

On a semilinear evolution equation

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1 Problem

We shall consider a semilinear parabolic evolution equation arising from fluid mechanics. More specifically, we deal with the following system of equations of the motion of incompressible micropolar fluids.

$$\begin{aligned}\frac{\partial u}{\partial t} - \mu \Delta u + (u \cdot \nabla)u + \nabla p &= f + 2\chi \nabla \times \left(\omega - \frac{1}{2} \nabla \times u \right), \\ \frac{\partial \omega}{\partial t} - \alpha \Delta \omega - \beta \nabla (\nabla \cdot \omega) + (u \cdot \nabla)\omega &= g - 4\chi \left(\omega - \frac{1}{2} \nabla \times u \right), \\ \nabla \cdot u &= 0,\end{aligned}$$

where the unknown functions are the velocity vector field u , the microrotation vector ω and the pressure p , and f and g are given vector fields, α, β, μ, χ are constants satisfying $\alpha > 0$, $\alpha + \beta > 0$, $\mu > 0$ and $\chi > 0$.

Micropolar fluid model is one of generalizations of the classical Navier-Stokes model. We shall discuss the solvability of the initial-boundary value problem of this system of equations in a bounded domain Ω with smooth boundary $\partial\Omega$ in $\mathbb{R}^3$.

As for the boundary conditions, we assume that

$$u|_{\partial\Omega} = 0, \quad \omega|_{\partial\Omega} = \frac{\theta}{2} \nabla \times u \Big|_{\partial\Omega},$$

where θ is a constant belonging to the interval $[0, 1]$ (see [1], [3]).

For the case where $\theta = 0$, i.e., both u and ω yield the homogeneous Dirichlet boundary conditions, there have been many results concerning this system of equations [4]. On the other hand, the case where $\theta \neq 0$ is not fully pursued yet. Furthermore, this boundary condition is not based on any physical principles, the well-posedness of the initial-boundary value problem is not clear. That is why we consider this problem.

Let us introduce a new variable $v := \omega - \frac{\theta}{2} \nabla \times u$ and change the variables (u, ω) to (u, v) . Then the original system of equations for u and ω will be

rewritten in the following form.

$$\begin{aligned}
\frac{\partial u}{\partial t} - (\mu + \chi)\Delta u + (u \cdot \nabla)u + \nabla p &= f + 2\chi\nabla \times v - \theta\chi\Delta u, \\
\frac{\partial v}{\partial t} - \alpha\Delta v - \beta\nabla(\nabla \cdot v) + (u \cdot \nabla)v + 4\chi v + \theta\chi\nabla \times (\nabla \times v) \\
&= g - \frac{\theta}{2}\nabla \times f - \frac{\theta}{2}\{(\mu + (1 - \theta)\chi - \alpha)\nabla \times \Delta u \\
&\quad + ((\nabla \times u) \cdot \nabla)u\} + 2(1 - \theta)\chi\nabla \times u, \\
\nabla \cdot u &= 0, \\
u|_{\partial\Omega} &= 0, \quad v|_{\partial\Omega} = 0, \\
u(\cdot, 0) &= u_0(\cdot), \quad v(\cdot, 0) = v_0(\cdot).
\end{aligned}$$

Notice that the boundary condition for the new unknown function v is reduced to the homogeneous boundary condition.

Our aim is to show the existence of a solution (u, v) local in time to this system of equations in the L^2 -framework.

2 Function spaces and operators

We refer the details of mathematical facts to be mentioned below to the books [2], [4] and [5].

