = \int_U Z_u \circ \varepsilon_u^- \tilde{N}(du). \quad (14)$$

It is known that this operator satisfies the adjoint property:

$$E[X \tilde{\delta}(Z)] = E\left[\int_U \tilde{D}_u X Z_u \hat{N}(du)\right], \quad (15)$$

for any bounded $\mathcal{F}$ -measurable random variable X . ([23], Lemma 1.4).

We shall next introduce linear maps Q and $\tilde{Q}_\rho$ by

$$QY = \int_T (D_t F) D_t Y dt, \quad (16)$$

$$\tilde{Q}_\rho Y = \frac{1}{\varphi(\rho)} \int_{A(\rho)} (\tilde{D}_u F) \tilde{D}_u Y \hat{N}(du). \quad (17)$$

Lemma *The adjoints of Q and $\tilde{Q}_\rho$ exist and are equal to*

$$Q^* X = \delta((DF)^T X), \quad (18)$$

$$\tilde{Q}_\rho^* X = \tilde{\delta}_\rho((\tilde{D}F)^T X), \quad (19)$$

respectively, where

$$\tilde{\delta}_\rho(Z) = \frac{1}{\varphi(\rho)} \tilde{\delta}(Z 1_{A(\rho)}) = \frac{1}{\varphi(\rho)} \int_{A(\rho)} Z_u \circ \varepsilon_u^- \tilde{N}(du). \quad (20)$$

Let $f(x)$ be a C^2 -function with bounded derivatives. We claim a modified formula of integration by parts. Note that $D_t(f(F)) = f(F) D_t F = (D_t F) \partial f(F)$. Then we get

$$Qf(F) = \int_T (D_t F) D_t(f(F)) dt = R \partial f(F). \quad (21)$$

Concerning the difference operator $\tilde{D}_u$, we have by the mean value theorem,

$$\tilde{D}_u(f(G)) = (\tilde{D}_u G)^T \int_0^1 \partial f(G + \theta \tilde{D}_u G) d\theta, \quad (22)$$

for a random variable G on the Poisson space. This implies

$$\begin{aligned} \tilde{Q}_\rho f(F) &= \tilde{R}_\rho \partial f(F) \\ &+ \frac{1}{\varphi(\rho)} \int_{A(\rho)} \tilde{D}_u F (\tilde{D}_u F)^T \left(\int_0^1 \{\partial f(F + \theta \tilde{D}_u F) - \partial f(F)\} d\theta \right) \hat{N}(du). \end{aligned} \quad (23)$$

Here

$$\tilde{R}_\rho = \frac{1}{\varphi(\rho)} \int_{A(\rho)} \tilde{D}_u F (\tilde{D}_u F)^T \hat{N}(du).$$

Sum up (21) and (23) and then take the inner product of this with $S_\rho X$. Its expectation yields the following.

Proposition [12] (*Analogue of the formula of integration by parts*) For any X we have

$$E[(X, \partial f(F))] = E[(Q + \tilde{Q}_\rho)^*(S_\rho X) f(F)] \quad (24)$$

$$- \frac{1}{\varphi(\rho)} E \left[\left(X, S_\rho \int_{A(\rho)} \tilde{D}_u F (\tilde{D}_u F)^T \left(\int_0^1 \{ \partial f(F + \theta \tilde{D}_u F) - \partial f(F) \} d\theta \right) \hat{N}(du) \right) \right].$$

Here $S_\rho = (R + \tilde{R}_\rho)^{-1}$.

Remark. If there is no Poisson part in (15), then the formula is written as

$$E[(X, \partial f(F))] = E[Q^*(R^{-1}X) f(F)] = E[\delta((R^{-1}X, DF)) f(F)]. \quad (25)$$

On the other hand, if $\tilde{R}_\rho$ is not zero or equivalently $\tilde{Q}_\rho$ is not zero, we have a remaining term (the last term of (15)). We have this term even if Z_t is a simple Poisson process N_t or its sums. However, if we take $f(x) = e^{i(w,x)}$, $w \in \mathbb{R}^d \setminus \{0\}$, we have $\partial f(x) = ie^{i(w,x)}w$ and

$$e^{i(w, F + \theta \tilde{D}_u F)} - e^{i(w, F)} = e^{i(1-\theta)(w, F)} \tilde{D}_u (e^{i(w, \theta F)}).$$

Hence we have an expression of the integration-by-parts for the functional

$$E[(X, w) \partial_x (e^{i(w, F)})] = E[(Q^* + \tilde{Q}_\rho^* + R_{\rho, w}^*) S_\rho X \cdot e^{i(w, F)}], \quad \forall w. \quad (26)$$

Here

$$R_{\rho, w}^* Y = - \frac{i}{\varphi(\rho)} \int_0^1 \left(\tilde{\delta}(e^{i(1-\theta)(w, F)}) \chi_\rho \tilde{D} F (\tilde{D} F)^T Y, e^{i(\theta-1)(w, F)} w \right) d\theta.$$

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