

# Continuous semigroup structures on $\mathbb{R}$

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Let  $I$  be a real interval. A *semigroup*  $S$  on  $I$  is a semigroup  $S = (I, *)$  such that the operation  $* : I \times I \rightarrow I$  is continuous with respect to the ordinary topology and compatible with the ordinary order in  $\mathbb{R}$ ;

$$x \leq y \Rightarrow x * z \leq y * z, z * x \leq z * y$$

for  $x, y, z \in S$ .

The following is classical.

**Theorem 1** (Abel 1826 [1], Aczél 1949 [2]). *Any group on  $\mathbb{R}$  is isomorphic to  $(\mathbb{R}, +)$ .*

Two topological ordered semigroups  $(S, *)$  and  $(S', *')$  are *equivalent* if there is a homeomorphism  $f : S \rightarrow S'$  which is a homomorphism or an anti-homomorphism and is order-preserving or order-reversing.

**Question 1.** *How many non-equivalent semigroups on  $\mathbb{R}$  ?*

For  $a \in \mathbb{R} \cup \{-\infty\}$  and  $b \in \mathbb{R} \cup \{+\infty\}$ ,  $I(a, b)$  (resp.  $I[a, b]$ ,  $I(a, b]$ ,  $I[a, b]$ ) denotes the open (resp. closed, half-open) interval between  $a$  and  $b$  in  $\mathbb{R}$ .

The following result means that there are exactly three non-equivalent cancellative semigroups on  $\mathbb{R}$ .

**Theorem 2** (Craig and Pales 1989 [5]). *There are exactly three non-equivalent cancellative semigroups on  $\mathbb{R}_+ = I(0, +\infty)$ . They are  $(\mathbb{R}_+, \times)$ ,  $(\mathbb{R}_+, +)$  and  $(\mathbb{R}_+, \star)$ , where  $\star$  is given by*

$$x \star y = x + y + 1.$$

Let  $S = (S, *)$  be a semigroup. For  $x \in S$  and  $n \in \mathbb{N}$ ,  $x^{n*}$  denotes the  $n$ -th power of  $x$  with respect to  $*$ . The *order* of  $x$  (denoted by  $\text{ord}(x)$ ) is the least  $n$  such that  $x^{n*} = x^{(n+1)*}$ . If there is no such  $n$ ,  $\text{ord}(x) = \infty$ .

A semigroup  $S$  is *nil* if it has a zero  $z$  ( $z * x = x * z = 0$  for all  $x \in S$ ) and for every  $x \in S$  there is  $n > 0$  such that  $x^{n*} = z$ .

Let  $S$  be an ordered semigroup. An element  $x \in S$  is *positive* (resp. *negative*, *idempotent*) if

$$x * x > x \quad (\text{resp. } x * x < x, x * x = x).$$

A subset of  $S$  is *positive* (resp. *negative*) if all its elements are positive (resp. negative). Let  $P$  (resp.  $Q$ ,  $E$ ) denotes the set of positive (resp. negative, idempotent) elements of  $S$ . Clearly,

$$S = P \cup Q \cup E \quad (\text{disjoint union}).$$

$S$  is *positively* (resp. *negatively*) *Archimedean*, if it is positive (resp. negative) and for any  $x, y \in S$  there is  $n > 0$  such that  $y < x^{n*}$  (resp.  $y > x^{n*}$ ).

A nil semigroup is *positive* (resp. *negative*), if all elements other than zero are positive (resp. negative).

From now on,  $S = (\mathbb{R}, *)$  is a semigroup on  $\mathbb{R}$ .

**Lemma 3.**  $P$  and  $Q$  are open subsets of  $\mathbb{R}$  and  $E$  is a closed subset of  $\mathbb{R}$ .

**Lemma 4.** For  $x \in P$  (resp.  $x \in Q$ ), the limit  $\lim_{n \rightarrow \infty} x^{n*}$  is  $+\infty$  (resp.  $-\infty$ ) if  $\text{ord}(x) = \infty$ , and the limit converges to an idempotent of  $S$  if  $\text{ord}(x) < \infty$ .

Let  $e \in E \cup \{-\infty\}$  and  $f \in E \cup \{+\infty\}$  such that  $e < f$ .