Let $\mathbb{C}_\sigma^\infty(\Omega)$ be the set of all smooth solenoidal vector functions in Ω with compact support, and $\mathbb{L}_\sigma^2(\Omega)$ and $\mathbb{H}_\sigma^1(\Omega)$ the closure of $\mathbb{C}_\sigma^\infty(\Omega)$ in $\mathbb{L}^2(\Omega)$ and in $\mathbb{H}^1(\Omega)$ respectively. We denote by $|\cdot|$ the norm of $\mathbb{L}^2(\Omega)$, $(\cdot, \cdot)$ the inner product of $\mathbb{L}^2(\Omega)$; $|\cdot|_\sigma$ the norm of $\mathbb{L}_\sigma^2(\Omega)$, $(\cdot, \cdot)_\sigma$ the inner product of $\mathbb{L}_\sigma^2(\Omega)$; $\|\cdot\|$ the norm of $\mathbb{H}_0^1(\Omega)$, $((\cdot, \cdot))$ the inner product of $\mathbb{H}_0^1(\Omega)$; $\|\cdot\|_\sigma$ the norm of $\mathbb{H}_\sigma^1(\Omega)$, $((\cdot, \cdot))_\sigma$ the inner product of $\mathbb{H}_\sigma^1(\Omega)$. Due to the Poincaré inequality, we equip $\mathbb{H}_\sigma^1(\Omega)$ and $\mathbb{H}_0^1(\Omega)$ with the norm

$$\begin{aligned}
\|u\|_\sigma &:= \left(\int_\Omega |\nabla u|^2 dx \right)^{1/2}, \\
\|v\| &:= \left(\int_\Omega |\nabla v|^2 dx \right)^{1/2}
\end{aligned}$$

for $u \in \mathbb{H}_\sigma^1(\Omega)$ and $v \in \mathbb{H}_0^1(\Omega)$, respectively.

The orthogonal projection from $\mathbb{L}^2(\Omega)$ to $\mathbb{L}_\sigma^2(\Omega)$ is denoted by P . The Stokes operator A is defined by $A := -P\Delta$ with $D(A) = \mathbb{H}^2(\Omega) \cap \mathbb{H}_\sigma^1(\Omega)$.

Let $\langle \cdot, \cdot \rangle$ denote the duality pairing between $\mathbb{H}^{-1}(\Omega)$ and $\mathbb{H}_0^1(\Omega)$. The differential operator L , which maps $\mathbb{H}_0^1(\Omega)$ to $\mathbb{H}^{-1}(\Omega)$, is defined by

$$\langle Lv, w \rangle := \alpha \int_\Omega \nabla v \cdot \nabla w dx + \beta \int_\Omega (\nabla \cdot v)(\nabla \cdot w) dx + 4\chi \int_\Omega v w dx \quad (1)$$

for all $v, w \in \mathbb{H}_0^1(\Omega)$. By integration by parts, we obtain

$$\int_\Omega |\nabla w|^2 dx = \int_\Omega |\nabla \cdot w|^2 dx + \int_\Omega |\nabla \times w|^2 dx \quad (2)$$

for $w \in \mathbb{H}_0^1(\Omega)$. Hence there are positive constants α_* and α^* such that

$$\alpha_* |\nabla w|^2 + 4\chi |w|^2 \leq \langle Lw, w \rangle \leq \alpha^* |\nabla w|^2 + 4\chi |w|^2 \quad (3)$$

for $w \in \mathbb{H}_0^1(\Omega)$.

For $v \in \mathbb{L}^2(\Omega)$, we also define $\nabla \times v \in \mathbb{H}^{-1}(\Omega)$ by

$$\langle \nabla \times v, w \rangle := \int_{\Omega} v \cdot \nabla \times w dx \quad (4)$$

with $w \in \mathbb{H}_0^1(\Omega)$.

We identify the space $\mathbb{L}^2(\Omega)$ with its dual. Then $\|\nabla \times v\|_* \leq |v|$ holds for $v \in \mathbb{L}^2(\Omega)$, where $\|\cdot\|_*$ stands for the norm of the space $\mathbb{H}^{-1}(\Omega)$.

3 Main result

In the above settings, our system can be regarded as a system of abstract evolution equations of the following form.

$$\begin{aligned} \frac{du}{dt} + (\mu + \chi)Au &= Pf + b_1(u, v) \quad \text{in } \mathbb{L}_{\sigma}^2(\Omega), \\ \frac{dv}{dt} + Lv &= g - \frac{\theta}{2} \nabla \times f + b_2(u, v) \quad \text{in } \mathbb{H}^{-1}(\Omega), \\ u(0) &= u_0, \quad v(0) = v_0, \end{aligned}$$

where $b_1(u, v)$ and $b_2(u, v)$ are defined by

$$\begin{aligned} b_1(u, v) &:= 2\chi \nabla \times v + \theta \chi Au - P(u \cdot \nabla)u, \\ b_2(u, v) &:= -\frac{\theta}{2} \{ (\mu + (1 - \theta)\chi - \alpha) \nabla \times \Delta u + ((\nabla \times u) \cdot \nabla)u \} \\ &\quad - (u \cdot \nabla)v + 2(1 - \theta)\chi \nabla \times u. \end{aligned}$$

