

Sufficient conditions for Teichmüller modular groups to be of the second kind

Ege Fujikawa

Department of Mathematics, Tokyo Institute of Technology

藤川 英華 (東京工業大学大学院 理工学研究科数学専攻)

1 Introduction

We consider the action of the reduced Teichmüller modular group $\text{Mod}^\#(R)$ for a hyperbolic Riemann surface R , which is a group of automorphisms of the reduced Teichmüller space $T^\#(R)$. If R is of analytically finite type, the reduced Teichmüller modular group is nothing but the ordinary Teichmüller modular group $\text{Mod}(R)$, and it is well known that $\text{Mod}(R)$ acts properly discontinuously on $T(R)$. However, if R is of infinite type, $\text{Mod}^\#(R)$ does not act properly discontinuously on $T^\#(R)$, in general. On the basis of this fact, in [1], we have introduced new notions, the limit set and the region of discontinuity for a Teichmüller modular group as an analogy to the theory of the Kleinian groups acting on the Riemann sphere.

Definition 1 We say that a point p in $T^\#(R)$ is a *limit point* for a subgroup G of $\text{Mod}^\#(R)$ if there exist a point $q \in T^\#(R)$ and a sequence $\{\chi_n\}$ of distinct elements of G such that $\lim_{n \rightarrow \infty} d_T(\chi_n(q), p) = 0$. The set of the limit points is called the *limit set* of G , and denoted by $\Lambda(G)$. The complement $T^\#(R) - \Lambda(G)$ of the limit set is denoted by $\Omega(G)$, and called the *region of discontinuity* of G .

Similarly, for a subgroup G of the ordinary modular group $\text{Mod}(R)$, we can define $\Lambda(G)$ and $\Omega(G)$ in $T(R)$. For a Riemann surface R of analytically finite type, we have $\Lambda(\text{Mod}(R)) = \Lambda(\text{Mod}^\#(R)) = \emptyset$. On the other hand, for a Riemann surface R whose Fuchsian model is of the second kind, we always have $\Omega(\text{Mod}(R)) = \emptyset$. This is the reason why we consider the reduced modular group $\text{Mod}^\#(R)$, not the ordinary modular group $\text{Mod}(R)$, for a Riemann surface R of infinite type.

Definition 2 For a subgroup G of $\text{Mod}^\#(R)$, we say that G is of the first kind if $\Omega(G) = \emptyset$, and otherwise of the second kind.

In this note, we focus our attention on normal covering surfaces of Riemann surfaces, and consider sufficient conditions for Teichmüller modular groups to be of second kind.

2 Sufficient conditions

Throughout this note, we assume that a Riemann surface R has the non-abelian fundamental group. In [1] and [2], we have shown sufficient conditions of Riemann surfaces R for $\text{Mod}^\#(R)$ to be of the first kind or of the second kind, which are stated below.

Definition 3 For a given $M > 0$, we say that a point p of R belongs to a subset R_M of R if there exists a non-trivial simple closed curve c_p containing p such that the hyperbolic length of c_p is less than M .

Definition 4 We say that R satisfies the *lower bound condition* if there exists an $\epsilon > 0$ such that R_ϵ consists only of cusp neighborhoods. Further we say that R satisfies the *upper bound condition* if there exist a constant $M > 0$ and a connected component R_M^* of R_M such that a homeomorphism of $\pi_1(R_M^*)$ to $\pi_1(R)$ that is induced by the inclusion map of R_M^* into R is surjective.

Theorem 1 ([1]) *If R does not satisfy the lower bound condition, then $\text{Mod}^\#(R)$ is of the first kind.*

Theorem 2 ([1]) *If R satisfies the lower and upper bound conditions, then $\text{Mod}^\#(R)$ is of the second kind.*

In connection with these results, we have stated the following conjecture in [1].

Conjecture *If R satisfies the lower bound condition, then $\text{Mod}^\#(R)$ is of the second kind. That is, considering Theorem 1, we conjecture that $\text{Mod}^\#(R)$ is of the second kind if and only if R satisfies the lower bound condition.*

In [2], we have proved a partial solution of this conjecture, giving a weaker condition than the upper bound condition.

Theorem 3 ([2]) *Let R be a Riemann surface that satisfies the following two conditions:*

1. *R satisfies the lower bound condition.*
2. *There exists a constant $M > 0$ such that, for any connected component V of the complement of R_M , V is either simply or doubly connected and $R - \bar{V}$ consists of finitely many connected components.*

Then $\text{Mod}^\#(R)$ is of the second kind.

