数理解析研究所講究録
969巻 1996年 86-93

Singularities of the Bergman kernel for certain weakly pseudoconvex domains

By *Joe Kamimoto* (Univ. of Tokyo)

Let Ω be a bounded domain with smooth boundary in $\mathbf{C}^n$, $B(\Omega)$ the set of holomorphic L^2-functions on Ω. It is well-known that $B(\Omega)$ is a closed linear subspace of the Hilbert space $L^2(\Omega)$. The *Bergman kernel* $K^B(z)$ of the domain Ω is defined by

$$K^B(z) = \sum_j |\phi_j(z)|^2,$$

where $\{\phi_j\}$ is a complete orthonormal basis for $B(\Omega)$. The above series converges uniformly on any compact subset of Ω. It is very important to investigate the singularities of $K^B(z)$. This is mainly because they contain much information about the analytic and geometric invariants of the domain Ω.

First we consider the case where Ω is a strongly pseudoconvex domain. In this case C. Fefferman [13], L. Boutet de Monvel and J. Sjöstrand [5] obtained the following asymptotic expansion for $K^B(z)$:

$$K^B(z) = \frac{\varphi^B(z)}{r(z)^{n+1}} + \psi^B(z) \log r(z), \tag{1}$$

where r is a defining function of Ω, i.e., $\Omega = \{z \in \mathbf{C}^n; r(z) > 0\}$ and $\mathrm{grad} r(z) \neq 0$ on $\partial\Omega$. The functions $\varphi^B(z)$ and $\psi^B(z)$ can be expressed as a power series of r. If the boundary $\partial\Omega$ is real analytic, then their series are convergent. From the viewpoint of ordinary differential equations, this result may be interpreted that the Bergman kernel of a strongly pseudoconvex domain has the singularities of *regular singular type*.

Next we proceed to the case of weakly pseudoconvex domain of finite type (in the sense of J. J. Kohn [24] or J. P. D'Angelo [9]). In this case there is no such strong general result that is comparable with (1) in the strongly pseudoconvex case ; yet there are many detailed results for specific domains. We refer to [2],[8],[16],[10],[14],[17] for explicit computations, to [18],[35],[12],[6],[19],[20] for estimates of the size and to [3] for boundary limits on non tangential cone. Especially D. Catlin [6] and G. Herbort [20] gave precise estimates of $K^B(z)$ from above and below for certain class of domains whose degenerate rank of the Levi form equals one. In general, however, the singularities of $K^B(z)$ are so complicated that a unified treatment of them seems to be difficult.

In this article, we pick up the specific domains

$$\mathcal{E}_m = \left\{ z = (z_1, \dots, z_n) \in \mathbf{C}^n \,;\, \sum_{j=1}^{n} |z_j|^{2m_j} < 1 \right\}, \tag{2}$$

where $m = (m_1, \dots, m_n) \in \mathbf{N}^n$ and $m_n \neq 1$, to clarify what is happening for the weakly pseudoconvex domains of finite type. Since $\mathcal{E}_m$ is a Reinhardt domain, the set of (normalized) monomials forms a complete orthonormal basis for $B(\mathcal{E}_m)$. Hence $K^B(z)$ can be represented by a convergent power series of $(|z_1|^2, \dots, |z_n|^2)$, whose coefficients were explicitly computed in [21],[8],[4]. A. Bonami and N. Lohoué [4] gave an important integral representation for the Bergman kernel $K^B(z)$ of $\mathcal{E}_m$. From this representation they deduced a detailed information about the singularities of $K^B(z)$, though their result is yet to be improved.