Notice that $\nabla \times w \in \mathbb{L}_{\sigma}^2(\Omega)$ whenever $w \in \mathbb{H}_0^1(\Omega)$ since $C_c^{\infty}(\Omega)$ is dense in $\mathbb{H}_0^1(\Omega)$ and if $w_0 \in C_c^{\infty}(\Omega)$ and $q \in \mathbb{H}^1(\Omega)$, the integration by parts gives

$$(\nabla \times w_0, \nabla q) = (-\nabla \cdot (\nabla \times w_0), q) = 0.$$

Now our main result reads as follows.

Theorem *There exists a constant $\theta_0 \in (0, 1]$ satisfying the following property: Given $\theta \in [0, \theta_0]$, $T > 0$, $(u_0, v_0) \in \mathbb{H}_{\sigma}^1(\Omega) \times \mathbb{L}^2(\Omega)$, $f \in L^2(0, T; \mathbb{L}^2(\Omega))$ and $g \in L^2(0, T; \mathbb{H}^{-1}(\Omega))$, there exist a $T_* \in (0, T]$ and a unique solution (u, v) to our system on the time interval $(0, T_*)$ with*

$$\begin{aligned} u &\in C([0, T_*]; \mathbb{H}_{\sigma}^1(\Omega)) \cap W^{1,2}(0, T_*; \mathbb{L}_{\sigma}^2(\Omega)), \\ v &\in C([0, T_*]; \mathbb{L}^2(\Omega)) \cap W^{1,2}(0, T_*; \mathbb{H}^{-1}(\Omega)). \end{aligned}$$

4 Sketch of a proof of Theorem

Our idea for proof is the following: First, solve a linear problem

$$\begin{aligned} \frac{du}{dt} + (\mu + \chi)Au &= Pf + h \quad \text{in } \mathbb{L}_\sigma^2(\Omega), \\ \frac{dv}{dt} + Lv &= g - \frac{\theta}{2}\nabla \times f + k \quad \text{in } \mathbb{H}^{-1}(\Omega), \\ u(0) &= u_0, \quad v(0) = v_0, \end{aligned}$$

for given $h \in L^2(0, T; \mathbb{L}_\sigma^2(\Omega))$ and $k \in L^2(0, T; \mathbb{H}^{-1}(\Omega))$. It is well-known that there is a unique solution (u, v) such that

$$\begin{aligned} u &\in C([0, T]; \mathbb{H}_\sigma^1(\Omega)) \cap W^{1,2}(0, T; \mathbb{L}_\sigma^2(\Omega)), \\ v &\in C([0, T]; \mathbb{L}^2(\Omega)) \cap W^{1,2}(0, T; \mathbb{H}^{-1}(\Omega)). \end{aligned}$$

Then one can define a mapping S from $L^2(0, T; \mathbb{L}_\sigma^2(\Omega)) \times L^2(0, T; \mathbb{H}^{-1}(\Omega))$ into itself as $S(h, k) := (b_1(u, v), b_2(u, v))$. If S has a fixed point, then (u, v) is a solution of our problem. Since b_1 and b_2 are sums of terms which are linear or quadratic in u and v , it is natural to expect that S would be a contraction mapping. This conjecture turns out to be true with the following trick. Let $\eta \in (0, 1]$ be a constant to be fixed later and $U := \eta u$. Then our system becomes as follows.