Remark 1 The upper and lower bound conditions and the conditions in Theorem 3 are quasiconformally invariant. Then these conditions are regarded as conditions for the Teichmüller space.

3 Normal covering surfaces

Throughout this section, let $\tilde{R}$ be a normal covering surface of a Riemann surface R which is not a universal cover. First, we have given a sufficient condition for $\text{Mod}^\#(\tilde{R})$ to be of the second kind.

Proposition 1 ([1]) *If R is of analytically finite type, then $\tilde{R}$ satisfies the upper and lower bound conditions. Thus, by Theorem 2, $\text{Mod}^\#(\tilde{R})$ is of the second kind.*

A simple example of this proposition is as follows.

Example 1 *Let*

$$\tilde{R} = \mathbb{C} - \{n \mid n \in \mathbb{Z}\}.$$

Then $\text{Mod}^\#(\tilde{R})$ is of the second kind. Indeed, $\tilde{R}$ is a normal covering surface of $\mathbb{C} - \{0, 1\}$, which is an analytically finite Riemann surface.

In connection with Proposition 1, we have the following.

Proposition 2 ([3]) *If $\tilde{R}$ satisfies the lower and upper bound conditions, then R also satisfies these conditions.*

Remark 2 In [3], we have considered a weakened upper bound condition. The same proof can be applied also for Riemann surfaces that satisfy the (generalized) upper bound condition in Definition 4.

From Theorem 2 and Proposition 2, we have a condition of a normal covering surface $\tilde{R}$ for $\text{Mod}^\#(R)$ to be of the second kind.

Proposition 3 *If $\tilde{R}$ satisfies the lower and upper bound conditions, then $\text{Mod}^\#(\tilde{R})$ and $\text{Mod}^\#(R)$ are of the second kind.*

If $\tilde{R}$ does not satisfy the upper bound condition, then Proposition 3 is not necessarily true.

Example 2 Let

$$\tilde{R} = \mathbf{C} - \bigcup_{n=1}^{\infty} \bigcup_{m \in \mathbf{Z}} \left\{ \frac{m}{n} \pm n^2 \sqrt{-1} \right\},$$

and $R = \tilde{R}/\langle f \rangle$, where $f(z) = z + 1$. Although $\tilde{R}$ does not satisfy the upper bound condition, $\tilde{R}$ satisfies the conditions in Theorem 3. Then $\text{Mod}^\#(\tilde{R})$ is of the second kind. On the other hand, R satisfies the upper bound condition, but does not satisfy the lower bound condition. Then, from Theorem 1, $\text{Mod}^\#(R)$ is of the first kind.

That $\tilde{R}$ satisfies the upper bound condition is not a necessary condition for both $\text{Mod}^\#(\tilde{R})$ and $\text{Mod}^\#(R)$ to be of the second kind.

Example 3 Let

$$\tilde{R} = \mathbf{C} - \bigcup_{n=1}^{\infty} \bigcup_{m \in \mathbf{Z}} \left\{ \frac{m}{n} + (2n+1)\sqrt{-1} \right\},$$

and $R = \tilde{R}/\langle f \rangle$, where $f(z) = z + 1$. Although $\tilde{R}$ does not satisfy the upper bound condition, $\tilde{R}$ satisfies the conditions in Theorem 3. Then $\text{Mod}^\#(\tilde{R})$ is of the second kind. Further, R satisfies the lower and upper bound conditions. Then, by Theorem 2, $\text{Mod}^\#(R)$ is also of the second kind.

In the last of this note, we give a following problem.

Problem If $\text{Mod}^\#(\tilde{R})$ is of the first kind, then so is $\text{Mod}^\#(R)$.

It is clear that, if $\tilde{R}$ does not satisfy the lower bound condition, then neither does R . Hence, from Theorem 1, if $\tilde{R}$ does not satisfy the lower bound condition, then $\text{Mod}^\#(\tilde{R})$ and $\text{Mod}^\#(R)$ are of the first kind. Further, if the conjecture, which is stated in the previous section, is true, then the problem is solved affirmatively.

References

- [1] E. Fujikawa, Limit sets and regions of discontinuity of Teichmüller modular groups, preprint.
- [2] E. Fujikawa, Teichmüller modular groups with non-empty regions of discontinuity, preprint.
- [3] E. Fujikawa, The order of conformal automorphisms of Riemann surfaces of infinite type, preprint.