From our point of view, we briefly review the result of [4]. Let $z^0 = (z_1^0, \dots, z_n^0) \in \partial\mathcal{E}_m$ be any boundary point of $\mathcal{E}_m$, $k \in \mathbf{Z}_{\geq 0}$ the degenerate rank of the Levi form at z^0. We say that z^0 is a *strongly* (resp. *weakly*) *pseudoconvex point* if $k = 0$ (resp. if $k > 0$). Let I, P and Q be the subsets of $N = \{1, \dots, n\}$ defined by

$$\begin{cases} I &= \{j \in N; m_j = 1\}, \\ P &= \{j \in N; z_j^0 = 0\} \setminus I, \\ Q &= \{j \in N; z_j^0 \neq 0\} \cup I. \end{cases}$$

Then the degenerate rank k equals the cardinality $|P|$ of P. One of the main results in [4] (p.181) states that the restriction of $K^B(z)$ to the subset $V = \{z_j = 0; j \in P\}$ admits the following expression around z^0:

$$K^B(z) = C_P^B \, \frac{\prod_{j\in Q} m_j^2 |z_j|^{2m_j-2}}{\{1 - \sum_{j\in Q} |z_j|^{2m_j}\}^{|Q|+|\frac{1}{m}|_P+1}} \; + \; O(1), \tag{3}$$

where C_P^B is a positive constant and $|\frac{1}{m}|_P = \sum_{j\in P}\frac{1}{m_j}$. The formula (3) is quite explicit, but still weak in the sense that it is valid only on the thin set V, and that the error term $O(1)$ is somewhat too loose.

Besides [4] there are some studies on the Bergman kernel (or Szegö kernel) of the domain $\mathcal{E}_m$ ([2],[21],[8],[15]). In the case $m = (1,\ldots,1,m)$, explicit expressions for $K^B(z)$ are computed ([2],[8],[4]), while there seems to be no explicit one for general m. Recently, N. W. Gebelt [15] generalized the method of producing the asymptotic expansion (1) due to Fefferman [13] to the weakly pseudoconvex case of $\mathcal{E}_m$ and obtained the analogous results about $K^B(z)$ of $\mathcal{E}_m$ $(m = (1,\ldots,1,m))$.

Now we state our main results. Our essential idea is to introduce the new variables (t,r), which we call the *polar coordinates* around z^0. Here $t = (t_j)_{j\in P}$ is defined by

$$t_j(z)^{2m_j} = \frac{|z_j|^{2m_j}}{1-\sum_{j\in Q}|z_j|^{2m_j}} \qquad (j \in P),$$

and r is the defining function of $\mathcal{E}_m$, i.e.,

$$r(z) = 1 - \sum_{j=1}^{n}|z_j|^{2m_j}.$$

We call t the *angular variables* and r the *radial variable*, respectively. Then the map $F : z \mapsto (t,r)$ takes $\mathcal{E}_m$ onto the region:

$$D = \left\{(t,r) \in \mathbf{R}^{|P|}\times(0,1]\,;\, t_j \geq 0, \sum_{j\in P} t_j^{2m_j} \leq 1-r\right\}.$$

The accumulation points of $F(z)$ as $\mathcal{E}_m \ni z \to z^0$ are precisely those points which belong to the set $\{0\}\times\overline{\Delta}$, where $\overline{\Delta}$ is the closure of the *locally closed* simplex:

$$\Delta = \left\{t = (t_j)_{j\in P}\,;\, t_j \geq 0, \sum_{j\in P} t_j^{2m_j} < 1\right\}.$$

Let $G = U \cap K$ be a locally closed subset of an Euclidean space, where U is open and K is closed, respectively. Then we say that $f \in C^\omega(G)$ if f is a real analytic function on some open neighborhood V of G in U, where V may depend on f.

The following theorem asserts that the asymptotic behavior of K^B as $\mathcal{E}_m \ni z \to z^0$ can be expressed most conveniently in terms of the polar coordinates (t,r).

THEOREM 1 *There is a function* $\Phi^B(t) \in C^\omega(\Delta)$ *such that*

$$K^B(z) \equiv \frac{n!}{\pi^n} \prod_{j\in Q} m_j^2 |z_j|^{2m_j-2} \frac{\Phi^B(t(z))}{r(z)^{|Q|+|\frac{1}{m}|_P+1}} \qquad \text{modulo } C^\omega(\{z^0\}). \tag{4}$$

Here $\Phi^B(t)$ *satisfies* (i) *or* (ii).