$$\begin{aligned} \frac{dU}{dt} + (\mu + \chi)AU &= \eta Pf + B_1(U, v) \quad \text{in } \mathbb{L}_\sigma^2(\Omega), \\ \frac{dv}{dt} + Lv &= g - \frac{\theta}{2}\nabla \times f + B_2(U, v) \quad \text{in } \mathbb{H}^{-1}(\Omega), \\ U(0) &= \eta u_0, \quad v(0) = v_0, \end{aligned}$$

where $B_1(U, v)$ and $B_2(U, v)$ are defined by

$$\begin{aligned} B_1(U, v) &:= 2\eta\chi\nabla \times v + \theta\chi AU - \frac{1}{\eta}P(U \cdot \nabla)U, \\ B_2(U, v) &:= -\frac{\theta}{2\eta}(\mu + (1 - \theta)\chi - \alpha)\nabla \times \Delta U - \frac{\theta}{2\eta^2}((\nabla \times U) \cdot \nabla)U \\ &\quad - \frac{1}{\eta}(U \cdot \nabla)v + \frac{2(1 - \theta)\chi}{\eta}\nabla \times U. \end{aligned}$$

We are going to show this modified system of equations has a solution (U, v) .

Let data $T > 0$, u_0 , v_0 , f and g be given and take a positive number R satisfying

$$R \geq \max\{\|u_0\|_\sigma, |v_0|, \|f\|_{L^2(0, T; \mathbb{L}^2(\Omega))}, \|g\|_{L^2(0, T; \mathbb{H}^{-1}(\Omega))}\}.$$

Let τ be a positive number in $(0, T]$ and is also to be fixed later. Denote by $\mathcal{B}_R$ the set of functions (h, k) such that $h \in L^2(0, \tau; \mathbb{L}_\sigma^2(\Omega))$ and $k \in L^2(0, \tau; \mathbb{H}^{-1}(\Omega))$ with $\|h\|_{L^2(0, \tau; \mathbb{L}_\sigma^2(\Omega))} \leq R$ and $\|k\|_{L^2(0, \tau; \mathbb{H}^{-1}(\Omega))} \leq R$.

It is well-known that there is a unique solution (U, v) to the problem

$$\frac{dU}{dt} + (\mu + \chi)AU = \eta Pf + h \quad \text{in } \mathbb{L}_\sigma^2(\Omega), \tag{5}$$

$$\frac{dv}{dt} + Lv = g - \frac{\theta}{2}\nabla \times f + k \quad \text{in } \mathbb{H}^{-1}(\Omega), \tag{6}$$

$$U(0) = \eta u_0, \quad v(0) = v_0, \tag{7}$$

which satisfies

$$\begin{aligned} U &\in C([0, \tau]; \mathbb{H}_\sigma^1(\Omega)) \cap L^2(0, \tau; \mathbb{H}^2(\Omega) \cap \mathbb{H}_\sigma^1(\Omega)) \cap W^{1,2}(0, \tau; \mathbb{L}_\sigma^2(\Omega)), \\ v &\in C([0, \tau]; \mathbb{L}^2(\Omega)) \cap L^2(0, \tau; \mathbb{H}_0^1(\Omega)) \cap W^{1,2}(0, \tau; \mathbb{H}^{-1}(\Omega)). \end{aligned}$$

Multiplying (5) by U , we get

$$\frac{1}{2} \frac{d}{dt} \|U\|_\sigma^2 + (\mu + \chi) \|U\|_\sigma^2 \leq C(|f| + |h|_\sigma) \|U\|_\sigma.$$

Here and henceforth C or C_i (i is a positive number) denotes a constant which may depend only on $\mu, \chi, \alpha, \beta, \Omega$ and may take different values line by line. Then we have

$$\begin{aligned} \|U\|_{L^\infty(0, \tau; \mathbb{L}_\sigma^2(\Omega))} &\leq C_1 R, \\ \|U\|_{L^2(0, \tau; \mathbb{H}_\sigma^1(\Omega))} &\leq C_2 R. \end{aligned}$$

Multiplying (5) by AU , we have

$$\frac{1}{2} \frac{d}{dt} \|U\|_\sigma^2 + (\mu + \chi) |AU|_\sigma^2 \leq C(|f| + |h|_\sigma) |AU|_\sigma,$$

whence follows

$$\begin{aligned} \|U\|_{L^\infty(0, \tau; \mathbb{H}_\sigma^1(\Omega))} &\leq C_3 R, \\ \|U\|_{L^2(0, \tau; \mathbb{H}_\sigma^1(\Omega) \cap \mathbb{H}^2(\Omega))} &\leq C_4 R. \end{aligned}$$

We here use the estimate from the elliptic regularity theory:

$$\|w\|_{\mathbb{H}^2(\Omega)} \leq C_0 \|Aw\|_\sigma,$$

which holds for $w \in D(A)$.