**Lemma 5.**  $I[e, f] = \{x \in \mathbb{R} \mid e \leq x \leq f\}$  is a subsemigroup of  $S$ .

The open interval  $I(e, f)$  is called *tube* if it contains no idempotent.

**Lemma 6.** A tube is either positive or negative.

**Proposition 7.** Let  $I(e, f)$  be a tube with  $e \in E \cup \{-\infty\}$  and  $f \in E \cup \{+\infty\}$ .

(1) If it is positive, then either  $I(e, f)$  is a positively Archimedean semigroup, or  $f \in E$  and  $I(e, f) \cup \{f\}$  is a nil semigroup with zero  $f$ .

(2) If it is negative, then either  $I(e, f)$  is a negatively Archimedean semigroup, or  $e \in E$  and  $I(e, f) \cup \{e\}$  is a nil semigroup with zero  $e$ .

Suppose that  $I = I(e, f)$  is positively Archimedean, that is, for any  $x, y \in I$  there is  $n > 0$  such that  $y < x^{n*}$ .

For a fixed  $a \in S$  define a real function  $\phi_a : I \rightarrow \mathbb{R}$  by

$$\phi_a(x) = \inf\{m/n \mid m, n \in \mathbb{N}, m > 0, n > 0, x^{n*} \leq a^{m*}\}.$$

for  $x \in S$ . We call  $\phi_a$  the *standard function based on  $a$* , and is classical for Archimedean ordered semigroups (see Fuchs [6], Hölder [7]).

**Theorem 8.** The function  $\phi_a$  is an order-preserving continuous homomorphism of semigroups from  $(I, *)$  into  $(\mathbb{R}_+, +)$ .

Define

$$\mu_a = \inf\{\phi_a(x) \mid x \in I\}.$$

**Lemma 9.** We have

$$0 \leq \mu_a \leq 1.$$

**Lemma 10.**  $\mu_a = 0$  if and only if  $e \neq -\infty$  or  $\inf \{x * x \mid x \in I\} = -\infty$ .

Define

$$\tau = \inf \{x * x \mid x \in I, \phi_a(x) > \mu_a\}.$$

**Lemma 11.**  $\tau = e$  if and only if  $\mu_a = 0$ .

**Theorem 12.**  $\phi_a$  is strictly increasing on  $I(\tau, f)$ .

**Corollary 13.** If  $e \neq -\infty$ , or  $\inf \{x * x \mid x \in I\} = -\infty$ , then  $(I, *)$  is isomorphic to  $(\mathbb{R}_+, +)$ .

More generally, we can characterize Archimedean tubes using Theorems 8 and 12 (a different approach is given in Storey [8]). Using the characterization, we can show that there are uncountably many non-equivalent Archimedean semigroups on  $\mathbb{R}$ . There are uncountably many non-equivalent nil semigroups on  $\mathbb{R}$  too, but a complete characterization of nil semigroups on  $\mathbb{R}$  seems to be difficult.

A closed interval  $I = I[e, f]$  for  $e, f \in E$  with  $e \leq f$  is called a *joint* if it is connected, included in  $E$  and maximal (no such  $J$  strictly includes  $I$ ). If  $I$  is a joint,  $(I, *)$  is an idempotent semigroup (band).

$S = (\mathbb{R}, *)$  consists of positive tubes, negative tubes and joints. To classify semigroups on  $\mathbb{R}$ , we need to describe all possible combinations of them, but it seems a very hard problem. Similar problems for more general structures called threads are studied by Clifford [3, 4]. The following is a certain special result in this context.

An ordered group  $G$  with identity element  $e$  is *Archimedean*, if  $\{x \in G \mid x > e\}$  is a positively Archimedean semigroup and  $\{x \in G \mid x < e\}$  is a negatively Archimedean semigroup.

**Theorem 14.** Let  $e, f, g \in E \cup \{-\infty, +\infty\}$  with  $f < e < g$ . Suppose that  $I(e, g)$  is a positive tube and  $I(f, e)$  is a negative tube. Then,  $(I(f, g), *)$  is an Archimedean group with the identity element  $e$ , and it is isomorphic to  $(\mathbb{R}, +)$ .

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