(i) *If* z^0 *is a strongly pseudoconvex point (i.e.*$P = \emptyset$*),* $\Phi^B(t) = 1$ *identically.*

(ii) *If* z^0 *is a weakly pseudoconvex point (i.e.* $P \neq \emptyset$*), then* $\Phi^B(t)$ *is positive on* Δ *and is unbounded as* $t \in \Delta$ *approaches* $\overline{\Delta} \setminus \Delta$.

Remark. The function $\Phi^B(t)$ is essentially the Laplace transform of a certain auxiliary function expressible in terms of Mittag-Leffler's function, i.e.

$$\Phi^B(t) = \frac{1}{n!}\Big[1 - \sum_{j\in P} t_j^{2m_j}\Big]^{|Q|+|\frac{1}{m}|_P+1} \int_0^\infty e^{-s} \prod_{j\in P} F_{m_j}(t_j^2 s^{\frac{1}{m_j}}) s^{|Q|+|\frac{1}{m}|_P} ds, \tag{5}$$

where

$$F_m(u) = m \sum_{\nu=0}^{\infty} \frac{u^\nu}{\Gamma(\frac{\nu}{m} + \frac{1}{m})}.$$

We mention a few implications of the formula (4) in order to compare it with the known results stated previously. First, if z^0 is a strongly pseudoconvex point, i.e. $P = \emptyset$, then the angular variables t do not appear and $\Phi^B(t) = 1$ identically, and therefore (4) reproduces the asymptotic expansion (1) due to C. Fefferman [13], L. Boutet de Monvel and J. Sjöstrand [5]. We remark that the logarithmic term in (1) does not appear in the present case. Secondly, the restriction of (4) to the subset V is just the substitution $t(z) = 0$ into (4), which induces the formula (3) with the error term $O(1)$ replaced by a real analytic function. Thus the formula (4) improves that of Bonami and Lohoué [4] in the sense that it is valid in a wider domain and that the error term is more accurate.

From the above theorem we consider the behavior of $K^B(z)$ at a weakly pseudoconvex point from the following three angles: (a) *estimate*, (b) *boundary limit* and (c) *asymptotic formula*. We assume z^0 is a weakly pseudoconvex point and define the region $\mathcal{U}_\alpha(z^0)(= \mathcal{U}_\alpha)$ by

$$\mathcal{U}_\alpha = \left\{ z \in \mathcal{E}_m; \sum_{j\in P} t_j(z) = \frac{\sum_{j\in P} |z_j|^{2m_j}}{1 - \sum_{j\in Q} |z_j|^{2m_j}} < \frac{1}{\alpha} \right\} \quad (\alpha > 1).$$

(a) By the boundedness of $\Phi^B(t)$ in (ii), we can precisely estimate the size of $K^B(z)$ on $\mathcal{U}_\alpha$. The region $\mathcal{U}_\alpha$ reminds us of the *admissible approach regions* considered in [33],[34],[27],[28],[1] etc. (b) The limit of $K^B(z) \cdot r(z)^{|Q|+|\frac{1}{m}|p+1}$ as $z \to z^0$ on each $\mathcal{U}_\alpha$ is not determined uniquely but depends on the angular variables t. Note that this boundary limit is uniquely determined on any nontangential cone. (c) In view of (1.4) the polar coordinates (t, r) is necessary to understand the asymptotic formula of $K^B(z)$ at z^0. This fact may be interpreted that the Bergman kernel has a singularity of irregular singular type at a weakly pseudoconvex point. The degeneration from the strong pseudoconvexity to the weak pseudoconvexity corresponds to the process of confluence from the regular singularity to the irregular singularity ([30],[29]).

In more detail we investigate the structure of singularities of the Bergman kernel of $\mathcal{E}_m$. The singularities of $\Phi^B(t)$ at $\overline{\Delta} \setminus \Delta$. $\Phi^B(t)$ can also be expressed in a similar form to (4) by introducing new polar coordinates on the simplex Δ. Through the finite recursive procedure of this type we can completely understand the structure of the singularities of $K^B(z)$.

Acknowledgment I would like to express my deepest gratitude to Katsunori Iwasaki for his kind help during the preparation of this article.

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