Taking the duality pairing between (6) and v , we obtain

$$\frac{1}{2} \frac{d}{dt} |v|^2 + \alpha_* \|v\|^2 + 4\chi |v|^2 \leq C(\|g\|_* + |f| + |k|) \|v\|.$$

From this it follows that

$$\begin{aligned} \|v\|_{L^\infty(0, \tau; \mathbb{L}^2(\Omega))} &\leq C_5 R, \\ \|v\|_{L^2(0, \tau; \mathbb{H}_0^1(\Omega))} &\leq C_6 R. \end{aligned}$$

Now we shall show η , θ and τ can be chosen so that $(B_1(U, v), B_2(U, v))$ also belongs to the set $\mathcal{B}_R$.

Let ϕ and ψ be scalar functions. $D\phi$ denotes any one of the partial derivative of ϕ . We need the following well-known inequalities in order to estimate the nonlinear terms.

If $\phi \in H_0^1(\Omega)$, we have

$$\begin{aligned} \|\phi\|_{L^3(\Omega)} &\leq C \|\phi\|_{L^2(\Omega)}^{1/2} \|\phi\|_{H^1(\Omega)}^{1/2}, \\ \|\phi\|_{L^6(\Omega)} &\leq C \|\phi\|_{H^1(\Omega)}. \end{aligned}$$

If we assume further that $\phi \in H^2(\Omega)$, then $\phi \in L^\infty(\Omega)$ and

$$\|\phi\|_{L^\infty(\Omega)} \leq C\|\phi\|_{H^1(\Omega)}^{1/2}\|\phi\|_{H^2(\Omega)}^{1/2}.$$

If $\phi \in H_0^1(\Omega)$ and $\psi \in H^2(\Omega)$ or $\phi \in H^2(\Omega)$ and $\psi \in H_0^1(\Omega)$, the product $\phi D\psi$ belongs to $L^2(\Omega)$ and

$$\|\phi D\psi\|_{L^2} \leq \begin{cases} C\|\phi\|_{H^1}^{1/2}\|\phi\|_{H^2}^{1/2}\|\psi\|_{H^1} & \text{for } \phi \in H^1(\Omega), \psi \in H^2(\Omega), \\ C\|\phi\|_{H^1}\|\psi\|_{H^2}^{1/2}\|\psi\|_{H^1}^{1/2} & \text{for } \phi \in H^2(\Omega), \psi \in H^1(\Omega). \end{cases} \quad (8)$$

From these estimate, we obtain

$$\begin{aligned} \int_0^\tau |P(U \cdot \nabla)U(s)|_\sigma^2 ds &\leq \int_0^\tau |(U \cdot \nabla)U(s)|^2 ds \\ &\leq C_7 \int_0^\tau \|U(s)\|^3 \|U(s)\|_{\mathbf{H}^2} ds \\ &\leq C_3^3 C_7 R^3 \int_0^\tau \|U(s)\|_{\mathbf{H}^2} ds \\ &\leq C_3^3 C_7 R^3 \tau^{1/2} \|U\|_{L^2(0,\tau;\mathbf{H}^2(\Omega) \cap \mathbf{H}_\sigma^1(\Omega))} \\ &\leq C_3^3 C_4 C_7 R^4 \tau^{1/2}. \end{aligned}$$

Therefore

$$\begin{aligned} &\|B_1(U, v)\|_{L^2(0,\tau;\mathbf{L}_\sigma^2(\Omega))} \\ &\leq 2\eta\chi\|\nabla \times v\|_{L^2(0,\tau;\mathbf{L}_\sigma^2(\Omega))} + \theta\chi\|AU\|_{L^2(0,\tau;\mathbf{L}_\sigma^2(\Omega))} + \frac{1}{\eta}\|(U \cdot \nabla)U\|_{L^2(0,\tau;\mathbf{L}^2(\Omega))} \\ &\leq \left(2C_6\eta\chi + C_4\theta\chi + \frac{C_3^{3/2}C_4^{1/2}C_7^{1/2}R\tau^{1/4}}{\eta} \right) R. \end{aligned}$$

Suppose that $w_1, w_2, w_3 \in \mathbf{H}^1(\Omega)$, $\nabla \cdot w_1 = 0$ and at least one of these functions vanishes on the boundary $\partial\Omega$. Then $((w_1 \cdot \nabla)w_2, w_3)$ is well-defined and it holds that $((w_1 \cdot \nabla)w_2, w_3) = -((w_1 \cdot \nabla)w_3, w_2)$.

For $w \in \mathbf{H}_0^1(\Omega)$ we have

$$\begin{aligned} |\langle \nabla \times \Delta U, w \rangle| &= |(\Delta U, \nabla \times w)| \\ &\leq C_8 \|U\|_{\mathbf{H}^2(\Omega)} \|w\|, \\ |(\langle (\nabla \times U) \cdot \nabla \rangle U, w)| &= |(\langle (\nabla \times U) \cdot \nabla \rangle U, w)| \\ &= | - ((\nabla \times U) \cdot \nabla) w, U | \\ &\leq C \|\nabla U\|_{\mathbf{L}^3(\Omega)} \|\nabla w\|_{\mathbf{L}^2(\Omega)} \|U\|_{\mathbf{L}^6(\Omega)} \\ &\leq C_9 \|U\|_\sigma^{3/2} \|U\|_{\mathbf{H}^2}^{1/2} \|w\|, \\ |(\langle (U \cdot \nabla)v, w \rangle)| &= |(\langle (U \cdot \nabla)v, w \rangle)| \\ &= | - ((U \cdot \nabla)w, v) | \\ &\leq C \|U\|_{\mathbf{L}^6(\Omega)} \|\nabla w\|_{\mathbf{L}^2(\Omega)} \|v\|_{\mathbf{L}^3(\Omega)} \\ &\leq C_{10} \|U\|_\sigma \|v\|^{1/2} \|v\|^{1/2} \|w\| \\ |\langle \nabla \times U, w \rangle| &\leq |U|_\sigma \|w\|, \end{aligned}$$

and further

$$\begin{aligned} & \|B_2(U, v)\|_{L^2(0, \tau; \mathbb{H}^{-1}(\Omega))} \\ & \leq \left[\frac{\theta}{2\eta} C_4 C_8 (\mu + \chi + \alpha) + \frac{\theta}{2\eta^2} C_3^{3/2} C_7^{1/2} C_9 R \tau^{1/4} \right. \\ & \quad \left. + \frac{C_3 C_5^{1/2} C_6^{1/2} C_{10} R \tau^{1/4}}{\eta} + \frac{2\chi C_1 \tau^{1/2}}{\eta} \right] R. \end{aligned}$$

Next, let (h_i, k_i) ($i = 1, 2$) be taken from $\mathcal{B}_R$ and (U_i, v_i) ($i = 1, 2$) be the solution of

$$\begin{aligned} & \frac{dU_i}{dt} + (\mu + \chi) A U_i = \eta P f + h_i, \\ & \frac{dv_i}{dt} + L v_i = g - \frac{\theta}{2} \nabla \times f + k_i, \\ & U_i(0) = \eta u_0, \quad v_i(0) = v_0. \end{aligned}$$

Then it is easy to see that the differences $\tilde{U} := U_1 - U_2$ and $\tilde{v} := v_1 - v_2$ can be estimated as

$$\begin{aligned} & \|\tilde{U}\|_{L^\infty(0, \tau; \mathbb{L}_\sigma^2(\Omega))} \leq C_{11} \|\tilde{h}\|_{L^2(0, \tau; \mathbb{L}_\sigma^2(\Omega))}, \\ & \|\tilde{U}\|_{L^2(0, \tau; \mathbb{H}_\sigma^1(\Omega))} \leq C_{12} \|\tilde{h}\|_{L^2(0, \tau; \mathbb{L}_\sigma^2(\Omega))}, \\ & \|\tilde{U}\|_{L^\infty(0, \tau; \mathbb{H}_\sigma^1(\Omega))} \leq C_{13} \|\tilde{h}\|_{L^2(0, \tau; \mathbb{L}_\sigma^2(\Omega))}, \\ & \|\tilde{U}\|_{L^2(0, \tau; \mathbb{H}_\sigma^1(\Omega) \cap \mathbb{H}^2(\Omega))} \leq C_{14} \|\tilde{h}\|_{L^2(0, \tau; \mathbb{L}_\sigma^2(\Omega))}, \\ & \|\tilde{v}\|_{L^\infty(0, \tau; \mathbb{L}^2(\Omega))} \leq C_{15} \|\tilde{k}\|_{L^2(0, \tau; \mathbb{H}^{-1}(\Omega))}, \\ & \|\tilde{v}\|_{L^2(0, \tau; \mathbb{H}_0^1(\Omega))} \leq C_{16} \|\tilde{k}\|_{L^2(0, \tau; \mathbb{H}^{-1}(\Omega))}. \end{aligned}$$

Since

$$B_1(U_1, v_1) - B_1(U_2, v_2) = 2\eta \chi \nabla \times \tilde{v} + \theta \chi A \tilde{U} - \frac{1}{\eta} [P(U_1 \cdot \nabla) \tilde{U} + P(\tilde{U} \cdot \nabla) U_2],$$

then

$$\begin{aligned} & \|B_1(U_1, v_1) - B_1(U_2, v_2)\|_{L^2(0, \tau; \mathbb{L}_\sigma^2(\Omega))} \\ & \leq 2\eta \chi C_{16} \|\tilde{k}\|_{L^2(0, \tau; \mathbb{H}^{-1}(\Omega))} + \left[\theta \chi C_{14} + \frac{2C_3 C_{13}^{1/2} C_{14}^{1/2} R \tau^{1/4}}{\eta} \right] \|\tilde{h}\|_{L^2(0, \tau; \mathbb{L}_\sigma^2(\Omega))}. \end{aligned}$$

Similarly we obtain

$$\begin{aligned} & \|B_2(U_1, v_1) - B_2(U_2, v_2)\|_{L^2(0, \tau; \mathbb{H}^{-1}(\Omega))} \\ & \leq \left[\frac{\theta}{2\eta} C_{14} (\mu + \chi + \alpha) + \frac{C_2^{1/2} C_6^{1/2} C_{13} R \tau^{1/4}}{\eta} + \frac{\theta C_3^{1/2} C_4^{1/2} C_{13} R \tau^{1/4}}{\eta^2} \right. \\ & \quad \left. + \frac{2\chi C_{11} \tau^{1/2}}{\eta} \right] \|\tilde{h}\|_{L^2(0, \tau; \mathbb{L}_\sigma^2(\Omega))} + \frac{C_3 C_{15}^{1/2} C_{16}^{1/2} R \tau^{1/4}}{\eta} \|\tilde{k}\|_{L^2(0, \tau; \mathbb{H}^{-1}(\Omega))}. \end{aligned}$$

Now, set the number $\eta \in (0, 1]$ so that the following inequalities hold:

$$2C_6 \chi \eta \leq \frac{1}{2}, \quad 2C_{16} \chi \eta \leq \frac{1}{4}.$$

After that, chose $\theta \in (0, 1]$ and $\tau \in (0, T]$ so small that

$$\begin{aligned} \|B_1(U, v)\|_{L^2(0, \tau; \mathbf{L}_\sigma^2(\Omega))} &\leq R, \\ \|B_2(U, v)\|_{L^2(0, \tau; \mathbf{H}^{-1}(\Omega))} &\leq R, \\ \|B_1(U, v) - B_1(U, v)\|_{L^2(0, \tau; \mathbf{L}_\sigma^2(\Omega))} &\leq \frac{1}{2} \|\tilde{h}\|_{L^2(0, \tau; \mathbf{L}_\sigma^2(\Omega))}, \\ \|B_2(U, v) - B_2(U, v)\|_{L^2(0, \tau; \mathbf{H}^{-1}(\Omega))} &\leq \frac{1}{2} \|\tilde{k}\|_{L^2(0, \tau; \mathbf{H}^{-1}(\Omega))}. \end{aligned}$$

Thus the mapping $(h, k) \mapsto (B_1(U, v), B_2(U, v))$ turns out to be a contraction, and the existence of a solution to our problem follows. The uniqueness of solution (U, v) logically follows from the above argument.

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